



INSTITUT DE FRANCE  
Académie des sciences

# *Comptes Rendus*

---

# *Mathématique*

Robert E. Greene, Kang-Tae Kim and Nikolay V. Shcherbina

**Topological invariants and Holomorphic Mappings**

Volume 360 (2022), p. 829-844

<https://doi.org/10.5802/crmath.336>



This article is licensed under the  
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.  
<http://creativecommons.org/licenses/by/4.0/>



*Les Comptes Rendus. Mathématique* sont membres du  
Centre Mersenne pour l'édition scientifique ouverte  
[www.centre-mersenne.org](http://www.centre-mersenne.org)  
e-ISSN : 1778-3569



---

Complex analysis and geometry / *Analyse et géométrie complexes*

# Topological invariants and Holomorphic Mappings

Robert E. Greene<sup>a</sup>, Kang-Tae Kim<sup>\*,b</sup> and Nikolay V. Shcherbina<sup>c</sup>

<sup>a</sup> Department of Mathematics, University of California, Los Angeles, CA 90095 U.S.A.

<sup>b</sup> Department of Mathematics, Pohang University of Science and Technology, Pohang City 37673 South Korea

<sup>c</sup> Department of Mathematics, University of Wuppertal, 42119 Wuppertal, Germany

*E-mails:* [green@math.ucla.edu](mailto:green@math.ucla.edu), [kimkt@postech.ac.kr](mailto:kimkt@postech.ac.kr),  
[shcherbina@math.uni-wuppertal.de](mailto:shcherbina@math.uni-wuppertal.de)

**Abstract.** Invariants for Riemann surfaces covered by the disc and for hyperbolic manifolds in general involving minimizing the measure of the image over the homotopy and homology classes of closed curves and maps of the  $k$ -sphere into the manifold are investigated. The invariants are monotonic under holomorphic mappings and strictly monotonic under certain circumstances. Applications to holomorphic maps of annular regions in  $\mathbb{C}$  and tubular neighborhoods of compact totally real submanifolds in general in  $\mathbb{C}^n$ ,  $n \geq 2$ , are given. The contractibility of a hyperbolic domain with contracting holomorphic mapping is explained.

**Funding.** Research of the second author is partially supported by the NRF Grant 4.0021348 of The Republic of Korea.

*Manuscript received 7 October 2021, revised and accepted 22 January 2022.*

## 1. Introduction

It is an elegant and effective technique in Riemannian geometry to consider the minimum-length curve in each nontrivial free homotopy class of closed curves in a compact Riemannian manifold. Such a minimum-length curve always exists and is a smooth closed geodesic. This idea is, for example, the basic step in proving that a compact, orientable manifold of even dimension with everywhere positive sectional curvature is simply connected (cf. [20, p. 172, Theorem 26]). If the Riemannian manifold is not compact, then a minimal length curve may not exist, but it remains of interest to consider the infimum of the lengths of closed curves in the free homotopy class and also the infimum over all nontrivial free homotopy classes as well.

---

\* Corresponding author.

The purpose of this paper is to examine this general idea in the context of complex manifolds and also to consider the corresponding possibilities for the situation of maps of the  $k$ -sphere which are homotopically nontrivial, with length replaced by a suitable idea of  $k$ -dimensional measure.

For Riemann surfaces which are covered by the unit disc  $\Delta$  (that is, all Riemann surfaces except  $\mathbb{C}, \mathbb{C} \setminus \{0\}, \mathbb{C} \cup \{\infty\}$ , and compact surfaces of genus 1) the situation becomes one of Riemannian metrics: If  $\pi: \Delta \rightarrow M$  is a holomorphic covering map of the Riemann surface, then the Poincaré metric on the unit disc  $\Delta$  “pushes down” to a smooth Riemannian (Hermitian) metric on  $M$ , i.e., there is a unique metric on  $M$  such that  $\pi$  is a local isometry. This canonical push-down metric on such  $M$ 's has the property that, if  $F: M_1 \rightarrow M_2$  is a holomorphic map, then  $F$  is metric nonincreasing. This property will be exploited early in the paper to recover various classical results on maps of annular regions in  $\mathbb{C}$  from the viewpoint of minimization of lengths of curves in free homotopy classes.

Extension of these ideas to higher dimensional complex manifolds and to  $k$ -homotopy classes,  $k > 1$ , involves new features: The natural metric to consider is of course the “hyperbolic metric” introduced by S. Kobayashi, which defines a natural extension to all dimensions of the canonical metric construction for Riemann surfaces just discussed. But a new aspect arises: the Kobayashi metric on a hyperbolic manifold (in the sense of Kobayashi) need not be a Riemannian metric in dimension  $\geq 2$ . It is, however, a Finsler metric with an infinitesimal form known as the Kobayashi–Royden metric [24]. But for such an infinitesimal metric, which is only upper semi-continuous and does not satisfy the triangle inequality in general, it is necessary to exercise care in defining  $k$ -dimensional volume measure to the image of a  $k$ -sphere in the manifold. We shall consider primarily the details of this matter only for some subsets of  $\mathbb{C}^n$ ,  $n \geq 2$  (cf. Sections 9 and 10), although in principle greater generality would be possible with other hypotheses. This relevant general idea is  $k$ -dimensional Hausdorff measures. These ideas are applied in the final sections to obtain results for holomorphic mappings of tubular neighborhoods of totally real  $k$ -spheres in  $\mathbb{C}^n$ , analogous to the earlier results on maps of annular regions (cf. Section 3). It is also natural to consider the tubular neighborhoods of more general compact totally real submanifolds. For such a general case, the degree of maps, a homology invariant is more appropriate and has been investigated.

### Some Notation

- $F_U^{\text{Kob}}$ : the Kobayashi–Royden metric of an open set  $U$  in  $\mathbb{C}^n$
- $\mu_{X,d}^k(A)$ : the  $k$ -dimensional Hausdorff measure of the subset  $A$  in the metric space  $(X, d)$
- $\mu_{\text{Euc}}^k(B)$ : the  $k$ -dimensional Hausdorff measure of the subset  $B$  in  $\mathbb{C}^n$  with respect to the Euclidean norm
- $\mu_{\text{Kob},U}^k(C)$ : the  $k$ -dimensional Hausdorff measure of the subset  $C$  of an open set  $U$  in  $\mathbb{C}^n$  with respect to the Kobayashi distance

## 2. The $\ell_1$ -invariant

Let  $(X, \rho)$  be a metric space with the metric  $\rho$ . For a continuous curve  $\alpha: [a, b] \rightarrow X$ , and a partition of the interval  $[a, b]$

$$P := \{t_k: k = 0, 1, \dots, N, \text{ with } a = t_0 < t_1 < \dots < t_N = b\}$$

for some positive integer  $N$ , let

$$s(\alpha, P) := \sum_{k=1}^N \rho(\alpha(t_{k-1}), \alpha(t_k)).$$

If the set  $\{s(\alpha, P) : P \text{ a partition of } [a, b]\}$  is bounded above, we say that  $\alpha$  is *rectifiable* and define the *length*  $L(\alpha)$  of  $\alpha$  by

$$L(\alpha) = \sup\{s(\alpha, P) : P \text{ a partition of } [a, b]\}.$$

As usual, reparametrizations of rectifiable curves are rectifiable and the length is independent of parametrization.

**Definition 1.** A map  $F: X_1 \rightarrow X_2$  from a metric space  $(X_1, \rho_1)$  to  $(X_2, \rho_2)$  is called distance nonincreasing if  $\rho_2(F(x), F(y)) \leq \rho_1(x, y)$  for all  $x, y \in X_1$ .

Notice that, if a map  $F$  is distance nonincreasing, then the  $F$ -images of rectifiable curves are rectifiable and the  $F$ -images have length  $\leq$  the length of the original curve, i.e., length  $F(\gamma(t)) \leq$  length  $\gamma(t)$ , for all rectifiable curves  $\gamma(t)$  in  $X_1$ .

**Definition 2 (The  $\ell_1$ -invariant).** Consider a metric space  $(X, \rho)$  which is not simply connected. Then the  $\ell_1$ -invariant of the metric space  $X$  is defined to be

$$\ell_1(X) := \inf\{L(\alpha) : \alpha \text{ a non-contractible rectifiable loop in } X\},$$

where  $\ell_1 = +\infty$  if there are no non-contractible loops.

