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On finite totally 2-closed groups

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Abstract. An abstract group G is called totally 2-closed if $H = H^{(2),\Omega}$ for any set Ω with $G \cong H \leq \mathrm{Sym}(\Omega)$, where $H^{(2),\Omega}$ is the largest subgroup of $\mathrm{Sym}(\Omega)$ whose orbits on $\Omega \times \Omega$ are the same orbits of H. In this paper, we classify the finite soluble totally 2-closed groups. We also prove that the Fitting subgroup of a totally 2-closed group is a totally 2-closed group. Finally, we prove that a finite insoluble totally 2-closed group G of minimal order with non-trivial Fitting subgroup has shape $Z \cdot X$, with $Z = Z(G)$ cyclic, and X is a finite group with a unique minimal normal subgroup, which is nonabelian.

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1. Introduction and results

Let Ω be a set and G be a group with $G \leq \mathrm{Sym}(\Omega)$. Then G acts naturally on $\Omega \times \Omega$ by $(\alpha_1, \alpha_2)^g = (\alpha_1^g, \alpha_2^g)$, where $g \in G$ and $\alpha_1, \alpha_2 \in \Omega$. The 2-*closure* of G on Ω, denoted by $G^{(2),\Omega}$, is defined to be the subgroup of $\mathrm{Sym}(\Omega)$ leaving each orbit of G on $\Omega \times \Omega$ fixed. By [22, Theorem 5.6]

$$G^{(2),\Omega} = \left\{\theta \in \mathrm{Sym}(\Omega) \,\middle|\, \forall\, \alpha, \beta \in \Omega, \exists\, g \in G, \alpha^\theta = \alpha^g, \beta^\theta = \beta^g\right\}.$$

Furthermore, $G^{(2),\Omega}$ contains G and is the largest subgroup of $\mathrm{Sym}(\Omega)$ whose orbits on $\Omega \times \Omega$ are the same orbits of G [22, Definition 5.3 and Theorem 5.4]. Indeed $G^{(2),\Omega}$ is the automorphism group of the set of all 2-ary relations invariant with respect to the group $G \leq \mathrm{Sym}(\Omega)$. Also G is 2-closed on Ω, i.e $G = G^{(2),\Omega}$ if and only if there exists a complete colored digraph Γ with vertex set Ω such that $\mathrm{Aut}(\Gamma) = G$. The partition of the set of arcs of this graph induced by the coloring forms a set of relations which generates the Krasner algebra [10, p. 15].

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In 1969, Wielandt initiated the study of 2-closures of permutation groups to present a unified treatment of finite and infinite permutation groups, based on invariant relations and invariant functions [22]. After Wielandt's pioneering work, there was some progress on the subject achieved mostly in the case of primitive groups [12, 13, 16, 19, 20, 25] and the 2-closure was used as a tool in studying the graph isomorphism problem [9, 17, 18]; the isomorphism problem for Schurian coherent configurations [10, 21]; and in the study of automorphisms of vertex transitive graphs [7, 23, 24]. The latter of these led to the formulation of the Polycirculant conjecture [5], which remains open, and has garnered much recent attention [2].

Due to its widespread motivation, the 2-closure has been studied extensively. In particular, an interesting open question asks how far $G^{(2),\Omega}$ can be from G. This question was answered in [13] in the case where G is a primitive almost simple permutation group, but remains open in general. In this paper, we study those finite groups which have the property that $G^{(2),\Omega} = G$ for all faithful permutation representations $G \leq \mathrm{Sym}(\Omega)$. Motivated by the study of some combinatorial invariants in lattice theory, Monks proves in [15] that a finite cyclic group satisfies this property. Derek Holt in his answer to a question [14] proposed by the second author in Mathoverflow introduced a class of abstract groups called totally 2-closed groups consisting of all groups which are 2-closed in all of their faithful permutation representations. Significant progress on the question of classifying the finite totally 2-closed groups was achieved in [3]. There, the finite totally 2-closed groups with trivial Fitting subgroup were classified:

