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Algebraic geometry / *Géométrie algébrique*

Motives and homotopy theory in logarithmic geometry

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Abstract. This document is a short user's guide to the theory of motives and homotopy theory in the setting of logarithmic geometry. We review some of the basic ideas and results in relation to other works on motives with modulus, motivic homotopy theory, and reciprocity sheaves.

Résumé. Ce document est un petit guide d'utilisation de la théorie des motifs et de la théorie de l'homotopie dans le cadre de la géométrie logarithmique. Nous passons en revue certaines des idées de base et des résultats en relation avec d'autres travaux sur les motifs avec module, théorie de l'homotopie motivique, et les faisceaux de réciprocité.

Keywords. Logarithmic geometry, motives, motivic homotopy theory.

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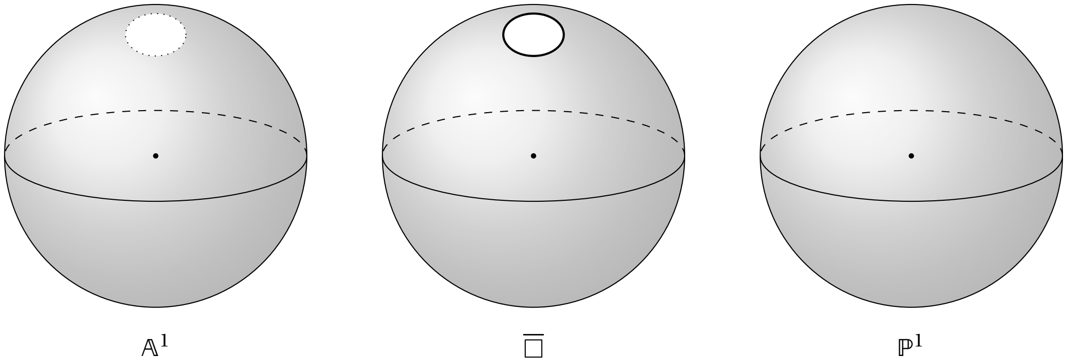

Figure 1. The logarithmic interval $\overline{\square}$ is contractible as $\mathbb{A}^1$ and compact as $\mathbb{P}^1$

1. Introduction

One of the basic ideas in Voevodsky's work on motives and motivic homotopy theory is to parameterize homotopies by the affine line $\mathbb{A}^1$, see [23, 32, 36]. This choice of a unit interval is adequate for many purposes, such as Rost and Voevodsky's solution of the Milnor and Bloch–Kato conjectures on Milnor K-theory and Galois cohomology [33,35]. Stable motivic homotopy theory has applications in the study of vector bundles over smooth affine schemes [2, 22], and opens up new vistas such as universal motivic invariants [16, 28]. By construction, all (co)homology theories in this setting are necessarily insensitive to $\mathbb{A}^1$. For example, motivic cohomology or higher Chow groups do not distinguish between $\mathbb{A}^1$ and the base field.

Many important algebro-geometric invariants, however, witness the difference between $\mathbb{A}^1$ and its field of definition: for example, the abelianized étale fundamental group in characteristic p (an object of study in geometric Class Field Theory), the additive group scheme $\mathbb{G}_a$, absolute Kähler differentials Ω^i, Hodge–Witt sheaves $W_n\Omega^i$, and crystalline and Hodge cohomology groups. Such examples bring the following question into focus: is there a convenient framework for non-$\mathbb{A}^1$-invariant (co)homology theories? Our attempt at answering this question invokes logarithmic geometry developed by Fontaine, Illusie, Kato, and many others [26]. In our approach we replace the affine line by its logarithmic counterpart

$$\overline{\square} := (\mathbb{P}^1, \infty) \tag{1}$$

Here we view the point at infinity as a boundary of the projective line $\mathbb{P}^1$, see Figure 1. Starting from this definition, we have developed theories of motives and homotopies in logarithmic geometry. The purpose of this note is to give a concise overview of [8, 9].

2. Background on logarithmic geometry

Our primary reference for logarithmic geometry is Ogus's book [26]. We let k be a field for simplicity, considered a log scheme with a trivial log structure.

2.1. *Logarithmic schemes and finite log correspondences*

Suppose $X \in \mathrm{Sm}_k$ is a smooth k-scheme with a smooth proper compactification $j\colon X \to \overline{X}$ for which the complement ∂X is a strict normal crossings divisor on $\overline{X}$ (this is can be achieved under resolution of singularities). The sheaf of monoids $j_*\mathscr{O}_X^\times \cap \mathscr{O}_{\bar{X}}$ on the small Zariski site of $\overline{X}$ specifies the data of a *compactifying log structure*; this is a convenient tool for keeping track of the boundary of the compactification, cf. Deligne's work on the definition of the mixed Hodge structure on the Betti cohomology of smooth open varieties [12]. The "log interval" $\overline{\square}$ arises in this way for $X = \mathbb{A}^1$, $\overline{X} = \mathbb{P}^1$, $\partial X = \infty$. Similarly, (X, D), where D is an effective Cartier divisor on $X \in \mathrm{Sm}_k$, defines a (divisorial) log scheme. Manifolds with boundary is a close topological analog of such log schemes. In parallel with schemes, logarithmic forms, i.e., differential forms with poles of order at most one along the boundary of the compactification ∂X, are used to define notions of log-smoothness and log-étaleness [26]. We write $l\mathrm{Sm}_k$ for the category of *fine and saturated* (fs for short) log schemes that are log smooth over k and $\mathrm{Sm}\,l\mathrm{Sm}_k$ for its full subcategory of log schemes X whose underlying scheme $\underline{X}$ is a smooth k-scheme. By [9, A.5.10], every $X \in \mathrm{Sm}\,l\mathrm{Sm}_k$ is obtained from a smooth k-scheme $\underline{X}$ with compactifying log structure given by a strict normal crossing divisor ∂X. The difference between $l\mathrm{Sm}_k$ and $\mathrm{Sm}\,l\mathrm{Sm}_k$ is intuitively measured by (singular) toric varieties, which are geometric objects defined by combinatorial data [13].