**Proposition 3.** If  $f: (X_1, \rho_1) \rightarrow (X_2, \rho_2)$  is a distance nonincreasing map, and if the induced homomorphism  $f_*: \pi_1(X_1) \rightarrow \pi_1(X_2)$  is injective, then  $\ell_1(X_2) \leq \ell_1(X_1)$ .

**Proof.** For a loop  $\alpha$  in  $X_1$  denote by  $[\alpha]$  the set of all loops that are free homotopic to  $\alpha$  in  $X_1$ . Set

$$\ell_1([\alpha]) = \inf\{L(\beta) : \beta \in [\alpha]\}.$$

Then

$$\ell_1(X_1) = \inf\{\ell_1([\alpha]) : \alpha \text{ non-contractible in } X_1\}.$$

Let  $\epsilon > 0$ . Take a noncontractible loop  $\beta$  in  $X_1$  with  $\ell_1([\beta]) < \ell_1(X_1) + \epsilon$ . Then  $f \circ \beta$  is noncontractible in  $X_2$ , since  $f_*$  is injective. Then  $\ell_1([f \circ \beta]) \geq \ell_1(X_2)$ . The distance nonincreasing property of  $f$  implies that

$$\ell_1(X_2) \leq \ell_1([f \circ \beta]) \leq \ell_1([\beta]) < \ell_1(X_1) + \epsilon.$$

Since this holds for any  $\epsilon > 0$ , the assertion follows immediately. □

Distance nonincreasing/length nonincreasing maps arise naturally in complex analysis. The classical Schwarz Lemma is equivalent to the fact that a holomorphic function  $f$  from the unit disc  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  into itself with  $f(0) = 0$  has the property that  $d(0, f(z)) \leq d(0, z)$ , where  $d$  = the Poincaré distance. Since the action by holomorphic isometries of  $\Delta$  to itself relative to the Poincaré metric are transitive on  $\Delta$ , what is known as the Schwarz–Pick Lemma follows immediately: *If  $f: \Delta \rightarrow \Delta$  is holomorphic, then  $d(f(z_1), f(z_2)) \leq d(z_1, z_2)$  for all  $z_1, z_2 \in \Delta$ , where  $d$  is the Poincaré distance.*

This can be extended to Riemann surfaces as follows: If  $M$  is a Riemann surface that is holomorphically covered by the unit disc  $\Delta$ . say  $\pi: \Delta \rightarrow M$  is a holomorphic covering map, then the covering transformation of the covering  $\pi$  are holomorphic isometries for the Poincaré metric. It follows that  $M$  has a unique Riemannian (indeed Hermitian) metric for which  $\pi$  is locally isometric. Let us call this the *canonical metric* for  $M$ .

If  $M_1$  and  $M_2$  are two such Riemann surfaces, with canonical (Riemannian) metrics  $g_1$  and  $g_2$  respectively, and if  $F: M_1 \rightarrow M_2$  is a holomorphic map, then the pull back  $F^*g_2$  to  $M_1$  of the metric  $g_2$ , is less than or equal to  $g_1$ , i.e.,

$$F^*g_2|_p \leq g_1|_p$$

for each  $p \in M_1$ , where

$$F^*g_2|_p(v, w) = g_2|_{F(p)}(dF_p(v), dF_p(w)),$$

for all  $v, w$  in the real tangent space of  $M_1$  at  $p$ , so  $dF_p(v)$  and  $dF_p(w)$  are in the real tangent space of  $M_2$  at  $F(p)$ .

To see this distance nonincreasing property, note that if  $\pi_1: \Delta \rightarrow M_1$  and  $\pi_2: \Delta \rightarrow M_2$  are holomorphic covering maps, then  $F: M_1 \rightarrow M_2$  can be lifted to a holomorphic map  $\hat{F}: \Delta \rightarrow \Delta$  in such a way that the diagram

$$\begin{array}{ccc} \Delta & \xrightarrow{\hat{F}} & \Delta \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ M_1 & \xrightarrow{F} & M_2 \end{array}$$

commutes. The map  $\hat{F}$  is nonincreasing for the Poincaré metric by the Schwarz–Pick Lemma. (cf. [13])

It follows, since  $\pi_1$  and  $\pi_2$  are (local) isometries, in the sense indicated of the inequality on the pull-back metric, that  $F: M_1 \rightarrow M_2$  is length nonincreasing/distance nonincreasing for the canonical metrics of  $M_1$  and of  $M_2$ , respectively.

### 3. Holomorphic maps of Riemann surfaces of general type

In the case that  $M = \{z \in \mathbb{C}: \frac{1}{\sqrt{R}} < |z| < \sqrt{R}\}$ ,  $R > 1$ , which is covered by the unit disc  $\Delta$ , the canonical metric on  $M$  is straightforward to compute. The function  $F(z) = e^{iz}$  defines a covering map of the “strip”  $\{x + iy: -\ln R < y < \ln R\}$  onto  $M$ , while the strip is biholomorphic to the upper half plane via the composition of a linear map, exponentiation, and a linear fractional transformation. Tracing through gives the formula

$$\frac{(\pi/(2\ln R))^2}{r^2 \cos^2(\pi \ln r / (2\ln R))} dr^2 + \frac{(\pi/(2\ln R))^2}{\cos^2(\pi \ln r / (2\ln R))} d\theta^2$$

for the canonical metric on  $M$  in  $(r, \theta)$  polar coordinates. (cf. [7, p. 39])

The free homotopy classes of closed curves in  $M$  are characterized by winding number (around 0) and are nontrivial for all winding numbers except 0. Since winding number is  $\frac{1}{2\pi}$  times the total change in polar angle  $\theta$  around the curve, it follows easily that, for any rectifiable curve  $\gamma$  in  $M$ , the length of the curve in the canonical metric is  $\geq \pi^2 / \ln R$ . Thus the  $\ell_1$ -invariant of  $M$  in its canonical metric is  $\geq \pi^2 / \ln R$  and indeed

$$\ell_1(M) = \frac{\pi^2}{\ln R}, \tag{1}$$

with the infimum realized by once-around the curve  $r = 1$  ( $\theta$  goes from 0 to  $2\pi$ ), either clockwise or counterclockwise. (Throughout, we are using the Poincaré metric on the unit disc  $\Delta$  to be  $4(1 - z\bar{z})^{-2} dz d\bar{z}$  so that the curvature  $\equiv -1$ .)

Since  $\{z \in \mathbb{C}: A < |z| < B\}$  is linearly biholomorphic to  $\{z: \sqrt{A/B} < |z| < \sqrt{B/A}\}$ , the  $\ell_1$ -invariant of  $\{z \in \mathbb{C}: A < |z| < B\}$  is equal to  $\pi^2 / \ln(B/A)$ . The  $\ell_1$ -invariant is preserved by biholomorphic mapping, so the classical result follows:

**Theorem 4 (Hadamard).** *The region  $\{z: A_1 < |z| < B_1\}$  is biholomorphic to  $\{z: A_2 < |z| < B_2\}$  if, and only if,  $B_1/A_1 = B_2/A_2$ .*

This result, originally proved by Hadamard (cf. [16]), is usually proved by non-metric methods, e.g., Schwarz Reflection, [1, 8, 25] et al.