Theorem 1 ([3, Theorem 1.2 and Corollary 1.3]). *Let G be a non-trivial finite group with trivial Fitting subgroup. Then G is totally 2-closed if and only if each of the following holds:*

(1) *$G = T_1 \times \ldots \times T_r$, where the T_i are nonabelian finite simple groups and $r \leq 5$;*
(2) *$T_i \not\cong T_j$ for each $i \neq j$; and*
(3) *One of the following holds:*
  (a) *$T_i \in \{\mathrm{J}_1, \mathrm{J}_3, \mathrm{J}_4, \mathrm{Th}, \mathrm{Ly}\}$ for each $i \leq r$; or*
  (b) *$T_i \in \{\mathrm{J}_1, \mathrm{J}_3, \mathrm{J}_4, \mathrm{Ly}, \mathbb{M}\}$ for each $i \leq r$.*

In particular, there are precisely 47 totally 2-closed finite groups with trivial Fitting subgroup. In this paper, we classify the finite soluble totally 2-closed groups. Our main result reads as follows:

Theorem A. *Let G be a finite soluble group. Then G is totally 2-closed if and only if it is cyclic or a direct product of a cyclic group of odd order with a generalized quaternion group.*

This extends the main theorem in [1], which classifies the finite nilpotent totally 2-closed groups. As the reader can see, all the groups in Theorem A are nilpotent, so we deduce that there are no non-nilpotent soluble totally 2-closed groups.

As a by-product of our methods, we show that the Fitting subgroup of such a totally 2-closed group is also totally 2-closed:

Theorem B. *Let G be a totally 2-closed finite group. Then $F(G)$ is totally 2-closed. In particular, $F(G)$ is cyclic or a direct product of a generalized quaternion group with a cyclic group of odd order.*

Finally, with these results, and the results in [3] in mind, we were interested in obtaining structural information on a finite insoluble totally 2-closed group with non-trivial Fitting subgroup, of minimal order (such a group may not exist, of course). Our final theorem reads as follows:

Theorem C. *A finite insoluble totally 2-closed group G of minimal order with non-trivial Fitting subgroup has shape $Z.X$, where $Z = Z(G)$ is cyclic, and X has a unique minimal normal subgroup, which is nonabelian.*

The following question naturally arises about the existence of insoluble totally 2-closed groups with non-trivial Fitting subgroup.

Question 2. Classify all finite groups with the structure given in Theorem C.

1.1. *Preliminary results and notations*

In this section we collect some basic and elementary results and notations we need later. Our notations are standard and are mainly taken from [6], but for the reader's convenience we recall some of them as follows:

$\mathrm{Sym}(\Omega)$: The symmetric group on the set Ω.
α^g : The action of g on α.
G_α : The point stabilizer of α in G.
α^G : The orbit of α under G.
$Z(G)$: The center of G.
$F(G)$: The fitting subgroup of G.
$C_G(H)$: The centralizer of the subgroup H of a group G.
$O_p(G)$: The intersection of all Sylow p-subgroups of G.
$\mathrm{Soc}(G)$: The socle of a group G.
H_G : The core of the subgroup H of G, that is the intersection of all G-conjugates of H.

Lemma 3 ([1, Lemma 2.1]). *Let $G \le \mathrm{Sym}(\Omega)$, $A, B \le G$ and $[A,B] = 1$. Then $[A^{(2),\Omega}, B^{(2),\Omega}] = 1$. In particular, if G is abelian then $G^{(2),\Omega}$ is also abelian. Furthermore, if $H \le G$ then $(C_G(H))^{(2),\Omega} \le C_{G^{(2),\Omega}}(H^{(2),\Omega})$.*

Lemma 4 ([1, Lemma 2.2])). *Let $G \le \mathrm{Sym}(\Omega)$ and $H \le \mathrm{Sym}(\Gamma)$ be permutation isomorphic. Then $G^{(2),\Omega}$ and $H^{(2),\Gamma}$ are permutation isomorphic.*