Replacing maps of schemes by finite correspondences in the sense of Suslin–Voevodsky turns Sm_k into the additive category of correspondences Cor_k. This is the starting point in Voevodsky's theory of mixed motives, see [1, 20]. In the logarithmic setting, we introduce an analogous notion of *finite log correspondences*. One can intuitively regard finite log correspondences as multi-valued functions, but some subtleties are arising involving the log structure. For every $X, Y \in l\mathrm{Sm}_k$, an *elementary log correspondence* Z from X to Y consists of

- an integral closed subscheme $\underline{Z}$ of $\underline{X} \times \underline{Y}$ that is finite and surjective over a connected component of the underlying k-scheme $\underline{X}$
- a morphism $Z^N \to Y$ of fs log schemes, where Z^N denotes the fs log scheme whose underlying scheme is the normalization of $\underline{Z}$ and the log structure $\mathscr{M}_{Z^N}$ is given by the pullback $p^*_{log}\mathscr{M}_X$, where $p : \underline{Z^N} \to \underline{X}$ denotes the induced scheme morphism

We turn $l\mathrm{Sm}_k$ into an additive category $l\mathrm{Cor}_k$ of finite log correspondences over k. The objects of $l\mathrm{Cor}_k$ are fs log schemes that are log smooth over k. As morphisms, we take the free abelian group generated by elementary log correspondences, see [9, 2.1]. The above definitions coincide with the ones in [20] and [36] when X and Y have trivial log structures. Let Λ be a commutative unital ring. A presheaf of Λ-modules with log transfers is an additive presheaf of Λ-modules on $l\mathrm{Cor}_k$. We let $\Lambda_{\mathrm{ltr}}(X)$ denote the presheaf with log transfers represented by $X \in l\mathrm{Sm}_k$.

2.2. *Dividing Nisnevich coverings and admissible blow-ups*

To build a category of log motives, we single out a topology witnessing local properties of log schemes. We achieve this in two steps. First, we consider the *strict Nisnevich* topology; it is the Grothendieck topology associated to the cd-structure on $l\mathrm{Sm}_k$ given by cartesian squares

$$\begin{array}{ccc} Y' & \xrightarrow{g'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{g} & X \end{array} \tag{2}$$

Here, g is an open immersion, f is strict étale (a strict morphism [26, Definition III.1.2.3] of fs log schemes and an étale morphism on the underlying schemes), and $f^{-1}(\underline{X} - g(\underline{X'})) \to \underline{X} - g(\underline{X'})$ is an isomorphism when both sides are considered with the reduced scheme structure. Concretely, the diagram (2) says the log structure on X pulls back to Y, X' and Y', and the underlying square of schemes is a Nisnevich distinguished square. Moreover, (2) yields the Nisnevich distinguished square

$$\begin{array}{ccc} Y'-\partial Y' & \longrightarrow & Y-\partial Y \\ \downarrow & & \downarrow \\ X'-\partial X' & \longrightarrow & X-\partial X \end{array} \tag{3}$$

Here, for any fs log scheme X, ∂X denotes the closed subset where the log structure is non-trivial. We note that ∂X is a closed subset of X according to [26, Proposition III.1.2.8]. For $X \in l\mathrm{Sm}_k$, it is the support of an effective Cartier divisor on the underlying scheme $\underline{X}$.

The second input for the topology comes from the idea that the motive of a log scheme X over k should be as close as possible to the motive of the open complement $X - \partial X$. To make this precise, we introduce the *dividing* cd-structure on $l\mathrm{Sm}_k$ defined by proper log étale surjective monomorphisms, which we call dividing covers. Intuitively, one can think of such morphisms as blow-ups with a center in the boundary ∂X. An example of a dividing cover is the blow-up of the affine plane in the origin

$$(\mathrm{Bl}_{\{0\}}\mathbb{A}^2, E + H_1' + H_2') \to (\mathbb{A}^2, H_1 + H_2)$$

Here E is the exceptional divisor and H_i' is the strict transform of the ith coordinate axis $H_i \subset \mathbb{A}^2$.

The *dividing Nisnevich* cd-structure is the union of the strict Nisnevich cd-structure and the dividing cd-structure; these are complete and regular cd-structures, but not bounded in the sense of Voevodsky [34]. For our purposes, however, it suffices to verify the weaker condition of quasi-boundedness for a density structure. The resulting topology is called the dividing Nisnevich topology on $l\mathrm{Sm}_k$. As for Voevodsky's h-topology on schemes [31], it is not sub-canonical.

3. Logarithmic motives

3.1. *Construction and basic properties*

The ∞-category $\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ of effective log motives over k with Λ-coefficients is a localization of the stable ∞-category of (unbounded) chain complexes of presheaves with log transfers on $l\mathrm{Sm}_k$. One localization imposes descent for all dividing Nisnevich coverings, and the other localization imposes $\overline{\square}$-homotopy invariance. To every $X \in l\mathrm{Sm}_k$ we associate its motive

$$M(X) \in \log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$$

It is the image of $a_{\mathrm{dNis}}\Lambda_{\mathrm{ltr}}(X) \in \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda)$ in the category of dividing Nisnevich sheaves with log transfers. Here $a_{\mathrm{dNis}}(-)$ denotes the sheafification functor for the dividing Nisnevich topology on $l\mathrm{Sm}_k$. The Tate objects $\Lambda(n)$, $n \in \mathbb{N}$, are defined by shifted cofiber and derived tensor products

$$\Lambda(0) := M(k), \Lambda(1) := M(k \xrightarrow{i_0} \mathbb{P}^1)[-2], \Lambda(n) := \Lambda(1)^{\otimes n}$$