The nonincreasing property of the  $\ell_1$ -invariant gives an extension of this result:

**Theorem 5.** *If  $0 < A_1 < B_1$ ,  $0 < A_2 < B_2$  and  $B_1/A_1 > B_2/A_2$ , then every holomorphic mapping  $f: \{z: A_1 < |z| < B_1\} \rightarrow \{z: A_2 < |z| < B_2\}$  is homotopically trivial, that is, it is homotopic to a constant map.*

**Proof.** The map  $f$  is homotopically trivial if and only if the  $f$ -image of the curve  $\gamma(t) = \frac{1}{2}(A_1 + B_1)(\cos t, \sin t)$ , ( $t \in [0, 2\pi]$ ) is homotopic to a constant curve: this is an immediate consequence of covering space theory. If  $f$  is not homotopically trivial, then the free homotopy class of the  $f$ -image indicated is nontrivial. Then Proposition 3 gives that

$$\ell_1(\{z: A_2 < |z| < B_2\}) \leq \ell_1(\{z: A_1 < |z| < B_1\}),$$

and the result follows from the formula (1) for the  $\ell_1$ -invariant.  $\square$

This Theorem 5 implies in particular the historical result:

**Theorem 6 (de Possel [21]).** *There is a 1-1 holomorphic mapping  $g: \{z: 0 < A_1 < |z| < B_1\} \rightarrow \{z: 0 < A_2 < |z| < B_2\}$  whose image separates the boundary components of  $\{z: A_2 < |z| < B_2\}$  if and only if  $B_1/A_1 \leq B_2/A_2$ .*

Notice that the condition on  $g$  implies that  $g$  is injective on homotopy; For instance the  $g$ -image of the curve  $\gamma$  in the preceding proof will be a closed curve in  $\{z: A_2 < |z| < B_2\}$  that goes around the origin exactly once. Notice also that the “if” part of this theorem is obviously true by a complex linear map. (cf. [5, 10] for the original proof.)

A conclusion by the same argument holds for multiply connected domains and Riemann surfaces as well.

**Theorem 7.** *Let  $X$  and  $Y$  be Riemann surfaces holomorphically covered by the unit disc with  $\ell_1(X) < \ell_1(Y)$ . If  $f: X \rightarrow Y$  is a holomorphic map, then it cannot be injective on homotopy. In particular,  $f$  cannot be a homotopy equivalence.*

#### 4. Non-Riemannian Kobayashi hyperbolic case

The only properties of the “canonical metric” on Riemann surfaces covered by the disc that were crucial here were that holomorphic maps were distance nonincreasing and that the canonical metric was locally comparable (in both directions) with any metric derived from local coordinates. In this context, it is clear that the whole viewpoint has an immediate extension to complex manifolds which are “hyperbolic” in the sense introduced by S. Kobayashi. In this section, we consider only complex manifolds (or complex spaces) that are hyperbolic in the sense of Kobayashi, namely, those for which the Kobayashi pseudodistance is an actual distance function. [14, 15].

In this setting, one can define again the  $\ell_1$ -invariant  $\ell_1(M)$  of a hyperbolic manifold to be the infimum of the lengths of closed curves that are not freely homotopic to 0 (assuming  $M$  is not simply connected). Then following the pattern of before one gets the results:

**Theorem 8.** *If  $f: M_1 \rightarrow M_2$  is a holomorphic map of hyperbolic manifolds, and if  $f$  is injective on free homotopy classes of closed curves (in the sense that if  $\gamma(t)$  is a closed curve in  $M_1$  not freely homotopic to a constant, then  $f(\gamma(t))$  is not freely homotopic to a constant), then*

$$\ell_1(M_2) \leq \ell_1(M_1).$$

Thus the results discussed in the concrete instances of Riemann surfaces covered by the unit disc and of annular regions in  $\mathbb{C}$  in particular, can be extended to far more general settings. The idea of the  $\ell_1$ -invariant via free homotopy classes of closed curves can also be extended to higher dimensional homotopy classes of maps of the  $k$ -sphere into complex hyperbolic manifolds, and to some extent, into general length spaces. These methods will be explored in subsequent sections.

### 5. Holomorphic mappings of tubular domains

The previous discussion of annular regions in  $\mathbb{C}$  has a straightforward extension to *tubular domains* in  $\mathbb{C}^n$ ,  $n > 1$ . For this, define, for  $0 < r < 1$ ,

$$T^n(r) = \{\vec{z} = (z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| = 1, \|\vec{z} - (z_1, 0, \dots, 0)\| < r\},$$

where  $\|\cdot\|$  is the Euclidean norm. Set  $A(r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z| < r_2\}$  where  $0 < r_1 < r_2$ . The projection map  $P_n(z_1, \dots, z_n) = z_1$  takes  $T^n(r)$  onto  $A(1 - r, 1 + r)$  and there is a natural injection  $J_n(z) = (z, 0, \dots, 0)$  mapping  $A(1 - r, 1 + r)$  into  $T^n(r)$ . These maps are homotopy equivalences:  $J_n$  and  $P_n$  are homotopy inverses to each other. This observation together with Theorem 5 gives rise to a comparison result on these *tubular domains* (meaning tubular neighborhoods of a circle, where the dimensions need not be equal):

**Theorem 9.** *If  $f: T^n(r) \rightarrow T^m(s)$  with  $r, s$  between 0 and 1, is holomorphic and if  $s < r$ , then  $f$  is homotopic to a constant map.*

**Proof.** The map  $f$  is homotopic to a constant map if and only if the  $f$ -image of the closed curve  $\Gamma(t) = (e^{it}, 0, \dots, 0)$ ,  $t \in [0, 2\pi]$  in  $T^n(r)$  is homotopic to a constant curve in  $T^m(s)$ . This is again by the standard covering space theory. But, this happens if and only if the holomorphic map  $F: A(1 - r, 1 + r) \rightarrow A(1 - s, 1 + s)$  defined by  $F := P_m \circ f \circ J_n$  is homotopic to a constant. This last follows from the homotopy equivalences already noted. Now Theorem 5 implies that, if  $s < r$ , this map  $F$  is, and consequently  $f$  is also, homotopic to a constant. □

### 6. Higher homotopy invariants

It is natural to consider extending the previous introduced ideas about curve lengths in free homotopy classes of closed curves to higher dimensional homotopy classes. In particular, if  $(X, d)$  is a metric space (which we assume arc-wise connected for convenience), then consider continuous maps  $f: S^k \rightarrow X$ ,  $k > 1$ ,  $S^k = \{(x_0, \dots, x_k) \in \mathbb{R}^{k+1} : x_0^2 + \dots + x_k^2 = 1\}$  and define two such  $f_0, f_1: S^k \rightarrow X$  to be free-homotopic if there is a continuous function  $H: S^k \times [0, 1] \rightarrow X$  such that, for all  $p$ ,  $H(p, 0) = f_0(p)$  and  $H(p, 1) = f_1(p)$ . If one has a situation where, for suitably restricted  $f$ , one can define the measure of (the image of) such  $f: S^k \rightarrow X$ , then one could imitate the definition of the  $\ell_1$ -invariant corresponding to the  $k = 1$  case. But certain difficulties arise: One needs an idea of such a measure and one would want to know that each free homotopy class of continuous maps  $S^k \rightarrow X$  would contain at least one map  $f$  with the associated measure of the image of  $f$  being finite. This would correspond to rectifiable curves and their lengths, and free-homotopy classes of curves, as in earlier sections.

In the case where  $(X, d)$  is a Riemannian manifold and  $d$  its Riemannian metric distance, the right idea of measure is, for smooth maps  $f: S^k \rightarrow X$ , the measure of  $f|_A$ ,  $A \subset S^k$ ,  $A =$  the set of points of  $S^k$  where the differential  $df$  has rank  $k$ , and the measure of  $f|_A$  is the usual Riemann-metric induced measure on  $k$ -dimensional submanifolds. This measure is

$$\mu(f) = \int_{S^k} |\text{volume form}|$$

where the |volume form| on  $(v_1, \dots, v_k)$ , where  $v_1, \dots, v_k \in T_p S^k$ , is equal to the absolute value of the  $k$ -dimensional Riemannian volume at  $f(p)$  of  $f_*(v_1) \wedge \dots \wedge f_*(v_k)$ . It is easy to show (and well-known) that every free homotopy class of continuous maps  $S^k \rightarrow X$  in this case ( $X$  is a Riemannian manifold) contains a smooth (not necessarily everywhere nonsingular) map: If  $f$  is a continuous map in the free homotopy class then a sufficiently good smooth approximation of  $f$  will be in the same free homotopy class (by deformation along minimal geodesics). So every free homotopy class contains finite  $k$ -measure maps and hence one can define the infimum of measures over finite-measure (smooth, e.g.) maps in the class and also an  $\ell_k$  invariant (= infimum over all nontrivial free homotopy classes).