Lemma 5 ([1, Lemma 2.3])). *Let $G \le \mathrm{Sym}(\Omega)$ and $x \in \mathrm{Sym}(\Omega)$. Then $(x^{-1}Gx)^{(2),\Omega} = x^{-1}G^{(2),\Omega}x$. In particular, $N_{\mathrm{Sym}(\Omega)}(G) \le N_{\mathrm{Sym}(\Omega)}(G^{(2),\Omega})$.*

Lemma 6 ([1, Lemma 2.9])). *Let $G_i \le \mathrm{Sym}(\Omega_i)$, Ω be disjoint union of Ω_i's, $i = 1, \ldots, n$ and $G = G_1 \times \cdots \times G_n$. Then the natural action of G on Ω is faithful. Furthermore, if $G^{(2),\Omega} = G$ then for each $i = 1, \ldots, n$, $G_i^{(2),\Omega_i} = G_i$. In particular, if G is a totally 2-closed group then G_i is a totally 2-closed group, $i = 1, \ldots, n$.*

Lemma 7 ([1, Lemma 4.5])). *Let $G = H \times K \le \mathrm{Sym}(\Omega)$ be transitive and $\Omega = \alpha^G$. If $(|H|, |K|) = 1$, then the action of G on Ω is equivalent to the action of G on $\Omega_1 \times \Omega_2$, where $\Omega_1 = \alpha^H$, $\Omega_2 = \alpha^K$ and G acts on $\Omega_1 \times \Omega_2$ by the rule $(\alpha^h, \alpha^k)^g = (\alpha^{hh_1}, \alpha^{kk_1})$, where $g = h_1k_1$.*

Theorem 8 ([1, Theorem 1])). *The center of every finite totally 2-closed group is cyclic.*

Theorem 9 ([1, Theorem 2])). *A finite nilpotent group is totally 2-closed if and only if it is cyclic or a direct product of a generalized quaternion group with a cyclic group of odd order.*

Theorem 10 ([5, Theorem 5.1])). *Let G_1 and G_2 be transitive permutation groups on sets Ω_1 and Ω_2, respectively. Then in their action on $\Omega := \Omega_1 \times \Omega_2$, we have*

$$(G_1 \times G_2)^{(2),\Omega} = G_1^{(2),\Omega_1} \times G_2^{(2),\Omega_2}, \quad (G_1 \wr G_2)^{(2),\Omega} = G_1^{(2),\Omega_1} \wr G_2^{(2),\Omega_2}.$$

Hence the following are equivalent:

(a) *G_1 and G_2 are 2-closed on Ω_1 and Ω_2, respectively.*
(b) *$G_1 \times G_2$ is 2-closed on Ω.*
(c) *$G_1 \wr G_2$ is 2-closed on Ω.*

Theorem 11 ([22, Dissection Theorem 6.5]). *Let G act on a set Ω, and suppose that $\Omega = \Delta \cup \Gamma$ (disjoint union), where Γ is G-invariant. Then the following are equivalent:*

(1) $G^{\Gamma} \times G^{\Delta} \leq G^{(2),\Omega}$.
(2) $G = G_{\gamma} G_{\delta}$ *for all* $\gamma \in \Gamma$ *and* $\delta \in \Delta$.
(3) G_{δ} *is transitive on* γ^{G} *for all* $\gamma \in \Gamma$ *and* $\delta \in \Delta$.

Theorem 12 (Universal embedding theorem [6, Theorem 2.6 A]). *Let G be an arbitrary group with a normal subgroup N and put $K := G/N$. Let $\psi : G \to K$ be a homomorphism of G onto K with kernel N. Let $T := \{t_u \mid u \in K\}$ be a set of right coset representatives of N in G such that $\psi(t_u) = u$ for each $u \in K$. Let $x \in G$ and $f_x : K \to N$ be the map with $f_x(u) = t_u x t_{u\psi(x)}^{-1}$ for all $u \in K$. Then $\varphi(x) := (f_x, \psi(x))$ defines an embedding φ of G into $N \wr K$. Furthermore, if N acts faithfully on a set Δ then G acts faithfully on $\Delta \times K$ by the rule $(\delta, k)^x = (\delta^{f_x(k)}, k\psi(x))$.*