Here $i_0 : \operatorname{Spec} k \to \mathbb{P}^1$ is the 0-section, and both schemes are equipped with a trivial log structure. For the log scheme $\mathbb{A}_{\mathbb{N}} = (\mathbb{A}^1, 0)$, the left and outer squares of the commutative diagram

$$\begin{array}{ccccc} \mathbb{G}_m & \longrightarrow & \mathbb{A}_{\mathbb{N}} & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{A}^1 & \longrightarrow & \overline{\square} & \longrightarrow & \mathbb{P}^1 \end{array} \tag{4}$$

are strict Nisnevich distinguished squares. This implies the naturally induced equivalence

$$M(\mathbb{A}_{\mathbb{N}} \to \mathbb{A}^1) \xrightarrow{\simeq} M(\overline{\square} \to \mathbb{P}^1) \simeq \Lambda(1)[2]$$

In $\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ there is a monoidal equivalence $M(X \times Y) \simeq M(X) \otimes M(Y)$ and $\overline{\square}$-homotopy invariance $M(X \times \overline{\square}) \simeq M(X)$ for all $X, Y \in l\mathrm{Sm}_k$. Our log version of the Mayer-Vietoris property says that for every strict Nisnevich distinguished square in $l\mathrm{Sm}_k$, see (2), there is a naturally induced homotopy cartesian square of log motives

$$\begin{array}{ccc} M(Y') & \longrightarrow & M(Y) \\ \downarrow & & \downarrow \\ M(X') & \longrightarrow & M(X) \end{array}$$

Moreover, every dividing cover $f : Y \to X$ of fs log schemes log smooth over k induces an isomorphism of log motives $M(f)\colon M(Y) \simeq M(X)$.

To every fs log scheme $X \in l\mathrm{Sm}_k$ and vector bundle $\xi\colon \mathscr{E} \to X$ we associate the Thom motive

$$MTh_X(\mathscr{E}) \in \log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$$

Our construction of log Thom motives is slightly different from the one in the $\mathbb{A}^1$-invariant setting; this is related to the fact that localization fails for log motives. In the motivic theory, one forms the cofiber of the natural map $\mathscr{E} - Z \to \mathscr{E}$, where Z is the zero section. In the log setting, the open subset $\mathscr{E} - Z$ does not give the "correct" homotopy type. Instead, we use the log compactification $(B_Z\mathscr{E}, E)$, where $B_Z\mathscr{E}$ is the blow up of $\mathscr{E}$ along its 0-section and E is the exceptional divisor. We can then define $MTh_X(\mathscr{E})$ as the cofiber $M((B_Z\mathscr{E}, E) \to \mathscr{E})$. The Betti realization of $MTh_X(\mathscr{E})$ is homotopy equivalent to the quotient of the unit disk bundle by the unit sphere bundle for the Betti realization of $\xi\colon \mathscr{E} \to X$. In the presence of a Euclidean metric, the latter is one formulation of Thom spaces in topology [21]. We use Thom spaces of vector bundles to show a logarithmic Gysin triangle (see [9, Construction 7.5.3, Theorem 7.5.4]).

- Let $X \in Sm_k$ and let Z be a smooth closed subscheme of X. Let $Z_1, \ldots, Z_r$ be smooth divisors on X such that $D = \sum_{i=1}^r Z_i$ is a strict normal crossing divisor on X, and let Y be the divisorial log scheme $(X, D) \in l\mathrm{Sm}_k$. Assume that Z intersects transversally with D. Let E be the exceptional divisor of the blow-up $B_Z Y$ of Y along Z, and let $N_Z Y$ denote the normal bundle of Z in Y. Then there is a functorial cofiber sequence

$$M(B_Z Y, E) \to M(Y) \to MTh(N_Z Y)$$

This sequence is a motivic incarnation of the triangle induced by the residue map along a smooth closed divisor for the logarithmic de Rham–Witt complex due to Gros [14]. It is also compatible with the analogous Gysin sequence for the cohomology of reciprocity sheaves due to Binda–Rülling–Saito [10].

In motivic homotopy theory, affine space $\mathbb{A}^n$ is contractible. It is natural to ask whether its "canonical" log compactification $(\mathbb{P}^n, \mathbb{P}^{n-1})$, where $\mathbb{P}^{n-1}$ is the hyperplane at infinity in $\mathbb{P}^n$, is contractible in the logarithmic setting. Using dividing descent, we show that log motives are $(\mathbb{P}^n, \mathbb{P}^{n-1})$-invariant in the following sense (see [9, Proposition 7.3.1]).

- For every $X \in l\mathrm{Sm}_k$ and $n \geq 1$ there is a naturally induced equivalence of log motives

$$M(X \times (\mathbb{P}^n, \mathbb{P}^{n-1})) \simeq M(X)$$

Assuming resolution of singularities, we show a projective bundle theorem and a Thom isomorphism for log motives (see [9, Theorems 8.3.5, 8.3.7]).

- Suppose k is a perfect field admitting resolution of singularities and let $\mathscr{E}$ be a vector bundle of rank n over $X \in l\mathrm{Sm}_k$. Then there are canonical isomorphisms

$$M(\mathbb{P}(\mathscr{E})) \simeq \bigoplus_{i=0}^{n-1} M(X)(i)[2i]$$

and

$$MTh(\mathscr{E}) \simeq M(X)(n)[2n]$$

The following admissible blow-up property tells us how $M(X)$ depends on the boundary ∂X (see [9, Theorem 7.6.7]).

- Suppose k is a perfect field admitting resolution of singularities. Let $f : Y \to X$ be a proper morphism of fs log schemes that are log smooth over k. If the naturally induced morphism $Y - \partial Y \to X - \partial X$ is an isomorphism of k-schemes, then there is a naturally induced isomorphism
$$M(Y) \simeq M(X)$$

We also show a more familiar type of cofiber sequence for blow-ups (see [9, Theorem 7.3.3]). In this result, we do not assume resolution of singularities because we can reduce to the case of the zero section $Z \hookrightarrow Z \times \mathbb{A}^n$.