In the case of more general metric spaces, it is not in general clear that any useful concept of  $k$ -dimensional measure exists. But in the case of metrics which are locally equivalent to Riemannian metrics, the familiar general idea of Hausdorff  $k$ -dimensional measure, can be used.

**Definition 10 (cf. [15, p. 343]).** *Given a metric space  $(X, d)$ , for a nonnegative real number  $k$  and a subset  $A$ , the  $k$ -dimensional Hausdorff measure  $\mu_{X,d}^k(A)$  is defined as*

$$\mu_{X,d}^k(A) = \sup_{\epsilon > 0} \inf \left\{ \sum_{i=1}^{\infty} (\delta(A_i))^k : A = \bigcup_{i=1}^{\infty} A_i, \delta(A_i) < \epsilon \right\},$$

where  $\delta(A_i)$  is the diameter of  $A_i$  defined by

$$\delta(A_i) = \sup_{p,q \in A_i} d(p,q).$$

For a smooth map  $f: S^k \rightarrow M$  of the  $k$ -sphere  $S^k$  into a Riemannian manifold  $M$ , this  $\mu_M^k(f(S^k))$  gives essentially the same concept of the  $k$ -dimensional measure as the Riemannian measure already defined. But this new notion of measure has the advantage that smoothness is not involved.

Rather than exploring these matters further in generality, we restrict our attention now to the specific situation of the Kobayashi metric for a bounded connected open set  $U$  in  $\mathbb{C}^n$ . The open set  $U$  has its Kobayashi metric in the infinitesimal form, the Kobayashi–Royden metric, as it is usually defined by

$$F_U^{Kob}(p, v) = \inf \left\{ r > 0 : f \in \mathcal{O}(\Delta, U), f(0) = p, df(0) = \frac{v}{r} \right\}$$

where  $\Delta = \{z \in \mathbb{C} : |z| < 1\}$  and where  $\mathcal{O}(\Delta, U)$  denotes the set of holomorphic maps from  $\Delta$  to  $U$ . As is well-known [23, 24], the Kobayashi distance from  $p$  to  $q$  for  $p, q \in U$  is the infimum of the length of the  $C^1$  curves from  $p$  to  $q$  in  $U$  where the length is taken to be the integral  $\int_{\gamma} F_U^{Kob}(z, dz)$ . (The existence of this integral was also shown in [23, 24]). Moreover, it was shown in [2] that  $F_U^{Kob}(p, v)$  is locally comparable to the standard Euclidean norm of  $v$  in  $\mathbb{C}^n$ . This leads easily to the fact that for every free homotopy class of maps  $S^k \rightarrow U$ , there is a smooth map  $f: S^k \rightarrow U$  (as already noted) and the smooth map will have finite Hausdorff  $k$ -measure relative to the Kobayashi metric, denoted by  $\mu_{Kob,U}^k(f(S^k))$ . We call such a map  $k$ -rectifiable (meaning: Hausdorff  $k$ -measure is finite), and the  $k$ -measure just constructed will be called, in this paper, the Hausdorff–Kobayashi  $k$ -measure.

Thus, one can define an invariant

$$\ell_k(U) = \text{infimum of the Hausdorff–Kobayashi } k\text{-measures over all the Hausdorff–Kobayashi } k\text{-rectifiable representatives of the nontrivial free homotopy classes in } U$$

**Lemma 11.** *If  $F: U \rightarrow V$  is a holomorphic mapping from a bounded domain in  $\mathbb{C}^n$  to a bounded domain in  $\mathbb{C}^m$  and if it is injective on free homotopy in the sense that the  $F$ -image of every nontrivial  $k$ -class in  $U$  is nontrivial in  $V$ , then  $\ell_k(V) \leq \ell_k(U)$ .*



**Proof.** This follows immediately from the fact that  $F$  is distance nonincreasing and hence diameter nonincreasing with respect to the Kobayashi metrics of  $U$  and  $V$ , so the Hausdorff sums are not increased by composition with  $F$ . □

### 7. Illustration of the $k$ -homotopy invariant

The most natural way to find a bounded connected open set in  $\mathbb{C}^n$  which has a nontrivial  $k$ -homotopy class is to take a tubular neighborhood of an embedded  $k$ -sphere. This is particularly interesting from the viewpoint of complex analysis if one takes the  $k$ -sphere to be totally real as a submanifold. In particular, the unit  $k$ -sphere in  $\mathbb{R}^{k+1}$  is totally real in  $\mathbb{C}^{k+1}$ . Let  $T_r$  be the radius  $r > 0$  tubular neighborhood

$$T_r = \left\{ (x_0, \dots, x_k) + \lambda \vec{u} : x_0^2 + \dots + x_k^2 = 1, \vec{u} \in \mathbb{C}^{k+1}, \|u\| = 1, 0 \leq \lambda < r \right\}.$$

For small values of  $r$ , this is a strongly pseudoconvex domain with  $C^\infty$  boundary (actually  $0 < r < \frac{1}{2}$  suffices). According to [6], the Kobayashi–Royden metric near the boundary goes to  $+\infty$  relative to the Euclidean metric. Note that there is a natural smooth projection, say  $P: T_r \rightarrow S^k$ , defined by  $P(z) =$  the closest point to  $z$  in the Euclidean sense in the set  $S^k$ . The injection  $S^k \hookrightarrow T_r$  is a homotopy equivalence with  $P$  its homotopy inverse. And the identity (injection) map  $S^k \rightarrow T_r$  is homotopically nontrivial. There may be other smooth maps, say  $\Gamma: S^k \rightarrow T_r$ , homotopic to the injection which have the associated Hausdorff–Kobayashi  $k$ -measure assigned to them being less than the measure assigned to the injection. But the infimum will be realized among the family of maps with image lying in some set of the form

$$\left\{ \vec{x} + \lambda \vec{u} : \vec{x} \in S^k, \vec{u} \in \mathbb{C}^{k+1}, \|u\| = 1, 0 \leq \lambda \leq r_1 \right\}$$

for some  $r_1 < r$ : this follows easily from the estimates on the growth of the Kobayashi metric near the boundary of  $T_r$  compared to the Euclidean metric. This will yield a conclusion similar to Theorem 4, once a technical point about the Kobayashi/Royden infinitesimal metric is established. This will be discussed further in the following sections.

### 8. Monotonicity of the Kobayashi metric

To find some analogue in this situation of Theorems 4, one needs to have an idea of strict monotonicity of the Kobayashi distance for the domains (open connected subsets) in  $\mathbb{C}^n$ . Specifically, if two domains  $\Omega_1$  and  $\Omega_2$  satisfy  $\Omega_1 \subset \Omega_2$ , then of course their respective Kobayashi–Royden metrics  $F_{\Omega_1}^{\text{Kob}}, F_{\Omega_2}^{\text{Kob}}$  satisfy  $F_{\Omega_2}^{\text{Kob}}(p, v) \leq F_{\Omega_1}^{\text{Kob}}(p, v)$  for all  $p \in \Omega_1$  and  $v \in \mathbb{C}^n$ . But we would like to obtain a strict comparison between them.

**Lemma 12.** *If  $\Omega_1$  and  $\Omega_2$  are bounded domains in  $\mathbb{C}^n$  satisfying  $\Omega_1 \subset\subset \Omega_2$  (relatively compact), then for any compact subset  $K$  of  $\Omega_1$  there exists a constant  $c_K$  with  $0 < c_K < 1$  such that*

$$F_{\Omega_2}^{\text{Kob}}(p, v) < c_K F_{\Omega_1}^{\text{Kob}}(p, v),$$

for any  $p \in K$  and any  $v \in \mathbb{C}^n$ .

**Proof.** Let the bounded domains  $\Omega_1$  and  $\Omega_2$  in  $\mathbb{C}^n$  satisfy  $\Omega_1 \subset\subset \Omega_2$ , i.e.,  $\Omega_1$  is a relatively compact subdomain of  $\Omega_2$ .

Then there exists  $\delta > 0$  such that the Euclidean distance between  $\Omega_1$  and  $\mathbb{C}^n \setminus \Omega_2$  is at least  $\delta$ .

