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Partial differential equations / *Équations aux dérivées partielles*

A bifurcation-type result for Kirchhoff equations

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Abstract. This paper deals with a class of Kirchhoff type elliptic Dirichlet boundary value problems where the combined effects of Kirchhoff term and nonlinear term allow us to establish a bifurcation-type result as the positive parameter varies.

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1. Introduction and main result

Let Ω be a smooth bounded domain in \mathbb{R}^N , $N \geq 1$, and $2^* = \begin{cases} \frac{2N}{N-2}, & N \geq 3, \\ +\infty, & N = 1, 2. \end{cases}$ We consider the following Kirchhoff type boundary value problems

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

where $a > 0, b > 0$ and $g: \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R}$ is a suitable continuous function.

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When $N = 1$, $\Omega = (0, L)$, $a = 1$ and $g(x, u) = g(x)$, solutions of Eq. (1) are related to the stationary states of the hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - \left(1 + b \int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right) \frac{\partial^2 u}{\partial x^2} = g, \quad u(0, t) = u(L, t) = 0,$$

which was proposed by Kirchhoff in 1883 to describe the small transversal oscillations of an elastic clamped string. For this reason, Eq. (1) is often called Kirchhoff equation.

Over the past 20 years Eq. (1) has been extensively studied by using variational methods, see for example, [1, 2, 6–10, 12] and the references therein. See also [11] and references therein for a broad survey.

In particular, Ambrosetti and Arcoya investigated the following special case of Eq. (1)

$$\begin{cases} -\left(a + b \int_{\Omega} |\nabla u|^2 dx\right) \Delta u = |u|^{p-2} u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $2 < p < \min\{4, 2^*\}$, and got that

- (I) there exists $\gamma_0 > 0$ such that for any $b \in (0, \gamma_0)$, I_b has a mountain pass critical point u_1 with $I_b(u_1) > 0$, as well as a global minimum u_2 with $I_b(u_2) < 0$, see [2, Theorem 4.4], where I_b is the Euler functional associated with Eq. (2);
- (II) if b is large, then Eq. (2) has no nontrivial solution, see [2, Remark 4.5 (ii)].

Conclusions (I) and (II) motivate us to study bifurcation-type results on Eq. (2) as the parameter b varies. This is the object of this paper to which we give a positive answer.

Define

$$S_p = \inf_{u \in H \setminus \{0\}} \frac{\|u\|^2}{|u|_p^2}.$$

where $\|u\|^2 = \int_{\Omega} |\nabla u|^2 dx$ and $|u|_p^p = \int_{\Omega} |u|^p dx$ denote the usual norms of u in $H := H_0^1(\Omega)$ and $L^p(\Omega)$ respectively. Then

$$\int_{\Omega} |u|^p dx \leq A_p \|u\|^p \text{ for all } u \in H, \tag{3}$$

where $A_p = S_p^{-\frac{p}{2}}$. Define

$$\Lambda = \frac{p-2}{2} A_p^{\frac{2}{p-2}} \left(\frac{4-p}{2a}\right)^{\frac{4-p}{p-2}}.$$

By using the variational methods, we obtain the following bifurcation-type theorem.

Theorem 1. *Suppose that $a > 0, b > 0, 2 < p < \min\{4, 2^*\}$. Then*

- (i) *Eq. (2) has no nontrivial solutions for any $b > \Lambda$;*
- (ii) *Eq. (2) has at least two positive solutions for any $b < \Lambda$;*
- (iii) *Eq. (2) has at least a positive solution for $b = \Lambda$.*

According to Remark 7 below, we can give the rough graphs of the mountain pass value c_b and the local minimum value m_b .

From [2] we know that for any $b \in (0, \gamma_0)$, I_b possesses a global minimum. But if $b \in (0, \Lambda)$ is close to Λ , then 0 is a global minimum. So we need to select an appropriate constraint in order to obtain a local minimum.