- Suppose $X \in \mathrm{Sm}_k$ and let X' be the blow-up of X along a smooth center Z. Then there is a cofiber sequence
$$M(Z \times_X X') \to M(X') \oplus M(Z) \to M(X)$$

3.2. *Comparison with Voevodsky's triangulated category of effective motives*

It is a natural question to relate our construction to Voevodsky's category of derived motives. A formal argument shows the existence of an adjoint pair

$$\omega_\sharp : \log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda) \rightleftarrows \mathscr{DM}^{\mathrm{eff}}(k,\Lambda) : \omega^* \tag{5}$$

Here $\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ is the ∞-category of Voevodsky's effective motives introduced in [36] (as a triangulated category, see also [20]), and the adjunction $(\omega_\sharp,\omega^*)$ is induced by the functor $\omega\colon l\mathrm{Sm}_k \to \mathrm{Sm}_k$ that sends X to $X - \partial X$. We write $\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)$ for the homotopy category of $\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ and $\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)$ for the homotopy category of $\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$.

The following results hold when k is a perfect field admitting resolution of singularities.

- If $X, Y \in l\mathrm{Sm}_k$ and X is proper, then for every $i \in \mathbb{Z}$, there is a naturally induced isomorphism of abelian groups
$$\mathrm{Hom}_{\mathbf{logDM}^{\mathrm{eff}}(k,\Lambda)}(M(Y)[i], M(X)) \simeq \mathrm{Hom}_{\mathbf{DM}^{\mathrm{eff}}(k,\Lambda)}(M(Y-\partial Y)[i], M(X-\partial X))$$
- The right adjoint functor ω^* is fully faithful.

Next we describe the essential image of ω^*. Recall that $\mathscr{F} \in \log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ is $\mathbb{A}^1$*-local* if, for every $X \in l\mathrm{Sm}_k$, the projection $X \times \mathbb{A}^1 \to X$ induces an equivalence of mapping spaces

$$\mathrm{Map}_{\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)}(M(X),\mathscr{F}) \to \mathrm{Map}_{\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)}(M(X\times\mathbb{A}^1),\mathscr{F})$$

It turns out that every $\mathbb{A}^1$-local effective log motive is in the essential image of ω^*. That is, the $\mathbb{A}^1$*-localization* of $\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ is equivalent to $\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$. Alternatively, we can describe $\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ as the full subcategory $\log\mathscr{DM}^{\mathrm{eff}}_{\mathrm{prop}}(k,\Lambda)$ of $\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$ generated by $M(X)$ for all proper schemes $X \in l\mathrm{Sm}_k$. We may identify the latter with the essential image of ω^*, and hence there is an equivalence

$$\log\mathscr{DM}^{\mathrm{eff}}_{\mathrm{prop}}(k,\Lambda) \simeq \mathscr{DM}^{\mathrm{eff}}(k,\Lambda) \tag{6}$$

Note that ω^* is clearly *not* essentially surjective: for example, the sheaf of logarithmic differentials $\Omega^i_{/k}$, considered as a dividing Nisnevich sheaf with log transfers, is not $\mathbb{A}^1$-local. Thus it does not belong to the essential image of ω^*.

3.3. *Log étale motives*

It is also interesting to study the construction of log motives in other topologies on $l\mathrm{Sm}_k$, such as the log étale and the dividing étale topologies [9, Definition 3.1.5], none of which arises from a cd-structure. Log motives with $\Lambda = \mathbb{Q}$-coefficients are invariant under change of the dividing Nisnevich and dividing étale topologies [9, Proposition 8.4.4] (this is unsurprising: for motivic sheaves and rational coefficients, there is no difference between the Nisnevich and the étale theories). On the other hand, (6) is false with $\Lambda = \mathbb{Z}$-coefficients if we replace the dividing Nisnevich topology with its étale counterparts (see [9, Example 9.7.3] — the constant sheaf $\mathbb{Z}/p$ is non-trivial for the dividing étale topology, where p is the characteristic of the field k).

3.4. *Relation with other works*

A significant interest in $\log\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ concerns representability of non-$\mathbb{A}^1$-invariant theories such as Hochschild and cyclic homology, (log)-crystalline and Hodge cohomology, and log de Rham–Witt theory. *Reciprocity sheaves*, see Kahn–Saito–Yamazaki–Rülling [18], provide such examples, and similarly for the closely related theory of *modulus sheaves with transfers* due to Kahn–Miyazaki–Saito–Yamazaki [17]. The category $\mathbf{RSC}_{\mathrm{Nis}}$ of (Nisnevich) reciprocity sheaves is a full subcategory of Voevodsky's category of Nisnevich sheaves with transfers $\mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\mathbb{Z})$; it contains $\mathbb{A}^1$-invariant Nisnevich sheaves with transfers, but also the additive group scheme $\mathbb{G}_a$, the sheaf of absolute Kähler differentials Ω^i, and the de Rham–Witt sheaves $W_m\Omega^i$. Binda–Rülling–Saito [10] established the existence of Gysin sequences, blow-up formula, and projective bundle formula in this setting.

Using the existence of proper push-forward established in [10], Saito [29] has related $\mathbf{RSC}_{\mathrm{Nis}}$ to dividing Nisnevich sheaves with log transfers by showing there exists a fully faithful exact functor

$$\mathscr{L}\mathrm{og}\colon \mathbf{RSC}_{\mathrm{Nis}} \to \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\mathbb{Z})$$

such that $\mathscr{L}\mathrm{og}(F)$ is strictly $\overline{\square}$-invariant for every $F \in \mathbf{RSC}_{\mathrm{Nis}}$. For each $X \in Sm/k$, there is a natural isomorphism

$$H^i_{\mathrm{Nis}}(X,F_X) \simeq \mathrm{Hom}_{\log\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)}(M(X),\mathscr{L}\mathrm{og}(F)[i])$$

In particular, the above shows that Nisnevich cohomology of reciprocity sheaves (at least for reduced modulus) is representable in $\log\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$.