Let  $p \in \Omega_1$ . Notice that there exist positive numbers  $b$  and  $B$  such that

$$b \leq F_{\Omega_1}^{\text{Kob}}(p, v) \leq B, \quad \forall v \in \mathbb{C}^n \text{ with } \|v\| = 1,$$

where  $\|\cdot\|$  denotes the standard Euclidean norm. Fix  $p \in \Omega_1$  and  $v \in \mathbb{C}^n$  with  $\|v\| = 1$ . Let  $\epsilon > 0$  be given arbitrarily, and then choose  $h \in \mathcal{O}(\Delta, \Omega_1)$  satisfying  $h(0) = p$ ,  $h'(0) = v/r$  for some  $r > 0$  and

$$F_{\Omega_1}^{\text{Kob}}(p, v) \leq r < F_{\Omega_1}^{\text{Kob}}(p, v) + \epsilon.$$

Take

$$\tilde{h}(z) = h(z) + \delta z v.$$

Then

$$\tilde{h}(0) = h(0) = p, \quad \tilde{h}'(0) = h'(0) + \delta v = \left(\frac{1}{r} + \delta\right) v$$

and

$$\tilde{h}(z) \subset \Omega_2, \quad \forall z \in \Delta,$$

since  $\|\tilde{h}(z) - h(z)\| = \|\delta z v\| < \delta$  for every  $z \in \Delta$ . So  $\tilde{h} \in \mathcal{O}(\Delta, \Omega_2)$ .

This implies

$$\begin{aligned} F_{\Omega_2}^{\text{Kob}}(p, v) &\leq \frac{1}{1/r + \delta} \\ &= \frac{r}{1 + r\delta} \\ &\leq \frac{F_{\Omega_1}^{\text{Kob}}(p, v) + \epsilon}{1 + \delta F_{\Omega_1}^{\text{Kob}}(p, v)} \\ &\leq \frac{1}{1 + \delta b} \left( F_{\Omega_1}^{\text{Kob}}(p, v) + \epsilon \right). \end{aligned}$$

Since  $\epsilon > 0$  is arbitrary, we obtain that

$$F_{\Omega_2}^{\text{Kob}}(p, v) \leq \frac{1}{1 + \delta b} F_{\Omega_1}^{\text{Kob}}(p, v)$$

for any  $v \in \mathbb{C}^n$ , due to the homogeneity of the Kobayashi–Royden metric.

Note that  $b$  depends on the location of  $p$ . But on a compact set it stays bounded away from zero, and the desired conclusion follows immediately.  $\square$

The restriction in Lemma 12 to a compact set  $K$  can be removed.

**Lemma 13.** *If  $U$  is a bounded open set in  $\mathbb{C}^n$  with its closure contained in another bounded open set  $V$ , then there is a constant  $c \in (0, 1)$  such that*

$$F_V^{\text{Kob}}(p, v) \leq c F_U^{\text{Kob}}(p, v) \quad \forall (p, v) \in U \times \mathbb{C}^n,$$

where  $F_U^{\text{Kob}}$  and  $F_V^{\text{Kob}}$  are the infinitesimal Kobayashi–Royden metrics of  $U$  and  $V$ , respectively.

**Proof.** Denote by  $\text{cl}(A)$  the closure in  $\mathbb{C}^n$  of the subset  $A$  of  $\mathbb{C}^n$ . There is a bounded open set  $W$  satisfying

$$\text{cl}(U) \subset W \subset \text{cl}(W) \subset V.$$

With  $W$  so chosen, we have:

$$F_W^{\text{Kob}}(q, v) \leq F_U^{\text{Kob}}(q, v) \quad \forall (q, v) \in U \times \mathbb{C}^n,$$

and

$$F_V^{\text{Kob}}(p, v) \leq F_W^{\text{Kob}}(p, v) \quad \forall (p, v) \in W \times \mathbb{C}^n.$$

Lemma 12 gives that there is a constant  $c$  with  $0 < c < 1$  such that

$$F_V^{\text{Kob}}(q, v) \leq c F_W^{\text{Kob}}(q, v) \quad \forall (q, v) \in \text{cl}(U) \times \mathbb{C}^n.$$

In particular, this yields that

$$F_V^{\text{Kob}}(p, v) \leq c F_W^{\text{Kob}}(p, v) \leq c F_U^{\text{Kob}}(p, v) \quad \forall (p, v) \in U \times \mathbb{C}^n,$$

as desired.  $\square$

### 9. The example from Section 7 concluded

The results of Section 6 and Section 7 together with the growth of the Kobayashi–Royden metric established in [6] can be combined to establish the results about the domains  $T_r$  defined in Section 6:

First we note that the  $\ell_k$ -invariants are nonzero in this case. As before, we assume that all  $r$ -values are small enough that  $T_r$  is strongly pseudoconvex.

**Lemma 14.** *With  $T_r$  as defined in Section 6, and  $\ell_k$  defined as before,*

$$\ell_k(T_r) > 0.$$

**Proof.** Since  $T_r$  has smooth strictly pseudoconvex boundary, Graham [6] gives that there is a compact subset  $K$  of  $T_r$  such that for each  $p \in T_r \setminus K$  and all  $v$ ,

$$F_{T_r}^{\text{Kob}}(p, v) \geq \|v\|,$$

where  $\|\cdot\|$  represents the usual Euclidean norm. For such a fixed compact set  $K$ , there is a constant  $c > 0$  such that

$$F_{T_r}^{\text{Kob}}(p, v) \geq c\|v\|$$

for all  $p \in K$  and all  $v \in \mathbb{C}^n$ .

Replace  $c$  by  $\min\{c, 1\}$ . It follows that  $\ell_k(T_r) \geq c^k L(T_r)$ , where  $L(T_r)$  represents the infimum of the Euclidean Hausdorff  $k$ -measure of the image of  $S^k$  in  $T_r$  not homotopic to a constant. This latter infimum is positive by elementary considerations.  $\square$

**Theorem 15.** *If  $0 < r < s$ , then  $\ell_k(T_r) > \ell_k(T_s)$ . In particular,  $T_r$  is not biholomorphic to  $T_s$ .*

**Proof.** Consider the inclusion map  $T_r \hookrightarrow T_s$ . By Graham [6], given any  $C > 0$ , there is a compact subset  $K$  of  $T_r$  such that  $F_{T_r}^{\text{Kob}} \geq C \cdot (\text{Euclidean metric})$  at every point  $p \in T_r \setminus K$ , since  $F_{T_r}^{\text{Kob}}$  goes to infinity at the boundary of  $T_r$  and hence, choosing  $C$  sufficiently large,  $F_{T_r}^{\text{Kob}} \geq 2F_{T_s}^{\text{Kob}}$ . This follows since  $F_{T_s}^{\text{Kob}}$  is bounded by some multiple of Euclidean metric on  $T_r$  since  $T_r \subset\subset T_s$ . By Lemma 12 (and its proof) there is an  $\epsilon$  with  $0 < \epsilon < 1$  such that  $F_{T_r}^{\text{Kob}} \geq (1 + \epsilon)F_{T_s}^{\text{Kob}}$  at every point of  $K$ . Then

$$F_{T_r}^{\text{Kob}} \geq (1 + \epsilon)F_{T_s}^{\text{Kob}}$$

at every point of  $T_r$ . It follows that  $\ell_k(T_r) \geq (1 + \epsilon)^k \ell_k(T_s)$  and, since  $\ell_k(T_s) > 0$ ,

$$\ell_k(T_r) > \ell_k(T_s).$$

This completes the proof.  $\square$

The arguments used to prove Theorem 5 can be extended in a straightforward way to prove a corresponding result for the domains of  $T_r$  type:

**Theorem 16.** *If  $r_1 > r_2$ , then every holomorphic mapping  $f: T_{r_1} \rightarrow T_{r_2}$  is homotopic to a constant map.*

### 10. Tubular neighborhoods in general

The analysis of tubular neighborhoods of totally real embeddings of spheres in the previous section can be extended to more general circumstances. But this extension involves what amounts to a shift from homotopy to homology: the role of being homotopically nontrivial is taken over by having degree not equal to 0.

