INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Mo Chen and Lionel Rosier

Reachable states for the distributed control of the heat equation

Volume 360 (2022), p. 627-639

<https://doi.org/10.5802/crmath.310>

CENTRE MERSENNE

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org

Comptes Rendus
Mathématique
2022, 360, p. 627-639
https://doi.org/10.5802/crmath.310

Control theory / *Théorie du contrôle*

Reachable states for the distributed control of the heat equation

Mo Chen [a] **and Lionel Rosier** [b]

[a] School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, 130024, P. R. China

[b] Université du Littoral Côte d'Opale, Laboratoire de Mathématiques Pures et Appliquées J. Liouville, BP 699, F-62228 Calais, France

E-mails: chenmochenmo.good@163.com, lionel.rosier@univ-littoral.fr

Abstract. We are concerned with the determination of the reachable states for the distributed control of the heat equation on an interval. We consider either periodic boundary conditions or homogeneous Dirichlet boundary conditions. We prove that for a L^2 distributed control, the reachable states are in the Sobolev space H^1 and that they have complex analytic extensions on squares whose horizontal diagonals are regions where no control is applied.

2020 Mathematics Subject Classification. 35K40, 93B05.

Funding. Most of this work was done when Mo Chen visited Université du Littoral Côte d'Opale in 2019. Mo Chen is supported by NSFC Grant (11701078) and China Scholarship Council. Lionel Rosier is partially supported by CAPES: n° 88881.520205/2020-01 and MATH AMSUD: 21-MATH-03.

Manuscript received 25th June 2021, revised 12th November 2021, accepted 7th December 2021.

1. Introduction

The null controllability of the heat equation has been investigated for a long time [4], and sharp results in dimension N were obtained in the nineties by using Carleman estimates [5, 9]. By contrast, sharp results for the exact controllability of the heat equation were obtained only recently. In [12], the authors noticed that for the boundary control of the heat equation on a real interval, the reachable space was sandwiched between the set of analytic functions on a ball and the set of analytic functions on a square. These results were improved in [2, 6], where the reachable space was sandwiched between two spaces of analytic functions on squares. The sharp result, derived in [7, 13], tells that the reachable space for the heat equation on the interval $(-1,1)$ with two boundary controls at $x = \pm 1$ taken in the space $L^2(0,T)$, is exactly the Bergman space of the functions that are both analytic and square integrable on the square $\Omega = \{x + iy;\ |x| + |y| < 1\}$. See also [8] for the reachable spaces of semilinear parabolic equations, and [11] and [1] for the reachable spaces of the linear Korteweg-de Vries equation and the linear Zakharov–Kuznetsov equation, respectively.

The present paper is concerned with the determination of the reachable space for the *distributed control* of the heat equation on an interval. Roughly, it is proved that when using L^2 distributed controls, the reachable states are H^1 in the control region and analytic elsewhere. At the same time, we shall provide a short proof of the main result in [2] by using the method introduced in [10] to construct backward solutions of the heat equation.

Let us review the main results in this paper. Consider first periodic boundary conditions. Let $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let $\chi_{(l_1,l_2)}(x) = 1$ if $x \in (l_1, l_2)$, 0 otherwise. We consider the control problem:

$$\begin{cases} w_t - w_{xx} = \chi_{(l_1,l_2)} u & (x,t) \in \mathbb{T} \times (0,T), \\ w(x,0) = 0 & x \in \mathbb{T}. \end{cases} \tag{1}$$

For $0 < L_1 < L_2 < 2\pi$, we introduce the open set

$$\mathscr{S}(L_1, L_2) = \{x + iy \in \mathbb{C};\ |y| < x - L_2 \text{ and } |y| < L_1 + 2\pi - x\}$$

which is a square with the interval $(L_2, L_1 + 2\pi)$ as a diagonal. Define

$$\mathscr{H}(L_1, L_2) = \left\{f \in H^1(\mathbb{T});\ f|_{(L_2, L_1+2\pi)} \text{ can be extended as an analytic function in } \mathscr{S}(L_1, L_2)\right\},$$

$$\mathscr{A}(L_1, L_2) = \left\{f \in \mathscr{H}(L_1, L_2);\ \int_{\mathscr{S}(L_1,L_2)} |f(x+iy)|^2 \mathrm{d}x\mathrm{d}y < \infty\right\}.$$

Note that any function $f \in \mathscr{H}(L_1, L_2)$ can be extended as an analytic function on $\cup_{k\in\mathbb{Z}} \mathscr{S}(L_1, L_2) + 2k\pi$, by 2π-periodicity of f. The following result is the first main result in this paper.

Theorem 1. *Let $T > 0$ and $0 < l_1 < l_2 < 2\pi$. Then*

(i) *for any $u \in L^2(0, T; L^2(\mathbb{T}))$, the solution w of system* (1) *satisfies $w(\cdot, T) \in \mathscr{A}(l_1, l_2)$;*
(ii) *for any $0 < \varepsilon < (l_2 - l_1)/2$ and any $w_T \in \mathscr{H}(l_1 + \varepsilon, l_2 - \varepsilon)$, there exists a control input $u \in L^2(0, T; L^2(\mathbb{T}))$ such that the solution w of system* (1) *satisfies $w(\cdot, T) = w_T$ in $\mathbb{T}$.*

Remark 2.

(1) The above result is "almost sharp", for ε can be taken as small as desired in the inclusion

$$\mathscr{H}(l_1 + \varepsilon, l_2 - \varepsilon) \subset \mathscr{A}(l_1, l_2).$$

(2) Having in mind the characterisation of the reachable space for the boundary control of the heat equation in [7], it is natural to conjecture that the reachable space for system (1) is the Bergman space $\mathscr{A}(l_1, l_2)$.

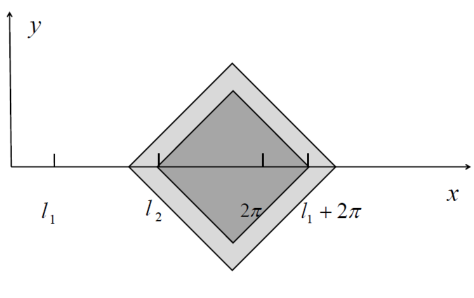

Figure 1. Reachable states for periodic boundary conditions.

Next, we consider distributed control systems on the interval $(0, 1)$ with homogeneous boundary conditions.[1]

[1] Homogeneous Neumann boundary conditions could be treated in a similar way.