The Euler functional associated with Eq. (2) is

$$I_b(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{p} \int_{\Omega} |u|^p dx.$$

Obviously, I_b is of class C^1 and

$$\langle I'_b(u), v \rangle = (a + b\|u\|^2) \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\Omega} |u|^{p-2} uv dx$$

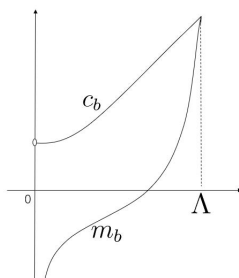


Figure 1.

for all $u, v \in H$.

2. Proof of Theorem 1

To prove Theorem 1, five lemmas are in order.

Lemma 2. *Suppose that $b > 0$ and $2 < p < \min\{4, 2^*\}$. Then I_b is coercive on H .*

Proof. From (3) we have

$$I_b(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{p} \int_{\Omega} |u|^p dx$$

$$\geq \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{A_p}{p} \|u\|^p.$$

Since $2 < p < \min\{4, 2^*\}$, I_b is coercive on H . □

Lemma 3. *Suppose that $a > 0, 0 < b < \Lambda, 2 < p < \min\{4, 2^*\}$ and*

$$T_b = \left[\frac{(p-2)A_p}{2b} \right]^{\frac{1}{4-p}}.$$

Then there exist $t_b \in (0, T_b)$ and $t'_b \in (T_b, +\infty)$ such that all nontrivial solution of Eq. (2) belongs to the set $\{u \in H : t_b \leq \|u\| \leq t'_b\}$. Moreover, if $0 < b_1 < b_2 < \Lambda$, then $t_{b_1} < t_{b_2}$ and $M_{b_2} \subset M_{b_1}$, where $M_b := \{u \in H : \|u\| \geq t_b\}$.

Proof. Suppose that $u \in H$ is a nontrivial solution of Eq. (2). Then combining with (3) we have

$$a\|u\|^2 + b\|u\|^4 = \int_{\Omega} |u|^p dx \leq A_p \|u\|^p.$$

Thus

$$b\|u\|^2 - A_p \|u\|^{p-2} + a \leq 0.$$

For $t > 0$, define $h_b(t) = bt^2 - A_p t^{p-2} + a$. Then $h'_b(T_b) = 2bT_b - (p-2)A_p T_b^{p-3} = 0$, $h'_b(t) < 0$ for all $t \in (0, T_b)$ and $h'_b(t) > 0$ for all $t \in (T_b, +\infty)$.

When $b < \Lambda$, $h_b(T_b) = T_b^{p-2}(bT_b^{4-p} - A_p) + a < 0$. Combining with $h_b(0) = a > 0$ and $\lim_{t \rightarrow +\infty} h_b(t) = +\infty$ implies that there exist $t_b \in (0, T_b)$ and $t'_b \in (T_b, +\infty)$ such that $h_b(t_b) = h_b(t'_b) = 0$, $h_b(t) > 0$ for all $t \in (0, t_b) \cup (t'_b, +\infty)$ and $h_b(t) < 0$ for all $t \in (t_b, t'_b)$. So $t_b \leq \|u\| \leq t'_b$.

If $0 < b_1 < b_2 < \Lambda$, then $h_{b_1}(t_{b_2}) < h_{b_2}(t_{b_2}) = 0$. So there exists $t_{b_1} \in (0, t_{b_2})$ such that $h_{b_1}(t_{b_1}) = 0$ and $h_{b_1}(t) > 0$ for all $t \in (0, t_{b_1})$. Hence $M_{b_2} \subset M_{b_1}$. □

From Lemma 2, $m_b := \inf_{u \in M_b} I_b(u)$ is well defined. We consider the constraint problem $m_b = \inf_{u \in M_b} I_b(u)$ in order to obtain a local minimum solution of Eq. (2).