The first step is to define the relevant concept of degree: Suppose that  $M$  is a smooth ( $C^2$  suffices) compact connected submanifold without boundary of a Euclidean space  $\mathbb{R}^N$  and let

$$T_r = \bigcup_{w \in M} \{v \in \mathbb{R}^N : \|v - w\| < r\},$$

where  $\|\cdot\|$  is the usual Euclidean norm. Notice that there exists  $R > 0$  such that  $T_r$ , for any  $r$  with  $0 < r < R$ , there exists a natural projection  $\pi: T_r \rightarrow M$  onto  $M$ , defined by

$$\pi(v) = \inf_{p \in M} \|p - v\|$$

so that  $\pi(v)$  is the closest point to  $v$  among points of  $M$ , with respect to the Euclidean distance. If  $g: M \rightarrow T_r$  is a continuous map of  $M$  into  $T_r$ , then it is natural to define the *degree* of  $g$ , denoted by  $\deg g$ , to be the degree of  $\pi \circ g: M \rightarrow M$ . (If  $M$  is orientable, this is to be the usual  $\mathbb{Z}$ -valued degree. If  $M$  is nonorientable, we take degree to be  $\mathbb{Z}_2$ -valued.) If  $G: T_{r_1} \rightarrow T_{r_2}$  is a continuous map, with  $r_1, r_2 \in (0, R)$ , we define

$$\deg G := \text{the degree of } G \circ i: M \rightarrow T_{r_2},$$

where  $i: M \rightarrow T_{r_1}$  is an injection. As well known, these concepts of degree are multiplicative: the degree of a composition is equal to the product of the degrees.

With these definitions in sight, the analogue of Theorem 16 is the following:

**Theorem 17.** *Let  $M$  be a smooth compact connected totally real submanifold of  $\mathbb{C}^n$ . Then there is a constant  $R > 0$  such that, if  $0 < r < s < R$  and if  $F: T_s \rightarrow T_r$  is holomorphic then,  $F$  has degree zero.*

The conditions on  $R$  here are such that

- $T_r$  has smooth strongly pseudoconvex boundary and
- the projection map  $\pi: T_r \rightarrow M$  is well defined (and continuous in particular),

whenever  $0 < r < R$ .

Note that this result is in fact an extension of Theorem 16 since, in the case that  $M$  is a sphere,  $F$  being homotopic to a constant is equivalent to  $\deg F = 0$ . But in general, of course,  $\deg F = 0$  does not imply that  $F$  is homotopic to a constant. The most obvious example may be the map of  $S^p \times S^p$  to itself,  $p \geq 1$ , identity on the first factor and constant on the second.

Proof of Theorem 17 follows the general pattern of the proofs of previous theorems, but requires some preparation.

First, we need

**Definition 18.** *Let  $k = \dim M$ . For  $r \in (0, R)$  as above, let*

$$V_r^k := \inf \left\{ \mu_{\text{Kob}, T_r}^k(g(M)) \mid g: M \rightarrow T_r \text{ continuous and rectifiable with } \deg g \neq 0 \right\}.$$

Now we need

**Lemma 19.**  $V_r^k > 0$  for any  $r \in (0, R)$ .

**Proof.** With  $g: M \rightarrow T_r$ ,  $\deg g \neq 0$ , the composition  $\hat{g} := \pi \circ g: M \rightarrow M$  has nonzero degree and is hence surjective.

The projection  $\pi: T_R \rightarrow M$  admits a constant  $C > 0$  such that  $\|\pi(x) - \pi(y)\| \leq C\|x - y\|$  with respect to the standard Euclidean norm. Thus the Euclidean Hausdorff  $k$ -volume of  $\pi \circ g(M)$  is at least as large as that of  $M$ , bounded away from 0.

On the other hand, for  $0 < r < R$ ,  $T_r$  is bounded strongly pseudoconvex. So, by [6], the Kobayashi–Royden metric  $F_{T_r}^{\text{Kob}}$  of  $T_r$  goes to infinity compared to the Euclidean metric as the base point approaches the boundary of  $T_r$ . Hence there is a constant  $c > 0$  such that  $F_{T_r}^{\text{Kob}}(q, w) \geq c\|w\|$  for any  $q \in T_r, w \in \mathbb{C}^n$ . Hence we obtain that

$$\mu_{\text{Kob}, T_r}^k(g(M)) \geq c^k \mu_{\text{Euc}}^k(g(M)).$$

Since

$$\mu_{\text{Euc}}^k(g(M)) \geq C^{-k} \mu_{\text{Euc}}^k(\pi(g(M))) \geq C^{-k} \mu_{\text{Euc}}^k(M),$$

it follows that  $\inf_g \mu_{\text{Kob}, T_r}^k(g(M)) \geq c^k C^{-k} \mu_{\text{Euc}}^k(M) > 0$ , as desired. □

**Lemma 20.**  $V_s^k < V_r^k$  if  $0 < r < s < R$ .

**Proof.** By Lemma 13, the strict inclusion of the closure of  $T_r$  into  $T_s$  implies that there is a constant  $c$  with  $0 < c < 1$  such that  $F_{T_s}^{\text{Kob}}(z, v) \leq cF_{T_r}^{\text{Kob}}(z, v)$  for any  $z \in T_r$  and  $v \in \mathbb{C}^n$ . Hence for each  $g: M \rightarrow T_r$  of nonzero degree,

$$\mu_{\text{Kob}, T_s}^k(g(M)) \leq c^k \mu_{\text{Kob}, T_r}^k(g(M)).$$

So  $V_s^k \leq c^k V_r^k$ . Since  $V_s > 0$  by Lemma 19, it follows that  $V_s^k < V_r^k$ . □

**Proof of Theorem 17.** Let  $0 < r < s < R$  as in the hypothesis. Suppose that  $F: T_s \rightarrow T_r$  is holomorphic. And suppose, for a proof by contradiction that  $\deg F \neq 0$ . If a continuous map  $g: M \rightarrow T_s$  has a nonzero degree, then

$$\deg(F \circ g) = (\deg F)(\deg g) \neq 0.$$

Thus

$$V_r^k \leq \mu_{\text{Kob}, T_r}^k(F \circ g(M)),$$

since  $F \circ g: M \rightarrow T_r$  is a continuous map with nonzero degree. But  $\mu_{\text{Kob}, T_r}^k(F \circ g(M)) \leq \mu_{\text{Kob}, T_s}^k(g(M))$ , since  $F$  is Kobayashi-metric nonincreasing. Thus

$$V_r^k \leq \mu_{\text{Kob}, T_s}^k(g(M)),$$

which implies that

$$V_r^k \leq V_s^k.$$

This contradicts Lemma 20, and the proof is complete. □

### 11. Contraction Mapping and Homotopy

The situation of a map  $f: U \rightarrow V$  with  $V \subset U$  with  $f$  distance nonincreasing for some metric  $d_U$  on  $U$  is a natural condition for considering the ideas associated to iterations of *contraction mappings*. It is a familiar and long-standing principle of analysis that in the case where  $f$  is distance nonincreasing by a factor  $0 \leq \alpha < 1$ , then  $f$  must have a fixed point if  $U$  is complete with respect to the metric  $d_U$ .

This is a natural way to prove the existence of a short-term solution of ordinary differential equations, the local surjectivity in the Inverse Function Theorem via Newton’s Method, and many other basic results (cf. e.g. [4]).

Such contraction mapping ideas were used in [3] to prove a fixed point theorem for holomorphic maps in Banach spaces, now known as the Earle–Hamilton Fixed Point Theorem. The finite dimensional version was proved earlier in [22]. In both papers, the Carathéodory metric rather than the Kobayashi metric was used. We point out also that the finite dimensional version was proved even earlier in [12, p. 83 (p. 92, in the 2nd ed.)], using the fact that any compact analytic set in the complex Euclidean space consists of finitely many points. On the other hand, the following theorem shows that the Kobayashi metric can also be used in the contraction mapping context in a way similar to [22] and [3].

