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Number theory / Théorie des nombres

# On a conjecture of Erdős

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Abstract. In this note, we confirm an old conjecture of Erdős.

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### 1. Introduction

Let *P* denote the set of all primes. In 1950, Erdős [5] made the following anecdotal conjecture:

**Conjecture 1 (Erdős Conjecture.).** *Let c be any constant and x sufficiently large,* 

$$a_1 < a_2 < \cdots < a_t \le x, \ t > \log x.$$

Then there exists an integer n so that the number of solutions of  $n = p + a_i$  ( $p \in \mathcal{P}, 1 \le i \le t$ ) is greater than c.

Erdős [5] himself proved this conjecture for the case  $a_i = 2^i$ , which gives an affirmative answer to a question of Turán. In a former note [3], the second author proved this conjecture for the case  $a_i | a_{i+1}$  with its quantitative form, which is a slight generalization of Erdős' result. In a subsequent note, the second author and Zhou [4] proved the conjecture for the case  $a_i = 2^{p_i}$ , where  $p_i$  is the *i*-th prime. This case was conjectured by the first author [1] years ago. Shortly after, the authors of the present note recognized that the complete proof of Erdős' conjecture actually follows directly from a new achievement of the distributions of the primes established by Maynard–Tao [7,8]. We keep record here as the closure of this longstanding conjecture.

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In this note, the following general results are proved. The Erdős Conjecture follows from Corollary 3.

**Theorem 2.** For any  $\ell$  distinct integers  $a_1, ..., a_\ell$ , there are infinitely many positive integers n such that the number of solutions of  $n = p + a_i$  ( $p \in \mathcal{P}, 1 \le i \le \ell$ ) is greater than

$$\frac{1}{8}\log\ell - 1.6.$$

From Theorem 2, we immediately have the following corollaries:

**Corollary 3.** Let  $x \ge 2$  and

$$a_1 < a_2 < \cdots < a_t \leq x, t > \log x$$

Then there exist infinitely many integers n so that the number of solutions of  $n = p + a_i$  ( $p \in \mathcal{P}, 1 \le i \le t$ ) is greater than

$$\frac{1}{8}\log\log x - 1.6.$$

**Corollary 4.** Let  $\mathscr{A} = \{a_i\}_{i=1}^{\infty}$  be an infinite set of integers and let

$$f_{\mathscr{A}}(n) = \#\{(p, a) : n = p + a, p \in \mathscr{P}, a \in \mathscr{A}\},\$$

then

$$\limsup_{n \to +\infty} f_{\mathscr{A}}(n) = +\infty.$$

### 2. Proofs

A set  $\{b_1, ..., b_k\}$  is called an admissible set if there is no a fixed integer d > 1 such that  $d \mid (n + b_1) \cdots (n + b_k)$  for all integers *n*. It is equivalent that for any prime *p*,  $\{b_1, ..., b_k\}$  modulo *p* occupies at most p - 1 residues. We begin with the following deep result for the distribution of the primes due to Maynard–Tao ([8, Theorem 16]). We will use the following quantitative result which was given by Granville [6] basing on Maynard [7].

**Lemma 5.** [6, Theorem 6.2] For any given integer  $m \ge 2$ , let k be a positive integer with  $k \log k > e^{8m+4}$ . For any admissible set  $\{b_1, \ldots, b_k\}$ , there are infinitely many integers n such that at least m of  $n + b_1, \ldots, n + b_k$  are prime numbers.

Lemma 6 ([2, Lemma 3]). We have

$$\prod_{3 \le p \le x} \left( 1 - \frac{1}{p} \right)^{-1} \le 0.923 \log x, \quad x \ge 74,$$

where the product is taken over all primes p with  $3 \le p \le x$ .

**Proof of Theorem 2.** If  $\ell \leq e^{12}$ , then

$$\frac{1}{8}\log\ell - 1.6 \le 0,$$

and Theorem 2 is trivial. In the following, we assume that  $\ell > e^{12}$ .

Let  $p_i$  be the  $i^{\text{th}}$  prime. Assume that  $a_1, \ldots, a_\ell$  are  $\ell$  distinct integers. For  $p_1$ , one of residues modulo  $p_1$  contains at most  $\lfloor \ell/p_1 \rfloor$  of  $a_1, \ldots, a_\ell$ . So at least  $\ell - \lfloor \ell/p_1 \rfloor$  of  $a_1, \ldots, a_\ell$  occupy at most  $p_1 - 1$  residues modulo  $p_1$ . Let  $\ell_0 = \ell$  and  $\ell_1 = \ell - \lfloor \ell/p_1 \rfloor$ . Without loss of generality, we assume that  $a_1, \ldots, a_{\ell_1}$  occupy at most  $p_1 - 1$  residues modulo  $p_1$ . Similarly, without loss of generality, we may assume that  $a_1, \ldots, a_{\ell_2}$  occupy at most  $p_2 - 1$  residues modulo  $p_2$ , where  $\ell_2 = \ell_1 - \lfloor \ell_1/p_2 \rfloor$ . Continuing this process, at the  $t^{\text{th}}$  step, we may assume that  $a_1, \ldots, a_{\ell_1}$  occupy at most  $p_t - 1$  residues modulo  $p_t$ , where  $\ell_t = \ell_{t-1} - \lfloor \ell_{t-1}/p_t \rfloor$ . Since  $\ell \ge \ell_1 \ge \cdots$  and  $p_1 < p_2 < \cdots$ , there exists t with  $\ell_t < p_{t+1}$ . Let s be the least integer with  $\ell_s < p_{s+1}$ . It is clear that  $\{a_1, \ldots, a_{\ell_s}\}$  is an admissible