Lemma 4. *Suppose that $a > 0, 0 < b < \Lambda$ and $2 < p < \min\{4, 2^*\}$. Then I_b satisfies mountain pass geometry, i.e., there exist $\alpha_b > 0$ and $v_b \in H$ such that $\|v_b\| > t_b$ and $\inf_{\|u\|=t_b} I_b(u) \geq \alpha_b > I_b(v_b)$. Moreover, $I_b(v_b) \geq m_b$.*

Proof. From (3) we have

$$\begin{aligned} \inf_{\|u\|=t_b} I_b(u) &= \inf_{\|u\|=t_b} \left(\frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{p} \int_{\Omega} |u|^p dx \right) \\ &\geq \inf_{\|u\|=t_b} \left(\frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{A_p}{p} \|u\|^p \right) \\ &= \frac{at_b^2}{2} + \frac{bt_b^4}{4} - \frac{A_p t_b^p}{p}. \end{aligned}$$

For $t > 0$, define $f_b(t) = \frac{at^2}{2} + \frac{bt^4}{4} - \frac{A_p t^p}{p}$. Then $f'_b(t) = at + bt^3 - A_p t^{p-1} = t h_b(t) > 0$ for all $t \in (0, t_b) \cup (t'_b, +\infty)$ and $f'_b(t) < 0$ for all $t \in (t_b, t'_b)$. Thus $f_b(t_b) > f_b(0) = 0$ and $f_b(t_b) > f_b(T_b) > f_b(t)$ for all $t \in (T_b, t'_b]$.

The infimum S_p can be achieved by a positive function $e \in H$ and $\int_{\Omega} |e|^p dx = A_p \|e\|^p$. Let $v_b = \frac{T_b}{\|e\|} e$. Then $\|v_b\| = T_b > t_b$ and

$$I_b(v_b) = \frac{aT_b^2}{2} + \frac{bT_b^4}{4} - \frac{A_p T_b^p}{p} = f_b(T_b).$$

Hence $\inf_{\|u\|=t_b} I_b(u) \geq \alpha_b := f_b(t_b) > f_b(T_b) = I_b(v_b) \geq m_b$. □

Lemma 5. *Suppose that $a > 0, 0 < b < \Lambda$ and $2 < p < \min\{4, 2^*\}$. Then there exists $\{u_n\} \subset M_b$ such that $I_b(u_n) \rightarrow m_b$ and $I'_b(u_n) \rightarrow 0$.*

Proof. By Ekeland’s variational principle [5], there exists $u_n \in M_b$ such that

$$m_b \leq I_b(u_n) \leq m_b + \frac{1}{n}$$

and for any $v \in M_b$,

$$I_b(v) - I_b(u_n) \geq -\frac{1}{n} \|u_n - v\|.$$

Thus $I_b(u_n) \rightarrow m_b$. From Lemma 4 we know that for n large enough, $\|u_n\| > t_b$. For any $h \in H$ and $\|h\| = 1$, let $v = u_n + th$ for t small enough, we have

$$\langle I'_b(u_n), h \rangle = \lim_{t \rightarrow 0^+} \frac{I_b(u_n + th) - I_b(u_n)}{t} \geq -\frac{1}{n}.$$

Similarly,

$$\langle I'_b(u_n), -h \rangle = \lim_{t \rightarrow 0^+} \frac{I_b(u_n - th) - I_b(u_n)}{t} \geq -\frac{1}{n}.$$

Then

$$|\langle I'_b(u_n), h \rangle| \leq \frac{1}{n} = o(1).$$

According to the arbitrariness of h we have $I'_b(u_n) \rightarrow 0$. □

Lemma 6. *Suppose that $a > 0, b > 0$ and $2 < p < \min\{4, 2^*\}$. If $\{u_n\}$ satisfies $\sup_n I_b(u_n) < +\infty$ and $I'_b(u_n) \rightarrow 0$, then $\{u_n\}$ contains a convergent subsequence.*