2. The proof of Theorem A

In this section, we will prove that every finite soluble totally 2-closed group is nilpotent and so it is a cyclic group of a direct product of a cyclic group of odd order with a generalized quaternion group. Using the following lemma, one can eliminate lots of candidates of totally 2-closed groups:

Lemma 13. *Let $G = HK$ be a totally 2-closed group, where H, K are proper subgroups of G. If $H_G \cap K_G = 1$, then $G = H_G \times K_G$ and both H_G and K_G are totally 2-closed. In particular, if $H \cap K = 1$, then $G = H \times K$ and both H and K are totally 2-closed*

Proof. Let Γ and Δ be the set of right cosets of H and K, respectively and $\Omega = \Gamma \cup \Delta$. Then for all $x, y \in G$, we have $G = H^x K^y$ [11, Problem 1A.4]. Consider the actions of G on Γ, Δ and Ω by right multiplication. Since $H_G \cap K_G = 1$, G acts on Ω faithfully. Also, since $G_{Hx} = H^x$ and $G_{Ky} = K^y$ for all $x, y \in G$, the Wielandt's Dissection Theorem [22, Dissection Theorem 5.6] implies that $G/H_G \times G/K_G = G^{\Gamma} \times G^{\Delta} \leq G^{(2),\Omega}$. Now since G is totally 2-closed, $G^{(2),\Omega} = G$ and so $G = H_G K_G$, which implies that $G = H_G \times K_G$. Now 6 implies that H_G and K_G are both totally 2-closed groups. □

In the following corollary, we determine the structure of normal Sylow subgroups of totally 2-closed groups.

Corollary 14. *Let G be a finite totally 2-closed group. Then every normal Sylow subgroup of G is cyclic or a generalized quaternion group.*

Proof. Let P be a normal Sylow subgroup of G. Then $G = P \rtimes H$ [11, Theorem 3.8], for some subgroup H of G. Hence, by Lemma 13 and Theorem 9, P is cyclic or a generalized quaternion group. □

Proof of Theorem A. One direction is clear by Theorem 9. Conversely, suppose that G be a finite soluble totally 2-closed group. Let p be a prime divisor of $|G|$ and $P \in \mathrm{Syl}_p(G)$. Since G is soluble, it has a Hall p'-subgroup H. Hence $G = HP$, where $H \cap P = 1$. Since G is totally 2-closed, Lemma 13 implies that $P \trianglelefteq G$. This means that G is a nilpotent group. Now Theorem 9 implies that G is cyclic or a direct product of a cyclic group of odd order with a generalized quaternion group. □

Corollary 15. *Let G be a finite group of even order which has a cyclic Sylow 2-subgroup. Then G is totally 2-closed if and only if G is cyclic. In particular, if G is a group of order $2n$, where n is odd, then G is totally 2-closed if and only if G is cyclic.*

Proof. Since the Sylow 2-subgroup of G is cyclic, it has a normal 2-complement [11, Corollary 5.14]. Hence $G = H \rtimes P$, where $P \in \mathrm{Syl}_p(G)$. If G is totally 2-closed, then Lemma 13, implies that either $H = 1$ or $G = H \times P$, where H and P are both totally 2-closed. In the first case there is nothing to prove and in the latter case, G is cyclic by Theorem A and the Feit–Thompson odd-order Theorem. □

3. The proof of Theorem B

By the [22, Exercise 5.28] of Wielandt's book, a finite group $G \leq \mathrm{Sym}(\Omega)$ is a p-group if and only if $G^{(2),\Omega}$ is a p-group. First, for the completeness, we prove this exercise and generalize it to nilpotent groups.