set. Let *m* be largest integer with  $\ell_s \log \ell_s > e^{8m+4}$ . If  $m \ge 2$ , then by Lemma 5, there are infinitely many integers *n* such that at least *m* of  $n - a_1, ..., n - a_{\ell_s}$  are prime numbers. Since there are infinitely many primes, it follows that there are infinitely many integers *n* such that at least one of  $n - a_1, ..., n - a_{\ell_s}$  is prime number. So the conclusion is also true for  $m \le 1$ .

Now we establish an explicit relation between  $\ell$  and m.

Since

$$\ell_{i+1} = \ell_i - \left\lfloor \frac{\ell_i}{p_{i+1}} \right\rfloor \ge \ell_i - \frac{\ell_i}{p_{i+1}} = \ell_i \left( 1 - \frac{1}{p_{i+1}} \right), \quad i = 0, 1, \dots,$$

it follows from the definition of s that

$$p_{s+1} > \ell_s \ge \ell_{s-1} \left( 1 - \frac{1}{p_s} \right) \ge \dots \ge \ell \left( 1 - \frac{1}{p_1} \right) \dots \left( 1 - \frac{1}{p_s} \right)$$
$$> e^{12} \left( 1 - \frac{1}{p_1} \right) \dots \left( 1 - \frac{1}{p_s} \right).$$

This cannot hold for  $s \le 100$ . So  $p_s \ge p_{100} = 541$ . Thus, by Lemma 6,

$$\ell_{s} \ge \ell \left( 1 - \frac{1}{p_{1}} \right) \cdots \left( 1 - \frac{1}{p_{s}} \right) \ge \frac{\ell}{2} \cdot \frac{1}{0.923 \log p_{s}} = \frac{\ell}{1.846 \log p_{s}}$$

By the definition of *s*,  $p_s \le \ell_{s-1}$ . Thus,

$$\ell_s \ge \left(1 - \frac{1}{p_s}\right) \ell_{s-1} \ge \left(1 - \frac{1}{p_s}\right) p_s = p_s - 1.$$

It follows that

$$\ell_s \ge \frac{\ell}{1.846 \log p_s} \ge \frac{\log 540}{1.846 \log 541} \frac{\ell}{\log(p_s - 1)} > \frac{0.54\ell}{\log\ell_s}$$

So  $\ell_s \log \ell_s \ge 0.54 \ell$ . In view of the definition of *m*,

 $m \ge$ 

$$e^{8m+12} \ge \ell_s \log \ell_s \ge 0.54\ell.$$

So

$$\frac{1}{8}\log\ell - \frac{12}{8} + \frac{\log 0.54}{8} > \frac{1}{8}\log\ell - 1.6.$$

This completes the proof of Theorem 2.

Proof of Corollary 3. Assume that

$$1 \le a_1 < \dots < a_t \le x, \quad t > \log x$$

By Theorem 2, there are infinitely many positive integers *n* such that the number of solutions of  $n = p + a_i$  ( $p \in \mathcal{P}, 1 \le i \le t$ ) is greater than

$$\frac{1}{8}\log t - 1.6 > \frac{1}{8}\log\log x - 1.6$$

This completes the proof of Corollary 3.

**Proof of Corollary 4.** By Theorem 2, there is a positive integer *n* such that the number of solutions of  $n = p + a_i$  ( $p \in \mathcal{P}$ ,  $1 \le i \le \ell$ ) is greater than  $\frac{1}{8} \log \ell - 1.6$ . That is,  $f_{\mathcal{A}}(n) \ge \frac{1}{8} \log \ell - 1.6$ . Now Corollary 4 follows immediately.

#### 3. Remarks

It is known that there is a positive proportion of positive odd numbers that can be represented as  $p+2^k$  with  $k \in \mathbb{N}$  and  $p \in \mathcal{P}$  (See Romanoff [9]) and there is an arithmetical progression of positive odd numbers none of which can be represented as  $p+2^k$  with  $k \in \mathbb{N}$  and  $p \in \mathcal{P}$  (see Erdős [5]).

Since one can take  $k \le c \exp((4 - \frac{28}{157})m)$  for some positive constant *c* in Lemma 5 (see [8, Theorem 16]), it follows that  $\frac{1}{8}$  in Theorem 2 and Corollary 3 can be improved to any constant less than  $(4 - \frac{28}{157})^{-1}$ .

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