Let $0 < l_1 < l_2 < 1$. For any given $u \in L^2(0,T,L^2(0,1))$, let w denote the solution of the control system

$$\begin{cases} w_t - w_{xx} = \chi_{(l_1,l_2)} u & (x,t) \in (0,1) \times (0,T), \\ w(0,t) = w(1,t) = 0 & t \in (0,T), \\ w(x,0) = 0 & x \in (0,1). \end{cases} \tag{2}$$

For any $L > 0$, we introduce the set

$$\mathscr{S}(L) = \{x + iy \in \mathbb{C};\ |x| + |y| < L\},$$

and the spaces

$$\mathscr{H}(L) = \left\{ f \in H^1(0,L);\ f \text{ can be extended as an odd analytic function on } \mathscr{S}(L) \right\},$$
$$\mathscr{A}(L) = \left\{ f \in \mathscr{H}(L);\ \int_{\mathscr{S}(L)} |f(x+iy)|^2 \mathrm{d}x\mathrm{d}y < \infty \right\}.$$

The following result is the second main result in this paper.

Theorem 3. *Let $T > 0$ and $0 < l_1 < l_2 < 1$. Then*

(i) *for any $u \in L^2(0,T;L^2(0,1))$, the solution w of system (2) satisfies $w(\cdot,T) \in H_0^1(0,1)$, $w(\cdot,T) \in \mathscr{A}(l_1)$ and $w(1-\cdot,T) \in \mathscr{A}(1-l_2)$;*

(ii) *for any $0 < \varepsilon < (l_2 - l_1)/2$, for any $w_T \in H_0^1(0,1)$ with $w_T \in \mathscr{H}(l_1 + \varepsilon)$ and $w_T(1-\cdot) \in \mathscr{H}(1 - l_2 + \varepsilon)$, there exists a control function $u \in L^2(0,T;L^2(0,1))$ such that the solution w of system (2) satisfies $w(\cdot,T) = w_T$ in $(0,1)$.*

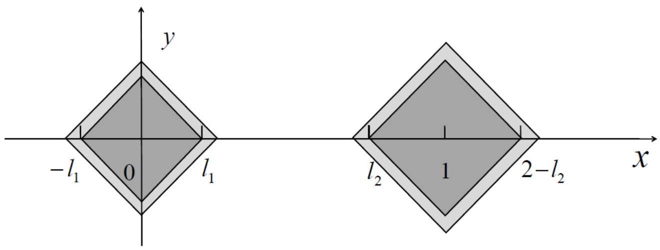

Figure 2. Reachable states for homogeneous Dirichlet boundary conditions.

Remark 4. The result is again "almost sharp". We conjecture that for $0 < l_1 < l_2 < 1$, the reachable space for system (2) is the set of functions $w_T \in H_0^1(0,1)$ such that $w_T \in \mathscr{A}(l_1)$ and $w_T(1-\cdot) \in \mathscr{A}(1-l_2)$.

The paper is outlined as follows. The proof of Theorem 1 (resp. Theorem 3) is given in Section 2 (resp. in Section 3). We provide in appendix a short proof of the main result in [2] which is used to prove Theorem 1.

2. Proof of Theorem 1

(i). Pick any $u \in L^2(0,T,L^2(\mathbb{T}))$ and let w denote the solution of (2). Using the regularity of solutions of the heat equations (see [3]), we see that

$$w \in C([0,T], H^1(\mathbb{T})), \tag{3}$$

and hence $w(\cdot,T)\in H^1(\mathbb{T})$. Introduce the function

$$v(x,t)=w(x,t)\quad \text{for } (x,t)\in[l_2,l_1+2\pi]\times[0,T].$$

Then v satisfies the following system

$$\begin{cases} v_t - v_{xx}=0 & (x,t)\in(l_2,l_1+2\pi)\times(0,T),\\ v(l_2,t)=w(l_2,t),\ v(l_1+2\pi,t)=w(l_1+2\pi,t) & t\in(0,T),\\ v(x,0)=0 & x\in(l_2,l_1+2\pi),\end{cases}$$

where the boundary controls $w(l_2,\cdot)$ and $w(l_1+2\pi,\cdot)$ are in $C([0,T])$, by (3). Then $v(\cdot,T)$ can be extended as an analytic function in $\mathscr{S}(l_1,l_2)$ by [12, Theorem 2.1]. Furthermore, $w(\cdot,T)\in L^2(\mathscr{S}(l_1,l_2))$ by [6, Proposition 1.2]. Therefore $w(\cdot,T)\in\mathscr{A}(l_1,l_2)$.

(ii). We introduce a partition of unity. Let $\psi_1,\psi_2\in C^\infty(\mathbb{T})$ be such that

$$\begin{aligned} &\psi_1+\psi_2=1,\quad 0\le\psi_i\le 1,\quad i=1,2\\ &\psi_1(x)=0,\quad x\in\mathbb{T}\setminus(l_1,l_2),\\ &\psi_2(x)=0,\quad x\in\left(l_1+\frac{\varepsilon}{2},l_2-\frac{\varepsilon}{2}\right).\end{aligned}$$

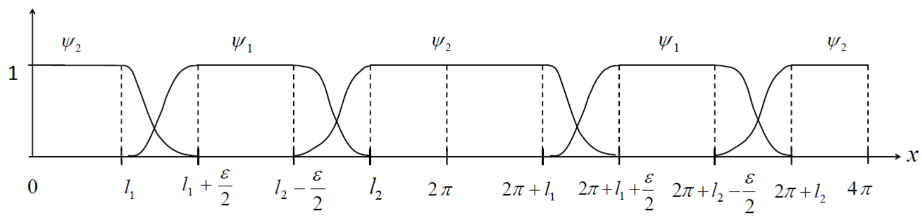

Figure 3. Partition of unity

Consider a control problem with a distributed control supported in $\mathbb{T}$:

$$\begin{cases} w_{1t}-w_{1xx}=\tilde{u}_1 & (x,t)\in\mathbb{T}\times(0,T),\\ w_1(x,0)=0 & x\in\mathbb{T},\end{cases}\tag{4}$$

where $\tilde{u}_1$ is the control.

Lemma 5. *For any $T>0$ and any $w_T\in H^1(\mathbb{T})$, there exists a function $\tilde{u}_1\in L^2(0,T;L^2(\mathbb{T}))$ such that the solution of* (4) *satisfies*

$$w_1(x,T)=w_T(x)\quad \forall\, x\in\mathbb{T}.$$

Proof. Expand the control input as a Fourier series $\tilde{u}_1(x,t)=\sum_{n\in\mathbb{Z}}u_n(t)e^{inx}$. Then by Duhamel formula

$$w_1(x,t)=\int_0^t\sum_{n\in\mathbb{Z}}e^{-n^2(t-\tau)}u_n(\tau)e^{inx}\mathrm{d}\tau.$$

Since $w_T\in H^1(\mathbb{T})$, we can expand w_T as $w_T(x)=\sum_{n\in\mathbb{Z}}a_ne^{inx}$ with $\sum_{n\in\mathbb{Z}}n^2|a_n|^2<\infty$.