In [6], Binda–Merici proved the log analog of Morel's connectivity theorem, together with a purity result for $\overline{\square}$-local complexes of sheaves with log transfers. Using this, and adapting an argument due to Ayoub and Morel, they construct a homotopy t-structure on $\log\mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$. The comparison functor $\log\mathcal{DM}^{\mathrm{eff}}(k,\Lambda) \to \mathcal{DM}^{\mathrm{eff}}(k,\Lambda)$ is t-exact. The heart of the said homotopy t-structure is the Grothendieck abelian category $\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}}$ of strictly $\overline{\square}$-invariant sheaves with log transfers. Under resolution of singularities, the purity theorem of [6] implies the composite

$$\mathbf{CI}^{\mathrm{ltr}}_{\mathrm{dNis}} \hookrightarrow \mathbf{Shv}^{\mathrm{ltr}}_{\mathrm{dNis}}(k,\Lambda) \xrightarrow{\omega_\sharp} \mathbf{Shv}^{\mathrm{tr}}_{\mathrm{Nis}}(k,\Lambda)$$

is fully faithful and exact (moreover, it is expected to be full). Its essential image contains $\mathbf{RSC}_{\mathrm{Nis}}$ as a full subcategory; one may view it as a replacement of reciprocity sheaves with better categorical properties. Binda–Merici–Saito [7] employs this to construct a logarithmic analog of the motivic higher Albanese sheaves of Ayoub–Barbieri-Viale [3] and Barbieri-Viale–Kahn [4].

4. Logarithmic motivic homotopy theory

4.1. *Log motivic spaces and spectra*

Logarithmic motivic homotopy theory is analogous to Morel–Voevodsky's motivic homotopy theory in [23]. The setups are very similar: let us begin with S a noetherian fs log scheme of finite

Krull dimension and the symmetric monoidal ∞-category $\mathscr{P}(l\mathrm{Sm}_S)$ of presheaves of spaces on $l\mathrm{Sm}_S$. Similarly to the case of log motives, we consider the accessible subcategories of $\overline{\square}$-local and dividing Nisnevich local presheaves, and denote the corresponding localization functors by

$$L_{\overline{\square}}\colon \mathscr{P}(l\mathrm{Sm}_S) \to L_{\overline{\square}}\mathscr{P}(l\mathrm{Sm}_S)$$

$$L_{\mathrm{dNis}}\colon \mathscr{P}(l\mathrm{Sm}_S) \to L_{\mathrm{dNis}}\mathscr{P}(l\mathrm{Sm}_S)$$

Here $F \in \mathscr{P}(l\mathrm{Sm}_S)$ is $\overline{\square}$-local if the naturally induced map $F(X) \to F(X \times \overline{\square})$ is an equivalence for every $X \in l\mathrm{Sm}_S$. Moreover, F is dividing Nisnevich local if and only if $F(\emptyset) \simeq *$, F turns every strict Nisnevich distinguished square (2) into a cartesian square, and every dividing cover into an equivalence. Note that representable presheaves are neither $\overline{\square}$-local nor dividing Nisnevich local in general.

The ∞-category of log motivic spaces $\log\mathscr{H}$ consists of presheaves that are both dividing Nisnevich local and $\overline{\square}$-local. Moreover, the inclusion $\log\mathscr{H} \subset \mathscr{P}(l\mathrm{Sm}_S)$ admits a left adjoint $L_{\mathrm{mlog}}\colon \mathscr{P}(l\mathrm{Sm}_S) \to \log\mathscr{H}$ called the logarithmic motivic localization functor. We write $\log\mathscr{H}_*$ for the symmetric monoidal ∞-category of pointed log motivic spaces.

There is an equivalence of pointed log motivic spaces

$$(\mathbb{P}^1, \emptyset) \simeq S^1 \otimes (\mathbb{P}^1, 0+\infty) \tag{7}$$

Here, the trivial log scheme $(\mathbb{P}^1, \emptyset)$ is pointed at ∞; the simplicial circle S^1 is a constant log motivic space; $\otimes$ is the symmetric monoidal product on pointed log motivic spaces; the log structure on $(\mathbb{P}^1, 0+\infty)$ is given by the divisor $0+\infty$, and as a log motivic space it is pointed at 1. We view (7) as the log motivic analogue of the Morel–Voevodsky equivalence $\mathbb{P}^1 \simeq S^1 \otimes \mathbb{G}_m$ in [23] obtained from the standard covering of $\mathbb{P}^1$ by two copies of $\mathbb{A}^1$. To deduce (7) we use the identification $\overline{\square} = (\mathbb{P}^1, i)$, $i = 0, \infty$, contractibility of the $\overline{\square}$-localization of $\overline{\square}$, and the cocartesian square

$$\begin{array}{ccc} (\mathbb{P}^1, 0+\infty) & \longrightarrow & \overline{\square} \\ \downarrow & & \downarrow \\ \overline{\square} & \longrightarrow & (\mathbb{P}^1, \emptyset) \end{array} \tag{8}$$

For integers $p \geq q \geq 0$, the (p,q)-motivic log sphere is defined by setting

$$S^{p,q} := S^{p-q} \otimes (\mathbb{P}^1, 0+\infty)^{\otimes q} \tag{9}$$

in $\log\mathscr{H}_*(S)$. Building on the discussion in 3.1, the *Thom space* of a rank d vector bundle $\mathscr{E} \to X$ with 0-section $Z \to \mathscr{E}$ is defined as the pointed log motivic space

$$\mathrm{Th}(\mathscr{E}/X) := \mathscr{E}/(\mathrm{Bl}_Z(\mathscr{E}), E) \tag{10}$$