Proof. Since $\{u_n\}$ satisfies $\sup_n I_b(u_n) < +\infty$, from Lemma 2 we have $\{u_n\}$ is bounded in H . Then there exists $u \in H$ such that up to a subsequence, $u_n \rightharpoonup u$ in H , $u_n \rightarrow u$ in $L^p(\Omega)$ and $u_n(x) \rightarrow u(x)$ a.e. in Ω . Combining with the Hölder inequality implies

$$\begin{aligned} \left| \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx \right| &\leq \int_{\Omega} |u_n|^{p-1} |u_n - u| dx \\ &\leq |u_n|_p^{p-1} |u_n - u|_p \\ &= o(1). \end{aligned}$$

Note that

$$\int_{\Omega} \nabla u_n \cdot \nabla u dx = \int_{\Omega} |\nabla u|^2 dx + o(1)$$

and

$$o(1) = \langle I'_b(u_n), u_n - u \rangle = (a + b \|u_n\|^2) \int_{\Omega} \nabla u_n \cdot \nabla (u_n - u) dx - \int_{\Omega} |u_n|^{p-2} u_n (u_n - u) dx.$$

Thus $\|u_n\| \rightarrow \|u\|$. Therefore, $u_n \rightarrow u$ in H . □

Proof of Theorem 1.

Conclusion (i). Eq. (2) has no nontrivial solutions for any $b > \Lambda$.

Suppose that $u \in H$ is a nontrivial solution of Eq. (2). Then using (3) and the Young inequality we have

$$\begin{aligned} a \|u\|^2 + b \|u\|^4 &= \int_{\Omega} |u|^p dx \\ &\leq A_p \|u\|^p \\ &= \left(\frac{2a}{4-p} \right)^{\frac{4-p}{2}} \|u\|^{4-p} A_p \left(\frac{4-p}{2a} \right)^{\frac{4-p}{2}} \|u\|^{2p-4} \\ &\leq \frac{4-p}{2} \frac{2a}{4-p} \|u\|^2 + \frac{p-2}{2} A_p^{\frac{2}{p-2}} \left(\frac{4-p}{2a} \right)^{\frac{4-p}{p-2}} \|u\|^4 \\ &= a \|u\|^2 + \frac{p-2}{2} A_p^{\frac{2}{p-2}} \left(\frac{4-p}{2a} \right)^{\frac{4-p}{p-2}} \|u\|^4. \end{aligned} \tag{4}$$

So $b \leq \Lambda$. Conclusion (i) holds.

Conclusion (ii). Eq. (2) has at least two positive solutions for any $b < \Lambda$.

From Lemma 5 and Lemma 6 we have that there exists $u \in M_b$ such that $I_b(u) = m_b$ and $I'_b(u) = 0$. Note that $I_b(|u|) = I_b(u) = m_b$ and $\| |u| \| = \|u\| > t_b$. For any $h \in H$,

$$\langle I'_b(|u|), h \rangle = \lim_{t \rightarrow 0^+} \frac{I_b(|u| + th) - I_b(|u|)}{t} \geq 0$$

and

$$\langle I'_b(|u|), -h \rangle = \lim_{t \rightarrow 0^+} \frac{I_b(|u| - th) - I_b(|u|)}{t} \geq 0.$$

Then $I'_b(|u|) = 0$. That $|u| > 0$ follows from the strong maximum principle. In fact, $|u|$ is a ground state solution. Indeed, from Lemma 3 one has $v \in M_b$ for any $v \in \{v \in H \setminus \{0\} : I'_b(v) = 0\}$ and then $I_b(v) \geq m_b = I_b(|u|)$.

According to Lemma 4, we define

$$c_b = \inf_{\gamma \in \Gamma_b} \max_{t \in [0,1]} I_b(\gamma(t)),$$

where

$$\Gamma_b = \{\gamma \in C([0, 1], H) : \gamma(0) = 0, \gamma(1) = v_b\}.$$

Then $c_b \geq \alpha_b > m_b$ and combining with Lemma 6, we know that c_b is a critical value by the mountain pass lemma [3]. From Theorem 10 in [4] we want to state positivity property of the

mountain pass solution. We take $p(u) = |u|$ in [4, Theorem 10] and obtain a critical point $u \geq 0$. The strong maximum principle implies $u > 0$. Conclusion (ii) holds.