Pick

$$u_n(t)=\begin{cases}\dfrac{a_0}{T} & \text{for } n=0,\\ \dfrac{2n^2e^{-n^2(T-t)}}{1-e^{-2n^2T}}a_n & \text{for } n\in\mathbb{Z}^*.\end{cases}$$

Then we obtain for $n \in \mathbb{Z}^*$

$$\int_0^T |u_n(t)|^2 \mathrm{d}t = \left(\frac{2n^2}{1-e^{-2n^2T}}\right)^2 |a_n|^2 \int_0^T e^{-2n^2(T-t)}\mathrm{d}t = \frac{2n^2}{1-e^{-2n^2T}}|a_n|^2 \sim 2n^2|a_n|^2 \text{ as } n \to +\infty.$$

It follows that

$$\|\widetilde{u}_1\|^2_{L^2(0,T;L^2(\mathbb{T}))} = 2\pi \sum_{n\in\mathbb{Z}} \int_0^T |u_n^2(t)|\mathrm{d}t < \infty.$$

On the other hand, we have $\int_0^T e^{-n^2(T-\tau)}u_n(\tau)\mathrm{d}\tau = a_n$ for all $n \in \mathbb{Z}$, and hence $w_1(\cdot, T) = w_T$. The proof of Lemma 5 is complete. □

It follows from (4) that the function $\psi_1 w_1$ satisfies the system:

$$\begin{cases} (\psi_1 w_1)_t - (\psi_1 w_1)_{xx} = u_1 & (x,t) \in \mathbb{T} \times (0,T), \\ (\psi_1 w_1)(x,0) = 0 & x \in \mathbb{T}, \\ (\psi_1 w_1)(x,T) = (\psi_1 w_T)(x) & x \in \mathbb{T}, \end{cases} \tag{5}$$

with

$$\begin{aligned} u_1 &= \psi_1(w_{1t} - w_{1xx}) - 2\psi_{1x}w_{1x} - \psi_{1xx}w_1 \\ &= \psi_1\widetilde{u}_1 - 2\psi_{1x}w_{1x} - \psi_{1xx}w_1. \end{aligned}$$

By (3) (still valid for $l_1 = 0$ and $l_2 = 2\pi$), we have that $w_1 \in C([0,T], H^1(\mathbb{T}))$. By construction of ψ_1, we have that $\partial_x^n \psi_1(x) = 0$ for all $x \in \mathbb{T} \setminus (l_1, l_2)$ and all $n \geq 0$, and hence

$$u_1 = \chi_{(l_1,l_2)} u_1 \qquad \text{in } L^2(0,T,L^2(\mathbb{T})).$$

Let $\mathrm{Hol}(\Omega)$ denote the space of (complex) analytic functions in Ω. A function $h \in C^\infty([0,T])$ is said to be *Gevrey of order 2*, and we write $h \in G^2([0,T])$, if there exist some positive constants C, R such that

$$|\partial_t^p h(t)| \leq C\frac{(p!)^2}{R^p} \quad \forall\, t \in [0,T],\ \forall\, p \geq 0.$$

The following result is needed.

Theorem 6. *Let $L > 1$, $T > 0$, and $\psi \in \mathrm{Hol}(S(L))$. Then there exist $h_{-1}, h_1 \in G^2([0,T])$ such that the solution $w = w(x,t)$ of the control system*

$$w_t - w_{xx} = 0, \qquad (x,t) \in (-1,1) \times (0,T), \tag{6}$$

$$w(-1,t) = h_{-1}(t),\ w(1,t) = h_1(t), \qquad t \in (0,T), \tag{7}$$

$$w(x,0) = 0, \qquad x \in (-1,1), \tag{8}$$

satisfies $w \in C^\infty([-1,1] \times [0,T])$ and $w(x,T) = \psi(x)$ for $x \in [-1,1]$. If, in addition, ψ is odd, then we can require that $w(\cdot,t)$ be odd for all $t \in [0,T]$, so that $h_{-1}(t) = -h_1(t)$ and $w(0,t) = 0$ for all $t \in [0,T]$.

Note that a similar result with $h_{-1}, h_1 \in C^\infty([0,T])$ was derived in [2, Theorem 5.2]. Here, we provide in Appendix a very short proof of Theorem 6 which is interesting in itself. Note that the control inputs can be made explicit and that they have the (sharp) time regularity G^2 corresponding to the space regularity G^1. That property is useful when dealing with nonlinear problems [8].

By Theorem 6, if $w_T \in \mathscr{H}(l_1+\varepsilon, l_2-\varepsilon)$, we can find $h_1, h_2 \in G^2([0,T])$ such that the solution w_2 of the following system

$$\begin{cases} w_{2t} - w_{2xx} = 0 & (x,t) \in (l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi) \times (0,T), \\ w_2(l_2 - \frac{\varepsilon}{2}, t) = h_1(t),\ w_2(l_1 + \frac{\varepsilon}{2} + 2\pi, t) = h_2(t) & t \in (0,T), \\ w_2(x,0) = 0 & x \in (l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi) \end{cases} \tag{9}$$

satisfies $w_2 \in C^\infty([l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi] \times [0, T])$ and

$$w_2(x, T) = w_T(x), \quad \forall\, x \in \left(l_2 - \frac{\varepsilon}{2}, l_1 + \frac{\varepsilon}{2} + 2\pi\right).$$

Extend w_2 as a function in $C^\infty(\mathbb{T} \times [0, T])$ (i.e. as a function smooth in (x, t) and 2π-periodic in x) and still denote this function by w_2. Then we have

$$\begin{cases} (\psi_2 w_2)_t - (\psi_2 w_2)_{xx} = u_2 & (x, t) \in \mathbb{T} \times (0, T), \\ (\psi_2 w_2)(x, 0) = 0 & x \in \mathbb{T} \\ (\psi_2 w_2)(x, T) = (\psi_2 w_T)(x) & x \in \mathbb{T} \end{cases} \tag{10}$$

where

$$u_2 = \psi_2(w_{2t} - w_{2xx}) - 2\psi_{2x} w_{2x} - \psi_{2xx} w_2.$$

It follows from the definition of ψ_2 and the first equation in (9) that $u_2(x, t) = 0$ for $x \in [l_2, l_1 + 2\pi]$ and $t \in [0, T]$. Thus we have

$$u_2 = \chi_{(l_1, l_2)} u_2 \quad \text{in } L^2(0, T, L^2(\mathbb{T})).$$

Since $w_2 \in C^\infty(\mathbb{T} \times [0, T])$, we have $u_2 \in C^\infty(\mathbb{T} \times [0, T])$. Combining (5) and (10), if we take

$$u = \chi_{(l_1, l_2)} u_1 + \chi_{(l_1, l_2)} u_2,$$

then $w = \psi_1 w_1 + \psi_2 w_2$ satisfies (1) and

$$w(x, T) = (\psi_1 w_T)(x) + (\psi_2 w_T)(x) = w_T(x), \quad \forall\, x \in \mathbb{T}.$$