Here E is the exceptional divisor on the blow-up $\mathrm{Bl}_Z(\mathscr{E})$. If $\mathscr{E}$ is the rank n trivial bundle over S and $O := (0, \dots, 0) \in \mathbb{A}^n$, there are equivalences

$$\mathbb{A}^n/(\mathrm{Bl}_O(\mathbb{A}^n), E) \simeq \mathbb{P}^n/(\mathrm{Bl}_O(\mathbb{P}^n), E) \simeq S^{2n,n} \tag{11}$$

Our preferred suspension coordinate is $(\mathbb{P}^1, \emptyset)$ pointed at ∞, see (7). The stable log motivic ∞-category $\log\mathscr{SH}(S)$ is defined by

$$\log\mathscr{SH}(S) := \lim\left(\cdots \xrightarrow{\Omega_{\mathbb{P}^1}} \log\mathscr{H}_*(S) \xrightarrow{\Omega_{\mathbb{P}^1}} \log\mathscr{H}_*(S)\right) \tag{12}$$

Here $\Omega_{\mathbb{P}^1}(-) = \mathrm{Map}_{\log\mathscr{H}_*(S)}(\mathbb{P}^1, -)$ and the limit is taken in the ∞-category of ∞-categories. The $\mathbb{P}^1$-suspension functor $\Sigma^\infty_{\mathbb{P}^1}\colon \log\mathscr{H}_*(S) \to \log\mathscr{SH}(S)$ is left adjoint to $\Omega^\infty_{\mathbb{P}^1}\colon \log\mathscr{SH}(S) \to \log\mathscr{H}_*(S)$. Since $\mathbb{P}^1$ is a symmetric object, i.e., the cyclic permutation on $(\mathbb{P}^1)^{\otimes 3} \in \log\mathscr{H}_*(S)$ is

$\overline{\square}$-homotopic to the identity, by appealing to [27, Corollary 2.22] we deduce that $\log\mathscr{SH}(S)$ is equivalent to the colimit

$$\log\mathscr{H}_*(S)[(\mathbb{P}^1)^{-1}] := \operatorname{colim}\left(\log\mathscr{H}_*(S) \xrightarrow{\Sigma_{\mathbb{P}^1}} \log\mathscr{H}_*(S) \xrightarrow{\Sigma_{\mathbb{P}^1}} \cdots\right) \tag{13}$$

If S has a trivial log structure, the left adjoint functor

$$\omega\colon l\mathrm{Sm}_S \to \mathrm{Sm}_S; X \mapsto X - \partial X \tag{14}$$

induces an adjoint functor pair of stable ∞-categories

$$\omega_\sharp\colon \log\mathscr{SH}(S) \rightleftarrows \mathscr{SH}(S) : \omega^* \tag{15}$$

The adjunction (15) shows in particular that every motivic spectrum has a naturally associated log motivic spectrum. In analogy with (5), one may ask whether ω^* is fully faithful when the base scheme is a perfect field admitting resolution of singularities.

Universality is an important philosophical aspect of motivic theories; log motivic homotopy types are supposed to capture the (co)homological essence of log schemes. Our characterization of $\log\mathscr{SH}(S)$ is analogous to Lurie's characterization of the stable ∞-category of spectra in [19] and Robalo's characterization of the stable ∞-category of motivic spectra in [27].

- The stable log motivic ∞-category $\log\mathscr{SH}(S)$ is the universal stable presentably symmetric monoidal ∞-category equipped with a monoidal functor $l\mathrm{Sm}_S \to \log\mathscr{SH}(S)$ from smooth fs log S-schemes and satisfying dividing Nisnevich excision, $\overline{\square}$-invariance, and $\mathbb{P}^1$-stability.

The universal property simplifies the problem of constructing realization functors such as the Kato–Nakayama realization functor into topological spaces (over fields of characteristic zero) and the log étale realization functor into ℓ-profinite spaces.

The stable ∞-category $\log\mathscr{SH}(S)$ enjoys many of the same fundamental properties as $\log\mathscr{DM}^{\mathrm{eff}}(k,\Lambda)$. For example, the following $(\mathbb{P}^n,\mathbb{P}^{n-1})$-invariance property is a key result.

- If $X \in l\mathrm{Sm}_S$ and $n \geq 1$, then $X \times (\mathbb{P}^n,\mathbb{P}^{n-1}) \to X$ induces a natural equivalence

$$\Sigma^\infty_{\mathbb{P}^1}(X \times (\mathbb{P}^n,\mathbb{P}^{n-1}))_+ \xrightarrow{\simeq} \Sigma^\infty_{\mathbb{P}^1} X_+$$

The log motivic sphere $\mathbf{1} = \Sigma^\infty_{\mathbb{P}^1} S_+$ is the unit object for the monoidal structure on $\log\mathscr{SH}(S)$. Understanding the properties of $\mathbf{1}$ is one of the most fundamental problems in $\log\mathscr{SH}$. While the canonical comparison functor $\omega^*\colon \mathscr{SH}(S) \to \log\mathscr{SH}(S)$ is lax monoidal, it is unclear how $\mathbf{1}$ relates to the unit in $\mathscr{SH}(S)$ because ω^* is not known to be monoidal.