Conclusion (iii). Eq. (2) has at least a positive solution for $b = \Lambda$.

Because S_p can be achieved by some positive normalized function $v \in H$, i.e., $S_p = \|v\|^2$ and $|v|_p = 1$, $S_p^{\frac{1}{p-2}}$ v is a positive solution of the following semi-linear elliptic equation

$$\begin{cases} -\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{5}$$

By scaling the function $w := \left(\frac{4-p}{2a}\right)^{\frac{1}{2-p}} S_p^{\frac{1}{p-2}} v$ solves the following equation

$$\begin{cases} -\frac{2a}{4-p}\Delta u = |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, by calculating one has $a + b\|w\|^2 = \frac{2a}{4-p}$. It shows that w is a positive solution of Eq. (2). Conversely, if w is a positive solution of Eq. (2), then $(a + b\|w\|^2)^{\frac{1}{2-p}} w$ is the one of Eq. (5). Then when the positive solution of Eq. (5) is unique, the one of Eq. (2) is also unique, for example, when Ω is ball or p is close to 2. Conclusion (iii) holds. \square

Remark 7. We prove some properties of critical values m_b and c_b .

- (i) m_b and c_b are monotone increasing functions on $(0, \Lambda)$;
- (ii) $m_\Lambda := \lim_{b \rightarrow \Lambda^-} m_b$ and $c_\Lambda := \lim_{b \rightarrow \Lambda^-} c_b$ are critical values of Eq. (2) with $b = \Lambda$;
- (iii) $m_\Lambda = c_\Lambda = \frac{p-2}{4p} \left(\frac{2}{4-p}\right)^{\frac{2}{p-2}} (aS_p)^{\frac{p}{p-2}} = I_\Lambda(u)$ for all $u \in \{u \in H \setminus \{0\} : I'_\Lambda(u) = 0\}$;
- (iv) $\lim_{b \rightarrow 0^+} m_b = -\infty$ and $c_0 := \lim_{b \rightarrow 0^+} c_b > 0$.

Proof of (i). For any $b_1, b_2 \in (0, \Lambda)$ and $b_1 < b_2$, from Lemma 3 we have $M_{b_2} \subset M_{b_1}$. Then

$$m_{b_1} = \inf_{u \in M_{b_1}} I_{b_1}(u) \leq \inf_{u \in M_{b_2}} I_{b_1}(u) \leq \inf_{u \in M_{b_2}} I_{b_2}(u) = m_{b_2}.$$

Fix $b_2 \in (0, \Lambda)$. For any $\gamma \in \Gamma_{b_2}$, $\max_{0 \leq t \leq 1} I_{b_2}(\gamma(t)) \geq f_{b_2}(t_{b_2}) > f_{b_2}(T_{b_2}) > f_{b_2}(t)$ for all $t \in (T_{b_2}, t'_{b_2}]$, where the symbols are from Lemma 3 and 4. According to the definition of T_b , there exists $b_1 \in (0, b_2)$ such that $T_{b_2} < T_{b_1} < t'_{b_2}$. For any $b \in (b_1, b_2)$, we define

$$\tilde{\gamma}(t) = \begin{cases} \gamma(2t), & 0 \leq t \leq \frac{1}{2}, \\ \frac{[T_{b_2} + (2t-1)(T_b - T_{b_2})]e}{\|e\|}, & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then $\tilde{\gamma} \in \Gamma_b$, $T_{b_2} \leq T_{b_2} + (2t-1)(T_b - T_{b_2}) \leq T_b < T_{b_1} < t'_{b_2}$ for all $t \in [\frac{1}{2}, 1]$. Thereby, one has

$$\max_{\frac{1}{2} \leq t \leq 1} I_{b_2}(\tilde{\gamma}(t)) = \max_{\frac{1}{2} \leq t \leq 1} f_{b_2}(T_{b_2} + (2t-1)(T_b - T_{b_2})) \leq f_{b_2}(T_{b_2}) < \max_{0 \leq t \leq 1} I_{b_2}(\gamma(t))$$

and then

$$\max_{0 \leq t \leq 1} I_{b_2}(\gamma(t)) = \max_{0 \leq t \leq \frac{1}{2}} I_{b_2}(\tilde{\gamma}(t)) = \max_{0 \leq t \leq 1} I_{b_2}(\tilde{\gamma}(t)) > \max_{0 \leq t \leq 1} I_b(\tilde{\gamma}(t)) \geq c_b.$$

In view of the arbitrariness of γ , we have $c_{b_2} \geq c_b$.