The proof of Theorem 1 is complete. □

3. Dirichlet boundary conditions

In this section, we prove Theorem 3. The necessary conditions in (i) are obtained as in the proof of Theorem 1. Indeed, introducing $v_1 = w_{|(0, l_1) \times (0, T)}$ and $v_2 = w_{|(l_2, 1) \times (0, T)}$ and applying [6, Proposition 5.1], we obtain the desired result.

To prove (ii), we use again a partition of unity. We pick some functions $\psi_1, \psi_2, \psi_3 \in C^\infty([0, 1])$ such that

$$\begin{aligned} \psi_1 + \psi_2 + \psi_3 = 1, &\quad 0 \le \psi_i \le 1, \quad i = 1, 2, 3, \\ \psi_1(x) = 0 &\quad x \in [0, l_1] \cup [l_2, 1], \\ \psi_2(x) = 0 &\quad x \in \left[l_1 + \frac{\varepsilon}{2}, 1\right], \\ \psi_3(x) = 0 &\quad x \in \left[0, l_2 - \frac{\varepsilon}{2}\right]. \end{aligned}$$

As in the proof of Lemma 5, we can show that for any $w_T \in H_0^1(0, 1)$, there exists a function $\widetilde{u}_1 \in L^2(0, T; L^2(0, 1))$ such that the solution w_1 of the system

$$\begin{cases} w_{1t} - w_{1xx} = \widetilde{u}_1 & (x, t) \in (0, 1) \times (0, T), \\ w_1(0, t) = w_1(1, t) = 0 & t \in (0, T), \\ w_1(x, 0) = 0 & x \in (0, 1), \end{cases}$$

satisfies

$$w_1(x, T) = w_T(x), \quad \forall\, x \in (0, 1).$$

Proceeding as for system (5), we see that there exists a control function $u_1 \in L^2(0,T;L^2(0,1))$ such that $\psi_1 w_1$ is the solution of the following system:

$$\begin{cases} (\psi_1 w_1)_t - (\psi_1 w_1)_{xx} = \chi_{(l_1,l_2)} u_1 & (x,t) \in (0,1)\times(0,T),\\ (\psi_1 w_1)(0,t) = (\psi_1 w_1)(1,t) = 0 & t\in(0,T),\\ (\psi_1 w_1)(x,0) = 0 & x\in(0,1),\\ (\psi_1 w_1)(x,T) = (\psi_1 w_T)(x) & x\in(0,1). \end{cases}$$

Since $w_T \in \mathscr{H}(l_1+\varepsilon)$, there exists by Theorem 6 a function $h_1 \in G^2([0,T])$ such that the solution w_2 of

$$\begin{cases} w_{2t} - w_{2xx} = 0 & (x,t)\in(0,l_1+\frac{\varepsilon}{2})\times(0,T),\\ w_2(0,t)=0,\ w_2(l_1+\frac{\varepsilon}{2},t) = h_1(t) & t\in(0,T),\\ w_2(x,0)=0 & x\in(0,l_1+\frac{\varepsilon}{2}) \end{cases}$$

satisfies $w_2 \in C^\infty([0,l_1+\frac{\varepsilon}{2}]\times[0,T])$ and

$$w_2(x,T) = w_T(x), \quad \forall\, x\in\left(0,l_1+\frac{\varepsilon}{2}\right).$$

We still denote by w_2 a smooth extension of w_2 to $[0,1]\times[0,T]$. Then there exists a function $u_2 \in L^2(0,T;L^2(0,1))$ such that $\psi_2 w_2$ solves

$$\begin{cases} (\psi_2 w_2)_t - (\psi_2 w_2)_{xx} = \chi_{(l_1,l_1+\frac{\varepsilon}{2})} u_2 & (x,t) \in (0,1)\times(0,T),\\ (\psi_2 w_2)(0,t) = (\psi_2 w_2)(1,t) = 0 & x\in(0,T),\\ (\psi_2 w_2)(x,0) = 0 & t\in(0,1),\\ (\psi_2 w_2)(x,T) = (\psi_2 w_T)(x) & x\in(0,1). \end{cases}$$

Similarly, since $w_T(1-\cdot) \in \mathscr{H}(1-l_2+\varepsilon)$, there exists a function $h_2 \in G^2([0,T])$ such that the solution $\widetilde{w}_3$ of the system

$$\begin{cases} \widetilde{w}_{3t} - \widetilde{w}_{3xx} = 0 & (x,t)\in(0,1-l_2+\frac{\varepsilon}{2})\times(0,T),\\ \widetilde{w}_3(0,t)=0,\ \widetilde{w}_3(1-l_2+\frac{\varepsilon}{2},t) = h_2(t) & t\in(0,T),\\ \widetilde{w}_3(x,0)=0 & x\in(0,1-l_2+\frac{\varepsilon}{2}) \end{cases}$$

satisfies $\widetilde{w}_3 \in C^\infty([0,1-l_2+\frac{\varepsilon}{2}]\times[0,T])$ and

$$\widetilde{w}_3(x,T) = w_T(1-x), \qquad x\in\left(0,1-l_2+\frac{\varepsilon}{2}\right).$$

Let

$$w_3(x,t) = \widetilde{w}_3(1-x,t), \quad x\in\left(l_2-\frac{\varepsilon}{2},1\right),\ t\in(0,T).$$

Then we have

$$\begin{cases} w_{3t} - w_{3xx} = 0 & (x,t)\in(l_2-\frac{\varepsilon}{2},1)\times(0,T),\\ w_3(l_2-\frac{\varepsilon}{2},t) = h_2(t),\ w_3(1,t)=0 & t\in(0,T),\\ w_3(x,0)=0 & x\in(l_2-\frac{\varepsilon}{2},1),\\ w_3(x,T) = w_T(x) & x\in(l_2-\frac{\varepsilon}{2},1). \end{cases}$$