4.2. *Log motivic cohomology theories*

In this section, we assume for simplicity that S has a trivial log structure. To every log motivic spectrum $\mathbf{E} \in \log\mathscr{SH}(S)$ and integers $p,q \in \mathbb{Z}$ we assign the homology theory

$$\mathbf{E}_{p,q}(X) = \mathrm{Hom}_{\log\mathscr{SH}(S)}(S^{p,q}, \mathbf{E} \otimes \Sigma^\infty_{\mathbb{P}^1} X_+)$$

and the cohomology theory

$$\mathbf{E}^{p,q}(X) = \mathrm{Hom}_{\log\mathscr{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, S^{p,q} \otimes \mathbf{E})$$

First we discuss the example of log cobordism **logMGL** with constituent spaces

$$\mathbf{logMGL}_m := L_{\overline{\square},\mathrm{dNis}} \operatorname{colim}_n \mathrm{Th}(\mathscr{T}_{m,n}) \tag{16}$$

Here $\mathrm{Th}(\mathscr{T}_{m,n})$ is the Thom space (computed in the logarithmic sense, as explained above) of the tautological bundle $\mathscr{T}_{m,n}$ over the Grassmannian $\mathrm{Gr}(m,n)$ with the trivial log structure. The inclusion $\mathscr{O}^n \to \mathscr{O}^n \oplus \mathscr{O}$ induces a closed immersion $\mathrm{Gr}(m,n) \to \mathrm{Gr}(m,n+1)$ and a morphism of Thom space $\mathrm{Th}(\mathscr{T}_{m,n}) \to \mathrm{Th}(\mathscr{T}_{m,n+1})$. The bonding maps in (16) are determined by the maps $\mathrm{Th}(\mathcal{T}_{m,n}) \otimes \mathrm{Th}(\mathcal{T}_{r,s}) \to \mathrm{Th}(\mathcal{T}_{m+r,n+s})$ in $\log\mathcal{H}_*(S)$. Our use of log Thom spaces mimics Voevodsky's approach to algebraic cobordism in $\mathcal{SH}(S)$ [32, Section 6.3].

Log cobordism **logMGL** is the universal oriented log motivic spectrum in the sense that there is a one-to-one correspondence between the ring maps $\mathbf{logMGL} \to \mathbf{E}$ and orientations on $\mathbf{E}$, i.e., classes $c_\infty \in \mathbf{E}^{2,1}(\mathbb{P}^\infty/\mathrm{pt})$ whose restriction to $\mathbb{P}^1/\mathrm{pt}$ is the class of $\mathbf{E}^{2,1}(\mathbb{P}^1/\mathrm{pt})$ given by

$$(\mathbb{P}^1/\mathrm{pt}) \simeq (\mathbb{P}^1/\mathrm{pt}) \otimes \mathbf{1} \to (\mathbb{P}^1/\mathrm{pt}) \wedge \mathbf{E} = \Sigma^{2,1}\mathbf{E}$$

We note there is a theory of characteristic classes for oriented log motivic spectra reminiscent of their topological namesakes. In particular, for every rank d vector bundle $\mathcal{E} \to X$ in $l\mathrm{Sm}_S$, there is a naturally induced Thom isomorphism

$$- \smile t(\mathcal{E}) \colon \mathbf{E}^{**}(\mathcal{E}) \xrightarrow{\simeq} \mathbf{E}^{*+2d,*+d}(\mathrm{Th}(\mathcal{E})) \tag{17}$$

We discuss two more examples beginning with a version of "log K-theory" which differs from Niziol's logarithmic K-theory in [24, 25]: our log K-theory is an oriented log motivic spectrum constructed as follows. We let $\mathbf{K}$ denote the Bass K-theory presheaf of spectra in [30]. Owing to [23, Propositions 3.9] there is a canonical equivalence

$$\Omega^\infty\mathbf{K} \simeq L_{\mathrm{Nis}}\Omega\left(\coprod_{n\geq 0} \mathrm{BGL}_n\right) \in \mathcal{H}_*(S) \tag{18}$$

Here L_{Nis} is the Nisnevich localization functor, and Ω^∞ is the infinite loop space functor. If S is regular, the *logarithmic K-theory spectrum* is

$$\mathbf{logKGL} := (\omega^*\Omega^\infty\mathbf{K}, \omega^*\Omega^\infty\mathbf{K}, \ldots) \in \log\mathcal{SH}(S) \tag{19}$$

Here $\omega^* \colon \mathcal{H}_*(S) \to \log\mathcal{H}_*(S)$ is induced by (14). We note that **logKGL** is Bott-periodic in the sense that

$$\mathbb{P}^1 \otimes \mathbf{logKGL} \simeq \mathbf{logKGL}.$$

Over a regular base scheme S, **logKGL** represents the K-theory of the open complement of the log structure in the sense that for every $X \in l\mathrm{Sm}_S$ there is a natural equivalence

$$\mathrm{K}(X - \partial X) \simeq \mathrm{map}_{\log\mathcal{SH}(S)}(\Sigma^\infty_{\mathbb{P}^1} X_+, \mathbf{logKGL})$$

When S is the spectrum of a perfect field admitting resolution of singularities, the infinite Grassmannian Gr with the trivial log structure yields the geometric model for log K-theory

$$\mathbf{logKGL} \simeq (L_{\overline{\square},\mathrm{dNis}}\mathbb{Z} \times \mathrm{Gr}, L_{\overline{\square},\mathrm{dNis}}\mathbb{Z} \times \mathrm{Gr}, \ldots) \tag{20}$$

Related to log K-theory we define the *logarithmic topological Hochschild spectrum* **logTHH** in $\log\mathcal{SH}(S)$, see also [5] for a related discussion of arbitrary log schemes. The torus group action gives rise to a refined invariant, the *logarithmic topological cyclic homology spectrum* **logTC** in the spirit of [15]. When S is the spectrum of a perfect field admitting resolution of singularities, we develop a logarithmic version $\mathbf{logKGL} \to \mathbf{logTC}$ of the cyclotomic trace due to Bökstedt–Hsiang–Madsen [11]. The cyclotomic log trace is orientation preserving in the sense that the formal group laws of **logKGL** and **logTC** coincide with the multiplicative one. We expect this map to play a major role in our understanding of **logKGL**, **logTHH**, and **logTC**.

References

[1] Y. André, *Une introduction aux motifs. Motifs purs, motifs mixtes, périodes*, Panoramas et Synthèses, vol. 17, Société Mathématique de France, 2004, xi+261 pages.

[2] A. Asok, J. Fasel, "Splitting vector bundles outside the stable range and $\mathbb{A}^1$ – homotopy sheaves of punctured affine spaces", *J. Am. Math. Soc.* **28** (2015), no. 4, p. 1031-1062.