Proof of (ii). For any $b \in (\frac{\Lambda}{2}, \Lambda)$, there exists a positive function $u_b \in H$ such that $I_b(u_b) = m_b$ and $I'_b(u_b) = 0$. Combining with (3) implies

$$\frac{\Lambda}{2} \|u_b\|^4 \leq a \|u_b\|^2 + b \|u_b\|^4 = \int_{\Omega} |u_b|^p dx \leq A_p \|u_b\|^p.$$

Thus $\|u_b\| \leq \left(\frac{2A_p}{\Lambda}\right)^{\frac{1}{4-p}}$. From

$$m_b = I_b(u_b) \leq \frac{a}{2}\|u_b\|^2 + \frac{\Lambda}{4}\|u_b\|^4$$

it follows that m_b has an upper bound on $(\frac{\Lambda}{2}, \Lambda)$. We set $m_\Lambda = \sup_{\frac{\Lambda}{2} < b < \Lambda} m_b$. For any $\varepsilon > 0$, there exists $b_\varepsilon \in (\frac{\Lambda}{2}, \Lambda)$ such that $m_\Lambda - \varepsilon < m_{b_\varepsilon} \leq m_\Lambda$. Take $\delta = \Lambda - b_\varepsilon$. When $\Lambda - \delta < b < \Lambda$, by using the monotonicity we have $m_\Lambda - \varepsilon < m_{b_\varepsilon} \leq m_b \leq m_\Lambda < m_\Lambda + \varepsilon$, i.e., m_b is left continuous at Λ ($m_\Lambda = \lim_{b \rightarrow \Lambda^-} m_b$). Let $\{b_n\} \subset (\frac{\Lambda}{2}, \Lambda)$ be an increasing sequence and $b_n \rightarrow \Lambda$. Then $m_{b_n} \rightarrow m_\Lambda$ and there exists a positive sequence $\{u_n\} \subset H$ such that $I_{b_n}(u_n) = m_{b_n}$ and $I'_{b_n}(u_n) = 0$. So $\|u_n\| \leq \left(\frac{2A_p}{\Lambda}\right)^{\frac{1}{4-p}}$ and then

$$I_\Lambda(u_n) = I_{b_n}(u_n) + \frac{\Lambda - b_n}{4}\|u_n\|^4 = m_{b_n} + \frac{\Lambda - b_n}{4}\|u_n\|^4 \rightarrow m_\Lambda,$$

$$\|I'_\Lambda(u_n)\|_* = \sup_{\|v\|=1} \langle I'_\Lambda(u_n), v \rangle = \sup_{\|v\|=1} (\Lambda - b_n)\|u_n\|^2 \int_\Omega \nabla u_n \cdot \nabla v \, dx \rightarrow 0.$$

From Lemma 6, we know that there exists a nonnegative $u \in H$ such that up to a subsequence, $u_n \rightarrow u$ in H . Hence $I_\Lambda(u) = m_\Lambda$ and $I'_\Lambda(u) = 0$. The strong maximum principle implies that u is positive.

The case of c_b is proved to be completely similar.