We still denote by w_3 a smooth extension of w_3 to $[0,1]\times[0,T]$. It follows that there exists a control function $u_3 \in L^2(0,T;L^2(0,1))$ such that $\psi_3 w_3$ solves

$$\begin{cases} (\psi_3 w_3)_t - (\psi_3 w_3)_{xx} = \chi_{(l_2-\frac{\varepsilon}{2},l_2)} u_3 & (x,t) \in (0,1)\times(0,T),\\ (\psi_3 w_3)(0,t) = (\psi_3 w_3)(1,t) = 0 & t\in(0,T),\\ (\psi_3 w_3)(x,0) = 0 & x\in(0,1),\\ (\psi_3 w_3)(x,T) = (\psi_3 w_T)(x) & x\in(0,1). \end{cases}$$

Finally, if we take

$$u = \chi_{(l_1,l_2)}u_1 + \chi_{(l_1,l_1+\frac{\varepsilon}{2})}u_2 + \chi_{(l_2-\frac{\varepsilon}{2},l_2)}u_3,$$

we infer that $w = \psi_1 w_1 + \psi_2 w_2 + \psi_3 w_3$ is the solution of (2). On the other hand, we have that

$$w(x,T) = (\psi_1 w_T)(x) + (\psi_2 w_T)(x) + (\psi_3 w_T)(x) = w_T(x), \quad \forall\, x \in (0,1).$$

The proof of Theorem 3 is complete. □

Appendix. Proof of Theorem 6

The proof is inspired by [10] where backward solutions of the heat equation were obtained by integrating the heat kernel along lines passing through the origin but different from the real line. Note that backward solutions of parabolic equations were considered for control purposes in [8, 14]. Introduce the following notations borrowed from [10]. For $\theta \in \mathbb{R}$ and $R > 0$, let

$$\mathscr{O}(\theta,R) := \left\{z \in \mathbb{C};\ \left|z - Re^{i\theta}\right| < R\right\},$$
$$\Omega(\theta,R) := \left\{z \in \mathbb{C};\ \operatorname{dist}\left(z, e^{i\theta}\mathbb{R}\right) < R\right\}.$$

(See Figure 2.)

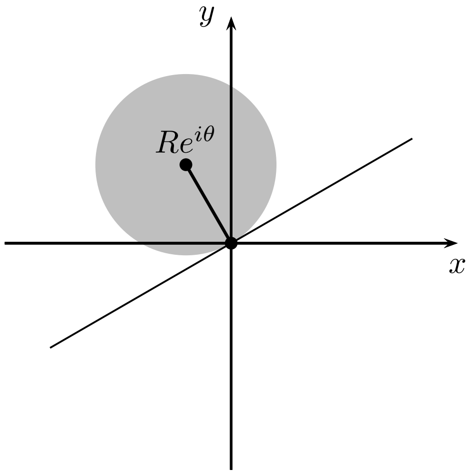

Figure 4. The set $\mathscr{O}(\theta,R)$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$.

Note that

$$S(1) = \Omega\left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right) \cap \Omega\left(\frac{3\pi}{4}, \frac{1}{\sqrt{2}}\right).$$

The first lemma is concerned with *separation of singularities.*

Lemma 7. *Let* $1 < l < L$ *and* $\psi \in \operatorname{Hol}(S(L))$. *Then there exist* $\theta_1 \in \left(\pi, \frac{3\pi}{2}\right)$, $\theta_2 \in \left(\frac{\pi}{2}, \pi\right)$, $r \in \left(\frac{1}{\sqrt{2}}, +\infty\right)$, $\psi_1 \in \operatorname{Hol}\left(\Omega\left(\frac{\theta_1}{2}, r\right)\right)$ *and* $\psi_2 \in \operatorname{Hol}\left(\Omega\left(\frac{\theta_2}{2}, r\right)\right)$ *such that*

$$\overline{S(l)} \subset \Omega\left(\frac{\theta_1}{2}, r\right) \cap \Omega\left(\frac{\theta_2}{2}, r\right),$$
$$\partial_z^j \psi_i \in L^\infty\left(\Omega\left(\frac{\theta_i}{2}, r\right)\right), \quad i = 1,2,\ j \in \mathbb{N},$$
$$\psi = \psi_1 + \psi_2 \qquad \text{in } S(l).$$

Proof of Lemma 7. Pick any $\widehat{l} \in (l, L)$. Let $\gamma_1(t) = (1-t+it)\widehat{l}$ for $t \in [0,1]$, and let

$$\big(\gamma_2(t), \gamma_3(t), \gamma_4(t)\big) = (i\gamma_1(t), -\gamma_1(t), -i\gamma_1(t)) \quad \text{for } t \in [0,1].$$

Let $\gamma : [0,4] \to \mathbb{C}$ be defined by

$$\gamma(t) = \gamma_i(t-i+1) \quad \text{for } i \in \{1,\ldots,4\}, t \in [i-1,i].$$

Note that $\gamma([0,4]) = \partial S(\widehat{l})$. We infer from Cauchy formula that for any $z \in S(\widehat{l})$

$$\begin{aligned}\psi(z) &= \frac{1}{2\pi i}\int_\gamma \frac{\psi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_1\cup\gamma_3} \frac{\psi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta + \frac{1}{2\pi i}\int_{\gamma_2\cup\gamma_4} \frac{\psi(\zeta)}{\zeta - z}\,\mathrm{d}\zeta \\ &=: \psi_1(z) + \psi_2(z)\end{aligned}$$

where $\psi_1 \in \mathrm{Hol}(\mathbb{C}\setminus(\gamma_1\cup\gamma_3))$ and $\psi_2 \in \mathrm{Hol}(\mathbb{C}\setminus(\gamma_2\cup\gamma_4))$. Since

$$\overline{S(l)} \subset S(\widehat{l}) = \Omega\left(\frac{\pi}{4}, \frac{\widehat{l}}{\sqrt{2}}\right) \cap \Omega\left(\frac{3\pi}{4}, \frac{\widehat{l}}{\sqrt{2}}\right),$$

there exist $r \in \left(\frac{l}{\sqrt{2}}, \frac{\hat{l}}{\sqrt{2}}\right)$ and $\theta_1 \in \left(\pi, \frac{3\pi}{2}\right)$, $\theta_2 \in \left(\frac{\pi}{2}, \pi\right)$ such that

$$\begin{aligned}&\overline{S(l)} \subset \Omega\left(\frac{\theta_1}{2}, r\right) \cap \Omega\left(\frac{\theta_2}{2}, r\right) \subset S(\widehat{l}),\\ &\psi_i \in \mathrm{Hol}\left(\Omega\left(\frac{\theta_i}{2}, r\right)\right) \qquad i = 1,2,\\ &\partial_z^j \psi_i \in L^\infty\left(\Omega\left(\frac{\theta_i}{2}, r\right)\right), \quad i = 1,2,\ j \in \mathbb{N}\end{aligned}$$