**Theorem 21.** *If  $U$  is a bounded domain in  $\mathbb{C}^n$  and  $f: U \rightarrow U$  is a holomorphic mapping such that  $f(U)$  is contained in a compact subset of  $U$ , then there is a unique point  $z_0 \in U$  such that  $f(z_0) = z_0$ . Moreover, this  $z_0$  is exactly the only point such that the iterates  $f^n(z) = f \circ \dots \circ f(z)$ ,  $n$ -times, of  $f$  converge uniformly on  $U$  to the constant map at  $z_0$  on  $U$ .*

**Proof.** Choose an open set  $V$  in  $\mathbb{C}^n$  with its compact closure  $\bar{V}$  in  $\mathbb{C}^n$  satisfying

$$f(U) \subset V \subset \bar{V} \subset U.$$

Let  $d_U$  be the Kobayashi distance on  $U$ . Lemma 13 shows that there exists  $0 \leq c < 1$  such that

$$F_U^{\text{Kob}}(f(p), df|_p(v)) \leq c \cdot F_V^{\text{Kob}}(f(p), df|_p(v)), \quad \forall (p, v) \in U \times \mathbb{C}^n,$$

which in turn implies that

$$\begin{aligned} F_U^{\text{Kob}}(f \circ \gamma(t), (f \circ \gamma)'(t)) &\leq c \cdot F_V^{\text{Kob}}(f \circ \gamma(t), (f \circ \gamma)'(t)) \\ &\leq c \cdot F_U^{\text{Kob}}(\gamma(t), \gamma'(t)) \end{aligned}$$

for any  $C^1$  curve  $\gamma: [0, 1] \rightarrow U$ . So we have

$$F_U^{\text{Kob}}(f \circ \gamma(t), (f \circ \gamma)'(t)) \leq c \cdot F_U^{\text{Kob}}(\gamma(t), \gamma'(t))$$

for any  $C^1$  curve  $\gamma: [0, 1] \rightarrow U$ , and consequently

$$d_U(f(p), f(q)) \leq c \cdot d_U(p, q), \quad \forall p, q \in U.$$

Namely,  $f$  is a strict contraction with the contraction factor  $c$ .

Now denote by  $f^n$  the  $n$ -th iterate of  $f$  defined inductively by

$$f^1 = f, \quad f^{n+1} = f \circ f^n \quad (n = 1, 2, \dots).$$

If  $p \in U$ , then it follows that the sequence  $f^n(p)$  is a Cauchy sequence with respect to  $d_U$ . Even if  $U$  were not necessarily  $d_U$ -complete, this Cauchy sequence still converges because, for  $n \geq 1$ ,  $f^n(p)$  belongs to  $\bar{V}$ , a compact set with  $\bar{V} \subset U$ . Thus  $z_0 := \lim_{n \rightarrow \infty} f^n(p)$  exists in  $\bar{V}$ , and consequently in  $U$ . And  $f(z_0) = z_0$ . Of course this is the unique fixed point: if  $f(z_1) = z_1$  then

$$d_U(z_0, z_1) = d_U(f(z_0), f(z_1)) \leq c \cdot d_U(z_0, z_1),$$

which implies  $d_U(z_0, z_1) = 0$  and  $z_0 = z_1$ .

The uniform convergence of  $f^n$  on  $U$  follows by

$$\begin{aligned} d_U(z_0, f^{k+1}(p)) &= d_U(f^{k+1}(z_0), f^{k+1}(p)) \\ &\leq c^k d_U(f(z_0), f(p)) \\ &\leq c^k d_U(z_0, f(p)), \end{aligned}$$

which implies

$$\sup_{p \in U} d_U(z_0, f^{k+1}(p)) \leq c^k \sup_{f(p) \in V} d_U(z_0, f(p)).$$

Note that  $\sup_{f(p) \in V} d_U(z_0, f^m(p))$  is bounded, since the Kobayashi distance is a continuous function by [2]. Moreover, there exists a positive integer  $m$  such that for some constant  $C > 0$  it holds that

$$\frac{1}{C} d_U(z_0, f^n(p)) \leq \|z_0 - f^n(p)\| \leq C d_U(z_0, f^n(p))$$

for any  $n > m$ . Altogether, we see that the convergence of  $f^k$  to the constant function at  $z_0$  is uniform on  $U$ , □

This recovers Theorem 16: With the notation therein, if  $f: T_{r_1} \rightarrow T_{r_2}$  with  $r_1 > r_2$  is holomorphic then, since  $T_{r_2}$  is relatively compact in  $T_{r_1}$ , there is a point  $z_0 \in T_{r_1}$  with  $f(z_0) = z_0$ . Consider  $f^n$  with  $n$  large. Then as mentioned before

$$\deg(f^n) = (\deg f)^n.$$

But  $f^n(T_{r_1})$  is contained in a small ball around  $z_0$ . So the composition  $P_{r_2} \circ f^n \circ i$  must have degree 0, and the rest of necessary arguments follows immediately.

A variant of this idea also produces this result about connected open subsets in  $\mathbb{C}^n$  and their images under holomorphic maps to themselves where the image is contained in a relatively compact subset.

**Theorem 22.** *If  $U$  is a bounded connected open set with smooth boundary in  $\mathbb{C}^n$ , and if there is a holomorphic map  $f: U \rightarrow U$  such that  $f(U)$  is contained in a compact subset of  $U$  and that, for some  $z \in U$  and for all  $k = 1, 2, \dots$ , the induced map  $f_*: \pi_k(U, z) \rightarrow \pi_k(U, f(z))$  is an isomorphism, then  $U$  is contractible.*

The theorem is illustrated by a bounded open sets  $U$  with smooth boundary that are star-shaped around the origin with  $f(z) = rz$  for a constant  $r$  with  $0 < r < 1$ .

The proof involves two lemmas:

**Lemma 23.** *Every bounded open set  $U$  in  $\mathbb{R}^N$  with smooth boundary has the homotopy type of a finite CW complex.*

**Lemma 24.** *With  $U$  and  $f$  as in the hypotheses of Theorem 22, and with  $z_0$  the fixed point of  $f$  (which has already been shown to exist), the  $k$ -th homotopy group  $\pi_k(U) = 0$  for any  $k = 1, 2, \dots$*

**Proof of Theorem 22.** It follows from these two lemmas and the Whitehead theorem ([26]. See also [11, Theorem 4.5 in p. 346]), since  $f$  and the constant map  $z_0$  have the same action on the homotopy groups of  $U$  at  $z_0$ . [The general result of Whitehead is that, if both  $X$  and  $Y$  have the homotopy type of connected finite CW complexes and if two continuous maps  $f, g: X \rightarrow Y$ , where  $f$  is a homotopy equivalence, satisfy the properties that  $f(x_0) = g(x_0)$  for some  $x_0 \in X$  and  $g_*, f_*: \pi_k(X, x_0) \rightarrow \pi_k(Y, f(x_0))$  are identical for every  $k = 1, 2, \dots$ , then  $f$  is homotopic to  $g$ .] In our case,  $f_* = 0$  and  $g$  is to be the constant map at  $z_0$ , so both  $f_*$  and  $g_*$  coincide as the trivial map of  $\pi_k$  for every  $k$ . □

**Remark 25.** The open set  $U$  is homotopically equivalent to a finite CW complex, but not homeomorphic to. Note for instance that any finite CW complex is compact. On the other hand, the fact that  $U$  is of the same homotopy type with a finite CW complex is sufficient for the preceding proof.

**Proof of Lemma 23.** Let  $d(z) = \text{dist}(z, \mathbb{C} \setminus U)$ , where  $\text{dist}(z, A) = \inf\{\|z - w\|: w \in A\}$ . Here of course  $\|\cdot\|$  is the Euclidean norm of  $\mathbb{C}^n$ . Since  $U$  is open,  $d(z) > 0$  for every  $z \in U$  and the map  $z \rightarrow d(z)$  is Lipschitz continuous. Also, since  $U$  has smooth boundary,  $d(z)$  is of class  $C^1$  at least ( $C^\infty$  if  $U$  has  $C^\infty$  boundary) on  $\mathcal{N}_\epsilon := \{z \in U: d(z) < \epsilon\}$  for sufficiently small a constant  $\epsilon > 0$ , and  $\|\text{grad}d\| = 1$  at every  $z \in U$  with  $d(z) < \epsilon$ , where  $\text{grad}d$  is the real Euclidean gradient.