Proof of (iii). Let $I'_\Lambda(u) = 0$, then from (4) we have

$$a\|u\|^2 + \Lambda\|u\|^4 = \int_\Omega |u|^p \, dx = A_p\|u\|^p.$$

Let $h(t) = \Lambda t^2 - A_p t^{p-2} + a$ for $t > 0$. Then $h'(T) = 0$, $h'(t) < 0$ for all $t \in (0, T)$ and $h'(t) > 0$ for all $t \in (T, +\infty)$, where

$$T = \left[\frac{(p-2)A_p}{2\Lambda} \right]^{\frac{1}{4-p}}.$$

Since $h(T) = T^{p-2}(\Lambda T^{4-p} - A_p) + a = 0$, T is a unique positive solution of equation $\Lambda t^2 - A_p t^{p-2} + a = 0$. Thereby, for any $u \in \{u \in H \setminus \{0\} : I'_\Lambda(u) = 0\}$ we have $\|u\| = T$ and

$$\begin{aligned} I_\Lambda(u) &= I_\Lambda(u) - \frac{1}{p} \langle I'_\Lambda(u), u \rangle \\ &= \frac{(p-2)a}{2p} T^2 - \frac{(4-p)\Lambda}{4p} T^4 \\ &= \frac{p-2}{4p} \left(\frac{2}{4-p} \right)^{\frac{2}{p-2}} (a S_p)^{\frac{p}{p-2}}. \end{aligned}$$

Then

$$m_\Lambda = c_\Lambda = \frac{p-2}{4p} \left(\frac{2}{4-p} \right)^{\frac{2}{p-2}} (a S_p)^{\frac{p}{p-2}}.$$

Proof of (iv). From Lemma 3 we know that t_b is a monotone increasing function on $(0, \Lambda)$. Fix $w \in H$, $\|w\| = 1$. For any $L > 0$ there exists $B > t_{\frac{\Lambda}{2}}$ such that

$$I_0(Bw) = \frac{aB^2}{2}\|w\|^2 - \frac{B^p}{p} \int_\Omega |w|^p \, dx < -L - 1.$$

Then there exists $b_0 \in (0, \frac{\Lambda}{2})$ such that

$$I_{b_0}(Bw) = I_0(Bw) + \frac{b_0 B^4}{4}\|w\|^4 < -L.$$

So for any $b \in (0, b_0)$, we have $\|Bw\| = B > t_{\frac{\Lambda}{2}} > t_{b_0} > t_b$ and then

$$m_b = \inf_{u \in M_b} I_b(u) \leq I_b(Bw) \leq I_{b_0}(Bw) < -L,$$

i.e., $\lim_{b \rightarrow 0^+} m_b = -\infty$.

Since c_b is a monotone increasing function on $(0, \Lambda)$ and $c_b > 0$ for all $b \in (0, \Lambda)$, we obtain that $\lim_{b \rightarrow 0^+} c_b$ exists. In the following, we prove $c_0 := \lim_{b \rightarrow 0^+} c_b > 0$. We adopt the symbols of Lemma 3 and Lemma 4. Define $t_0 = \left(\frac{a}{A_p}\right)^{\frac{1}{p-2}}$. Then $t_0 < T_b$ and $h_b(t_0) = bt_0^2 > 0$. Since $t_b < T_b$, $h_b(t_b) = 0$ and $h'_b(t) < 0$ for all $t \in (0, T_b)$, we have $t_0 < t_b$. Combining with $f'_b(t) > 0$ for all $t \in (0, t_b)$ implies $f_b(t_b) > f_b(t_0)$. Then we get

$$\inf_{\|u\|=t_b} I_b(u) \geq \alpha_b = f_b(t_b) > f_b(t_0) \geq t_0^2 \left(\frac{a}{2} - \frac{A_p}{p} t_0^{p-2} \right) = \frac{p-2}{2p} \frac{a^{\frac{p}{p-2}}}{A_p^{\frac{2}{p-2}}} > 0.$$

Thereby $c_b \geq \alpha_b \geq \frac{p-2}{2p} \frac{a^{\frac{p}{p-2}}}{A_p^{\frac{2}{p-2}}}$. Hence $c_0 \geq \frac{p-2}{2p} \frac{a^{\frac{p}{p-2}}}{A_p^{\frac{2}{p-2}}} > 0$.

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