(see Figure 5). The proof of Lemma 7 is complete. □

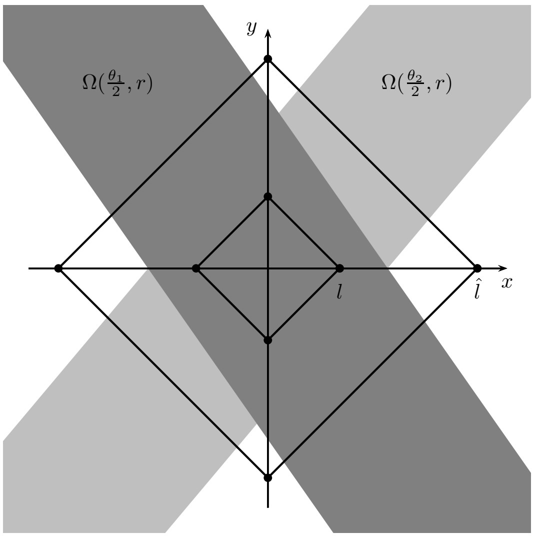

Figure 5. Separation of singularities.

The second lemma yields a backward solution of the heat equation for the initial data ψ_1 (resp. ψ_2). In the context of control theory, backward solutions of parabolic equations were considered in [8] for 1D semilinear heat equations, and in [14] for the linear heat equation on the unit ball of $\mathbb{R}^N$, using a trick due to Wick to derive backward solutions from forward solutions.

Lemma 8. *Let $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, $r > 1/\sqrt{2}$, and $\psi \in \mathrm{Hol}\left(\Omega\left(\frac{\theta}{2}, r\right)\right) \cap L^\infty\left(\Omega\left(\frac{\theta}{2}, r\right)\right)$. Then the function*

$$v(z,\tau) := \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}} \psi(z-\zeta)\,\mathrm{d}\zeta$$

is well-defined and analytic in z and τ for $z \in \Omega\left(\frac{\theta}{2}, r\right)$ and $\tau \in \mathscr{O}(\theta, R)$ for any $R > 0$.[2] *Furthermore, v satisfies*

$$v_\tau - v_{zz} = 0, \quad z \in \Omega\left(\frac{\theta}{2}, r\right),\ \tau \in \mathscr{O}(\theta, R), \tag{11}$$

$$\lim_{\tau \to 0^-} v(z,\tau) = \psi(z), \quad z \in \Omega\left(\frac{\theta}{2}, r\right). \tag{12}$$

If, in addition, $\partial_z^j \psi \in L^\infty\left(\Omega\left(\frac{\theta}{2}, r\right)\right)$ for all $j \in \mathbb{N}$ and if $\overline{S(1)} \subset \Omega\left(\frac{\theta}{2}, r\right)$, then $v \in C^\infty([-1,1] \times [-T,0])$ for all $T > 0$ with $-T \in \mathscr{O}(\theta, R)$.

Proof of Lemma 8. For $s \in \mathbb{R}$ and $\zeta = e^{i\frac{\theta}{2}} s$, we have

$$\left| e^{-\frac{\zeta^2}{4\tau}} \right| = \left| e^{-\frac{s^2}{4}\frac{e^{i\theta}}{\tau}} \right| = e^{-\frac{s^2}{4}\mathrm{Re}\,\frac{e^{i\theta}}{\tau}}.$$

But if $\tau \in \mathscr{O}(\theta, R)$, $\left|\frac{\tau}{e^{i\theta}} - R\right| < R$, so that $\mathrm{Re}\,\frac{e^{i\theta}}{\tau} = \mathrm{Re}\,\frac{\tau}{e^{i\theta}} > 0$. On the other hand

$$|\psi(z-\zeta)| \le \|\psi\|_{L^\infty(\Omega(\frac{\theta}{2},r))}, \quad \text{for } z \in \Omega\left(\frac{\theta}{2}, r\right) \text{ and } \zeta \in \mathbb{R}e^{i\frac{\theta}{2}}.$$

Straightforward calculations show that

$$\begin{aligned} v_{zz} &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}} \psi''(z-\zeta)\mathrm{d}\zeta \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} \frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left[e^{-\frac{\zeta^2}{4\tau}}\right] \psi(z-\zeta)\mathrm{d}\zeta \\ &= \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} \left(-\frac{1}{2\tau} + \frac{\zeta^2}{4\tau^2}\right) e^{-\frac{\zeta^2}{4\tau}} \psi(z-\zeta)\mathrm{d}\zeta, \end{aligned}$$

while

$$v_\tau = \frac{1}{\sqrt{4\pi\tau}}\left(-\frac{1}{2\tau}\right) \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}} \psi(z-\zeta)\mathrm{d}\zeta + \frac{1}{\sqrt{4\pi\tau}}\left(\frac{1}{4\tau^2}\right) \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} \zeta^2 e^{-\frac{\zeta^2}{4\tau}} \psi(z-\zeta)\mathrm{d}\zeta$$

and hence (11) holds in $\Omega\left(\frac{\theta}{2}, r\right) \times \mathscr{O}(\theta, R)$. It remains to show that (12) is fulfilled. First, we notice that for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ and $R > 0$, there exists $T > 0$ such that $\mathscr{O}(\theta, R) \cap \mathbb{R} = (-T, 0)$ (see Figure 2). Therefore, taking the limit of $v(z,\tau)$ as $\tau \to 0^-$ is meaningful.

Claim 1. *For $\theta \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $\tau \in (-\infty, 0)$, we have*

$$\frac{1}{\sqrt{4\pi\tau}} \int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}} \mathrm{d}\zeta = 1.$$

Indeed, if $\theta \in \left(\frac{\pi}{2}, \pi\right)$ (resp. $\theta \in \left(\pi, \frac{3\pi}{2}\right)$), we have for $\kappa \in \left(\frac{\theta}{2}, \frac{\pi}{2}\right)$ (resp. $\kappa \in \left(\frac{\pi}{2}, \frac{\theta}{2}\right)$), $s \in \mathbb{R}$, $\tau \in (-\infty, 0)$, and $\zeta := se^{i\kappa}$,

$$\left| e^{-\frac{\zeta^2}{4\tau}} \right| = \left| e^{-\frac{s^2}{4\tau}\cos(2\kappa)} \right| \le e^{-\frac{s^2}{4|\tau|}|\cos(\theta)|}.$$

[2] We pick a branch of the argument function defined in $\mathbb{C} \setminus \mathbb{R}_- i$ (resp. in $\mathbb{C} \setminus \mathbb{R}_+ i$) if $\frac{\pi}{2} < \theta \le \pi$ (resp. if $\pi < \theta \le \frac{3\pi}{2}$).