Now set  $\delta(z) = -\ln d(z)$ . Then  $\delta$  is smooth on  $\mathcal{N}_\epsilon$ , and  $\|\text{grad}\delta(z)\| \geq 1/\epsilon$  for every  $z \in \mathcal{N}_\epsilon$ . (In fact, the inequality is  $=$ , but we do not need it here.) Now, for notational convenience, let

$$K_1 = \{z \in U: d(z) \geq \epsilon\}, \quad K_m = \{z \in U: d(z) \geq \epsilon/m\}$$

for  $m = 1, 2, \dots$  (We shall need only the first few of these). The sets  $K_m$  defined as such are compact in  $\mathbb{C}^n$ .

Now  $d$  may not be smooth on  $U$ ; indeed,  $d$  cannot be smooth since  $d$  has a maximum in  $U$  but  $\|\text{grad}d\| = 1$  at every point at which  $d$  is of class  $C^1$ . So  $\delta$  is also nonsmooth at some points. However, by standard convolution smoothing arguments, there is a smooth function, say  $\Delta$ , such that  $\Delta$  is uniformly close to  $\delta$  on  $K_4$  but (uniformly)  $C^1$ -close to  $\delta$  on  $K_4 \setminus K_1$ . By usual Morse theoretic considerations ([18]. cf. [17, Section I.6]. See also [9].) there is a function  $\Delta_1$  which is uniformly  $C^1$ -close to  $\Delta$  on  $K_3$  and has only nondegenerate critical points. Since  $\Delta$  and  $\delta$  are  $C^1$ -close on  $K_4 \setminus K_1$  and  $\|\text{grad}\delta\| \geq 1/\epsilon$  on  $K_4 \setminus K_1$ , it follows by the usual partition of unity argument that  $\Delta$  on the interior of  $K_3$ , and  $\delta$  on  $U$  can be patched together to yield a function  $\Delta_2$ , say, such that

$$\Delta_2(z) = \Delta(z), \quad \forall z \in K_3$$

with  $\|\text{grad}\Delta_2\| \geq 1/(2\epsilon)$  on  $U \setminus K_1$ , and such that

$$\Delta_2(z) = \delta(z), \quad \forall z \in U \setminus K_1.$$

Then  $\Delta_2$  is an exhaustion function for  $U$  with only nondegenerate critical points, since  $\Delta_2$  has no critical points in  $U \setminus K_1$  and hence only nondegenerate critical points in  $U$ , which necessarily lie in the set where  $\Delta = \Delta_1$ .

The standard Morse theory gives now that  $U$  has the homotopy type of a finite CW complex, with cells given by the finite number of nondegenerate critical points of  $\Delta_2$ .  $\square$

**Proof of Lemma 24.** Let  $z_0$  be the fixed point of  $f$ . Suppose that  $\Gamma: S^k \rightarrow U$  is a representation of a  $k$ -homotopy class in  $\pi_k(U, z_0)$ . By Theorem 21, the iterates  $f^n \circ \Gamma$  converge uniformly on  $S^k$  to the constant map at  $z_0$ . In particular, if a positive constant  $r$  is such that  $B^n(z_0, r) = \{z: \|z - z_0\| < r\} \subset U$  and  $f^n \Gamma(S^k) \subset B^n(z_0, r)$ , then  $[f^n \circ \Gamma] = 0$  in  $\pi_k(U, z_0)$ . But by hypothesis,  $f_*^n: \pi_k(U, z_0) \rightarrow \pi_k(U, z_0)$  is an isomorphism. So  $[\Gamma] = 0$  in  $\pi_k(U, z_0)$ .  $\square$

### Acknowledgements

After this article was written and posted in arXiv.org, T. Pacini kindly informed us the relevance of [19]. We express our gratitude for this.

### References

- [1] L. V. Ahlfors, *Complex analysis. An introduction to the theory of analytic functions of one complex variable*, 3rd ed., International Series in Pure and Applied Mathematics, McGraw-Hill, 1979.
- [2] T. J. Barth, "The Kobayashi distance induces the standard topology", *Proc. Am. Math. Soc.* **35** (1972), p. 439-441.
- [3] C. J. Earle, R. S. Hamilton, "A fixed point theorem for holomorphic mappings", in *Global Analysis*, Proceedings of Symposia in Pure Mathematics, vol. 16, American Mathematical Society, 1968, p. 61-65.
- [4] T. W. Gamelin, R. E. Greene, *Introduction to topology*, Dover Publications, 1999, xiv+234 pages.
- [5] G. M. Goluzin, *Geometric theory of functions of a complex variable*, Translations of Mathematical Monographs, vol. 26, American Mathematical Society, 1969, vi+676 pages.
- [6] I. Graham, "Boundary behavior of the Carathéodory, Kobayashi, and Bergman metrics on strongly pseudoconvex domains in  $C^n$  with smooth boundary", *Trans. Am. Math. Soc.* **207** (1975), p. 219-240.
- [7] R. E. Greene, K.-T. Kim, S. G. Krantz, *The geometry of complex domains*, Progress in Mathematics, vol. 291, Birkhäuser, 2011, xiv+303 pages.
- [8] R. E. Greene, S. G. Krantz, *Function theory of one complex variable*, Graduate Studies in Mathematics, vol. 40, American Mathematical Society, 2006, xix+504 pages.
- [9] R. E. Greene, H.-H. Wu, " $C^\infty$  approximations of convex, subharmonic, and plurisubharmonic functions", *Ann. Sci. Éc. Norm. Supér.* **12** (1979), no. 1, p. 47-84.
- [10] H. Grötzsch, "Zur Theorie der Verschiebung bei schlichter konformer Abbildung", *Comment. Math. Helv.* **8** (1935), no. 1, p. 382-390.
- [11] A. Hatcher, *Algebraic topology*, Cambridge University Press, 2002, xii+544 pages.
- [12] M. Hervé, *Several complex variables. Local theory*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 1, Oxford University Press, 1963.
- [13] K.-T. Kim, H. Lee, *Schwarz's lemma from a differential geometric viewpoint*, IISc Lecture Notes Series, vol. 2, World Scientific, 2011, xvi+82 pages.
- [14] S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings*, Pure and Applied Mathematics, vol. 2, Marcel Dekker, 1970, ix+148 pages.
- [15] ———, *Hyperbolic complex spaces*, Grundlehren der Mathematischen Wissenschaften, vol. 318, Springer, 1998.
- [16] J. E. Littlewood, "Quelques conséquences de l'hypothèse que la fonction  $\zeta(s)$  de Riemann n'a pas de zeros dans le demi-plan  $\Re(s) > 1/2$ ", *C. R. Acad. Sci. Paris* **154** (1912), p. 263-266.
- [17] J. W. Milnor, *Morse theory. Based on lecture notes by M. Spivak and R. Wells*, Annals of Mathematics Studies, vol. 51, Princeton University Press, 1963, vi+153 pages.
- [18] M. Morse, "The critical points of a function of  $n$  variables", *Trans. Am. Math. Soc.* **33** (1931), p. 72-91.
- [19] T. Pacini, "Extremal length in higher dimensions and complex systolic inequalities", *J. Geom. Anal.* **31** (2021), no. 5, p. 5073-5093.



- [20] P. Petersen, *Riemannian geometry*, Graduate Texts in Mathematics, vol. 171, Springer, 2006, xvi+401 pages.
- [21] R. de Possel, "Sur quelques propriétés de la représentation conforme des domaines multiplement connexes, en relation avec le théorème des fentes parallèles", *Math. Ann.* **107** (1933), no. 1, p. 496-504.
- [22] H.-J. Reiffen, "Die Carathéodorysche Distanz und ihre zugehörige Differentialmetrik", *Math. Ann.* **161** (1965), p. 315-324.
- [23] H. L. Royden, *Remarks on the Kobayashi metric*, Lecture Notes in Mathematics, vol. 185, Springer, 1970, 125-137 pages.
- [24] ———, "Report on the Teichmüller metric", *Proc. Natl. Acad. Sci. USA* **65** (1970), p. 497-499.
- [25] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill, 1987.
- [26] J. H. C. Whitehead, "Combinatorial homotopy II", *Bull. Am. Math. Soc.* **55** (1949), p. 453-496.