[3] J. Ayoub, L. Barbieri-Viale, "1-motivic sheaves and the Albanese functor", *J. Pure Appl. Algebra* **213** (2009), no. 5, p. 809-839.

[4] L. Barbieri-Viale, B. Kahn, *On the derived category of 1-motives*, Astérisque, vol. 381, Société Mathématique de France, 2016, xi+254 pages.

[5] F. Binda, T. Lundemo, D. Park, P. A. Østvær, "A logarithmic Hochschild-Kostant-Rosenberg theorem", in preparation, 2022.

[6] F. Binda, A. Merici, "Connectivity and Purity for logarithmic motives", to appear in *J. Inst. Math. Jussieu*, 2022, https://arxiv.org/abs/2012.08361v3.

[7] F. Binda, A. Merici, S. Saito, "Derived log Albanese sheaves", 2022, https://arxiv.org/abs/2107.00984v2.

[8] F. Binda, D. Park, P. A. Østvær, "Logarithmic motivic homotopy theory", in preparation, 2022.

[9] ———, *Triangulated categories of logarithmic motives over a field*, Astérisque, vol. 433, Société Mathématique de France, 2022.

[10] F. Binda, K. Rülling, S. Saito, "On the cohomology of reciprocity sheaves", 2021, https://arxiv.org/abs/2010.03301v3.

[11] M. Bökstedt, W.-C. Hsiang, I. Madsen, "The cyclotomic trace and algebraic K-theory of spaces", *Invent. Math.* **111** (1993), no. 3, p. 465-539.

[12] P. Deligne, "Théorie de Hodge. II. (Hodge theory. II)", *Publ. Math., Inst. Hautes Étud. Sci.* **40** (1971), p. 5-57.

[13] W. Fulton, *Introduction to toric varieties. The 1989 William H. Roever lectures in geometry*, Annals of Mathematics Studies, vol. 131, Princeton University Press, 1993, xi+157 pages.

[14] M. Gros, "Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique", *Mém. Soc. Math. Fr., Nouv. Sér.* (1985), no. 21, p. 87.

[15] L. Hesselholt, P. Scholze, "Arbeitsgemeinschaft: Topological Cyclic Homology", *Oberwolfach Rep.* **15** (2018), no. 2, p. 805-940.

[16] D. Isaksen, P. A. Østvær, "Motivic stable homotopy groups", in *Handbook of homotopy theory*, CRC Press, 2020, p. 35.

[17] B. Kahn, S. Saito, T. Yamazaki, "Motives with modulus", 2019, https://arxiv.org/abs/1511.07124v6.

[18] B. Kahn, S. Saito, T. Yamazaki, K. Rülling, "Reciprocity Sheaves", *Compos. Math.* **152** (2016), no. 9, p. 1851-1898.

[19] J. Lurie, "Higher algebra", available at http://www.math.harvard.edu/~lurie/papers/HA.pdf., 2017.

[20] C. Mazza, V. Voevodsky, C. Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society; Clay Mathematics Institute, 2006, xiv+216 pages.

[21] J. W. Milnor, J. D. Stasheff, *Characteristic classes*, Annals of Mathematics Studies, vol. 76, Princeton University Press, 1974.

[22] F. Morel, $\mathbb{A}^1$*-algebraic topology over a field*, Lecture Notes in Mathematics, vol. 2052, Springer, 2012, x+259 pages.

[23] F. Morel, V. Voevodsky, "$\mathbf{A}^1$-homotopy theory of schemes", *Publ. Math., Inst. Hautes Étud. Sci.* (1999), no. 90, p. 45-143.

[24] W. Nizioł, "K-theory of log-schemes. I", *Doc. Math.* **13** (2008), p. 505-551.

[25] ———, "K-theory of log-schemes II: Log-syntomic K-theory", *Adv. Math.* **230** (2012), no. 4-6, p. 1646-1672.

[26] A. Ogus, *Lectures on logarithmic algebraic geometry*, Cambridge Studies in Advanced Mathematics, vol. 178, Cambridge University Press, 2018, xviii+539 pages.

[27] M. Robalo, "K-theory and the bridge from motives to noncommutative motives", *Adv. Math.* **269** (2015), p. 399-550.

[28] O. Röndigs, M. Spitzweck, P. A. Østvær, "The first stable homotopy groups of motivic spheres", *Ann. Math.* **189** (2019), no. 1, p. 1-74.

[29] S. Saito, "Reciprocity Sheaves and Logarithmic motives", 2021, https://arxiv.org/abs/2107.00381.

[30] R. W. Thomason, T. Trobaugh, "Higher algebraic K-theory of schemes and of derived categories", in *The Grothendieck Festschrift Vol. III*, Progress in Mathematics, vol. 88, Birkhäuser, 1990, p. 347-435.

[31] V. Voevodsky, "Homology of schemes", *Sel. Math., New Ser.* **2** (1996), no. 1, p. 111-153.

[32] ———, "$\mathbb{A}^1$-homotopy theory", in *Proceedings of the International Congress of Mathematicians, Vol. I (Berlin, 1998)*, vol. Extra Vol. I, 1998, p. 579-604.

[33] ———, "Motivic cohomology with $\mathbb{Z}/2$-coefficients", *Publ. Math., Inst. Hautes Étud. Sci.* (2003), no. 98, p. 59-104.

[34] ———, "Homotopy theory of simplicial sheaves in completely decomposable topologies", *J. Pure Appl. Algebra* **214** (2010), no. 8, p. 1384-1398.

[35] ———, "On motivic cohomology with $\mathbf{Z}/l$-coefficients", *Ann. Math.* **174** (2011), no. 1, p. 401-438.

[36] V. Voevodsky, A. Suslin, E. M. Friedlander, *Cycles, transfers, and motivic homology theories*, Annals of Mathematics Studies, vol. 143, Princeton University Press, 2000, vi+254 pages.