Thus

$$\lim_{|s|\to+\infty}\int_{\frac{\theta}{2}}^{\frac{\pi}{2}} e^{-\frac{(se^{i\kappa})^2}{4\tau}} sie^{i\kappa}\mathrm{d}\kappa = 0 \qquad \text{if } \theta\in\left(\frac{\pi}{2},\pi\right),$$

$$\text{(resp.}\quad \lim_{|s|\to+\infty}\int_{\frac{\pi}{2}}^{\frac{\theta}{2}} e^{-\frac{(se^{i\kappa})^2}{4\tau}} sie^{i\kappa}\mathrm{d}\kappa = 0 \qquad \text{if } \theta\in\left(\pi,\frac{3\pi}{2}\right).\text{)}$$

It follows from the residue theorem that

$$\frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\zeta^2}{4\tau}}\mathrm{d}\zeta = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\pi}{2}}}^{\infty e^{i\frac{\pi}{2}}} e^{-\frac{\zeta^2}{4\tau}}\mathrm{d}\zeta = \frac{1}{i\sqrt{4\pi|\tau|}}\int_{-\infty}^{\infty} e^{-\frac{s^2}{4|\tau|}} i\mathrm{d}s = 1,$$

which completes the proof of Claim 1.

Finally, letting $\zeta=\sqrt{|\tau|}\xi$, we see that for any $z\in\Omega\left(\frac{\theta}{2},r\right)$

$$v(z,\tau)-\psi(z) = \frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\left(\psi(z-\sqrt{|\tau|}\xi)-\psi(z)\right)\mathrm{d}\xi$$

tends to 0 as $\tau\to 0^-$, by dominated convergence.

Finally, assume that $\partial_z^j\psi\in L^\infty(\Omega(\frac{\theta}{2},r))$ for all $j\in\mathbb{N}$ and that $\overline{S(1)}\subset\Omega\left(\frac{\theta}{2},r\right)$.

Claim 2. *For all $j\in\mathbb{N}$, we have $\partial_z^j[v(z,\tau)-\psi(z)]\to 0$ as $\tau\to 0^-$ uniformly for $z\in[-1,1]$.*

Indeed, for all $j\in\mathbb{N}$, we can write

$$\partial_z^j[v(z,\tau)-\psi(z)] = \frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{-\infty e^{i\frac{\theta}{2}}}^{\infty e^{i\frac{\theta}{2}}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\left(\psi^{(j)}(z-\sqrt{|\tau|}\xi)-\psi^{(j)}(z)\right)\mathrm{d}\xi.$$

For given $\varepsilon>0$, we pick a number $A>0$ such that

$$\int_{|s|>A}\left|e^{-\frac{(se^{i\frac{\theta}{2}})^2}{4\frac{\tau}{|\tau|}}}\right|\mathrm{d}s<\varepsilon.$$

Then

$$\left|\frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{\substack{\xi\in(-\infty,\infty)e^{i\frac{\theta}{2}},\\ |\xi|>A}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\left(\psi^{(j)}(z-\sqrt{|\tau|}\xi)-\psi^{(j)}(z)\right)\mathrm{d}\xi\right| \le \frac{2\|\psi^j\|_{L^\infty(\Omega(\frac{\theta}{2},r))}\varepsilon}{\sqrt{4\pi}}.$$

On the other hand, it follows from the uniform continuity of $\psi^{(j)}$ on some open neighborhood of $[-1,1]$ that

$$\frac{1}{\sqrt{4\pi\frac{\tau}{|\tau|}}}\int_{\substack{\xi\in(-\infty,\infty)e^{i\frac{\theta}{2}},\\ |\xi|\le A}} e^{-\frac{\xi^2}{4\frac{\tau}{|\tau|}}}\left(\psi^{(j)}(z-\sqrt{|\tau|}\xi)-\psi^{(j)}(z)\right)\mathrm{d}\xi\to 0 \qquad \text{as } \tau\to 0^-,$$

uniformly for $z\in[-1,1]$. Claim 2 is proved.

Using the fact that $\partial_t^k\partial_z^j v = \partial_z^{2k+j}v$ for $k,j\in\mathbb{N}$, $\tau<0$, and $z\in[-1,1]$, we infer that $v\in C^\infty([-1,1]\times[-T,0])$ for any $T>0$ such that $-T\in\mathscr{O}(\theta,R)$. The proof of Lemma 8 is complete. □

Let us go back to the proof of Theorem 6. Pick ψ_1 and ψ_2 as given by Lemma 7, and let

$$v_j(z,\tau) = \frac{1}{\sqrt{4\pi\tau}}\int_{-\infty e^{i\frac{\theta_j}{2}}}^{\infty e^{i\frac{\theta_j}{2}}} e^{-\frac{\zeta^2}{4\tau}}\psi_j(z-\zeta)\mathrm{d}\zeta \quad \text{for } j=1,2,\ z\in\Omega\left(\frac{\theta_j}{2},r\right) \text{ and } \tau\in\mathscr{O}(\theta_j,R).$$

Then

$$v_{1,\tau} - v_{1,zz} = 0, \quad \text{in } \Omega\left(\frac{\theta_1}{2}, r\right) \times \mathcal{O}(\theta_1, R),$$
$$v_{2,\tau} - v_{2,zz} = 0, \quad \text{in } \Omega\left(\frac{\theta_2}{2}, r\right) \times \mathcal{O}(\theta_2, R).$$

Let $v(z,\tau) = v_1(z,\tau) + v_2(z,\tau)$. Then v is well-defined and analytic in $D := \left[\Omega\left(\frac{\theta_1}{2}, r\right) \cap \Omega\left(\frac{\theta_2}{2}, r\right)\right] \times [\mathcal{O}(\theta_1, R) \cap \mathcal{O}(\theta_2, R)]$, and it fulfills

$$v_\tau - v_{zz} = 0 \quad \text{in } D,$$
$$\lim_{\tau\to 0^-} v(z,\tau) = \psi_1(z) + \psi_2(z) = \psi(z) \quad \text{in } \Omega\left(\frac{\theta_1}{2}, r\right) \cap \Omega\left(\frac{\theta_2}{2}, r\right).$$

Furthermore, if $\widehat{T} > 0$ is such that $[-\widehat{T}, 0) \subset \mathcal{O}(\theta_1, R) \cap \mathcal{O}(\theta_2, R)$ and $\widehat{T} \le T$, then $v(\cdot, -\widehat{T})$ is analytic in the open set $\Omega\left(\frac{\theta_1}{2}, r\right) \cap \Omega\left(\frac{\theta_2}{2}, r\right)$ which contains $\overline{S(1)}$.

To complete the proof, we proceed as in [14], combining the above construction with a null controllability result. Pick any $s \in (1,2)$ and any function $\rho \in G^s([-\widehat{T}, 0])$ such that $\rho(t) = 1$ for $-\widehat{T} \le t \le -\frac{\widehat{T}}{2}$ and $\rho(t) = 0$ for $-\frac{\widehat{T}}{4} \le t \le 0$. Let $g_0(t) := \rho(t) v(0,t)$ and $g_1(t) := \rho(t)\partial_\tau v(0,t)$ for $t \in [-\widehat{T}, 0]$. Using the fact that $v(0,\cdot), \partial_\tau v(0,\cdot) \in \mathrm{Hol}(\mathcal{O}(\theta_1, R) \cap \mathcal{O}(\theta_2, R))$ and [12, Lemma 3.7], we infer that $g_0, g_1 \in G^s([-\widehat{T}, 0])$. Therefore, for any $R > 1$ there exists some constant $C > 0$ such that

$$|\partial_t^j g_0(t)| + |\partial_t^j g_1(t)| \le C\frac{(2j)!}{R^j} \quad \forall\, t \in [-\widehat{T}, 0],\ \forall\, j \in \mathbb{N}.$$

According to [12, Proposition 3.1], the problem

$$\widehat{v}_t - \widehat{v}_{xx} = 0, \quad t \in (-\widehat{T}, 0),\ x \in (-1,1),$$
$$\widehat{v}(0,t) = g_0(t),\ \partial_x \widehat{v}(0,t) = g_1(t), \quad t \in (-\widehat{T}, 0)$$

possesses a solution $\widehat{v} \in C^\infty([-1,1] \times [-\widehat{T}, 0])$. It follows then from the definition of ρ and Holmgren's theorem that

$$\widehat{v}(x,t) = 0, \qquad t \in \left(-\frac{\widehat{T}}{4}, 0\right),\ x \in (-1,1),$$
$$\widehat{v}(x,t) = v(x,t), \quad t \in \left(-\widehat{T}, -\frac{\widehat{T}}{2}\right),\ x \in (-1,1),$$

so that $\widehat{v}(\cdot, 0) = 0$ and $\widehat{v}(\cdot, -\widehat{T}) = v(\cdot, -\widehat{T})$. Let

$$h_{\pm 1}(t) := v(\pm 1, t) - \widehat{v}(\pm 1, t), \quad \text{for } t \in (-\widehat{T}, 0),$$
$$w(x,t) := v(x,t) - \widehat{v}(x,t), \qquad \text{for } t \in (-\widehat{T}, 0),\ x \in (-1,1).$$

Then w satisfies

$$\begin{cases} w_t - w_{xx} = 0, & t \in (-\widehat{T}, 0),\ x \in (-1,1), \\ w(x, -\widehat{T}) = 0, & x \in (-1,1), \\ w(\pm 1, t) = h_{\pm 1}(t), & t \in (-\widehat{T}, 0), \\ w(x, 0) = \psi(x), & x \in (-1,1). \end{cases}$$

Extending w and $h_{\pm 1}$ by 0 for $t \le -\widehat{T}$, we obtain the main result in Theorem 6 on the interval $[-T, 0]$. A simple translation in time gives the result on the interval $[0, T]$.

Assume in addition that ψ be odd. Then it is easy to see that both ψ_1 and ψ_2 are odd, and that v_1 and v_2 are odd with respect to z. It follows that v and $\widehat{v}$ are odd with respect to z. (Note that $g_0 \equiv 0$.) Therefore, w is odd with respect to z. □

Acknowledgements

The authors thank Camille Laurent for useful remarks.

References

[1] M. Chen, L. Rosier, “Exact controllability of the linear Zakharov–Kuznetsov equation”, *Discrete Contin. Dyn. Syst.* **25** (2020), no. 10, p. 3889-3916.

[2] J. Dardé, S. Ervedoza, “On the reachable set for the one-dimensional heat equation”, *SIAM J. Control Optim.* **56** (2018), no. 3, p. 1692-1715.

[3] L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, American Mathematical Society, 1998.

[4] H. O. Fattorini, D. L. Russell, “Exact controllability theorems for linear parabolic equations in one space dimension”, *Arch. Ration. Mech. Anal.* **43** (1971), no. 4, p. 272-292.

[5] A. V. Fursikov, O. Y. Imanuvilov, *Controllability of evolution equations*, Lecture Notes Series, Seoul, vol. 34, Seoul National University, 1996.

[6] A. Hartmann, K. Kellay, M. Tucsnak, “From the reachable space of the heat equation to Hilbert spaces of holomorphic functions”, *J. Eur. Math. Soc.* **22** (2020), no. 10, p. 3417-3440.

[7] A. Hartmann, M.-A. Orsoni, “Separation of singularities for the Bergman space and application to control theory”, *J. Math. Pures Appl.* **150** (2021), p. 181-201.

[8] C. Laurent, L. Rosier, “Exact controllability of semilinear heat equations in spaces of analytic functions”, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **37** (2020), no. 4, p. 1047-1073.

[9] G. Lebeau, L. Robbiano, “Contrôle exact de l’équation de la chaleur”, *Commun. Partial Differ. Equations* **20** (1995), no. 1, p. 1-2.

[10] D. A. Lutz, M. Miyake, R. Schäfke, “On the Borel summability of divergent solutions of the heat equation”, *Nagoya Math. J.* **154** (1999), p. 1-29.

[11] P. Martin, I. Rivas, L. Rosier, P. Rouchon, “Exact controllability of a linear Korteweg-de Vries equation by the flatness approach”, *SIAM J. Control Optim.* **57** (2019), no. 4, p. 2467-2486.

[12] P. Martin, L. Rosier, P. Rouchon, “On the reachable states for the boundary control of the heat equation”, *AMRX, Appl. Math. Res. Express* **2016** (2016), no. 2, p. 181-216.

[13] M.-A. Orsoni, “Reachable states and holomorphic function spaces for the 1-D heat equation”, *J. Funct. Anal.* **280** (2021), no. 7, article no. 108852 (18 pages).

[14] A. Strohmaier, A. Waters, “Analytic properties of heat equation solutions and reachable sets”, https://arxiv.org/abs/2006.05762v1, 2006.