Lemma 16. *Let $G \leq \mathrm{Sym}(\Omega)$ be a transitive p-group, $|\Omega| < \infty$. Then $G^{(2),\Omega}$ is a p-group.*

Proof. Since G is a transitive p-group, $|\Omega| = p^k$, for some integer $k \geq 1$. Also there exists $P \in \mathrm{Syl}_p(\mathrm{Sym}(\Omega))$ such that $G \leq P$. Let $\Delta = \{\alpha_1, \dots, \alpha_p\}$ be a subset of Ω of size p and $C = \langle(\alpha_1, \dots, \alpha_p)\rangle$. Define recursively: $P_1 = C$ acting on Δ; and $P_m = P_{m-1} \wr_\Delta C$ acting on Δ^m for $m \geq 2$. Then P_m acts faithfully on Δ^m. Now $P \leq \mathrm{Sym}(\Omega)$ is permutation isomorphic to $x^{-1}P_kx \leq \mathrm{Sym}(\Delta^k)$ for some $x \in \mathrm{Sym}(\Delta^k)$, for more details see [6, Example 2.6.1], and so it is permutation isomorphic to $P_k \leq \mathrm{Sym}(\Delta^k)$. Now Theorem 10 implies that $P_k^{(2)}$ is 2-closed on Δ^k. Hence P is 2-closed on Ω, by Lemma 4. Thus $G^{(2),\Omega} \leq P$ is a p-group. □

Corollary 17. *Let $G \leq \mathrm{Sym}(\Omega)$, $|\Omega| < \infty$ and p be a prime. Then G is a p-group if and only if $G^{(2),\Omega}$ is a p-group.*

Proof. Let G have m orbits $\Omega_1, \dots, \Omega_m$ on Ω. Since the orbits of $G^{(2)}$ on Ω are the same orbits of G, we have $G^{(2),\Omega} \leq G^{(2),\Omega_1} \times \cdots \times G^{(2),\Omega_m}$. On the other hand, by [22, Exercise 5.25], for each i we have $G^{(2),\Omega_i} \leq (G^{\Omega_i})^{(2),\Omega_i}$, $i = 1, \dots, m$. Since G^{Ω_i} is a transitive permutation p-group, Lemma 16 implies that for each i, $(G^{\Omega_i})^{(2),\Omega_i}$ is a p-group. Hence $G^{(2),\Omega}$ is a p-group. The converse direction is clear, and the proof is complete. □

Corollary 18. *Let P be a Sylow p-subgroup of a finite group $G \leq \mathrm{Sym}(\Omega)$. If $G^{(2),\Omega} = G$ then $P^{(2),\Omega} = P$ and $(O_p(G))^{(2),\Omega} = O_p(G)$.*

Corollary 19. *Let $G \leq \mathrm{Sym}(\Omega)$ and $|\Omega| < \infty$. Then G is nilpotent if and only if $G^{(2),\Omega}$ is nilpotent.*

Proof. Let $|G| = p_1^{n_1} \dots p_r^{n_r}$, where $p_1 < p_2 < \cdots < p_r$ be primes and $n_1, \dots, n_r \geq 1$ be integers. Let P_i be the Sylow p_i-subgroup of G. Then $G = P_1 \times \cdots \times P_r$. First suppose that G is transitive on Ω and $\alpha^G = \Omega$. Then Lemma 7 implies that the action of G on Ω is equivalent to the pointwise action of G on $\Omega_1 \times \cdots \times \Omega_r$, where $\Omega_i = \alpha^{P_i}$. Let $\Delta = \Omega_1 \times \cdots \times \Omega_r$. Since P_i acts transitively on Ω_i, Theorem 10 implies that, in the pointwise action on Δ, $G^{(2)} = P_1^{(2)} \times \cdots \times P_r^{(2)}$. On the other hand, by Corollary 17, $P_i^{(2)}$ is a p_i-group and so $P_i^{(2)}$ is a Sylow p_i-subgroup of $G^{(2)}$, which means that $G^{(2)}$ is a nilpotent group.

Now let G have m orbits $\Omega_1, \dots, \Omega_m$ on Ω. Then by a similar argument in the proof of Corollary 17, $G^{(2),\Omega} \leq (G^{\Omega_1})^{(2),\Omega_1} \times \cdots \times (G^{\Omega_m})^{(2),\Omega_m}$. Since for each i, G^{Ω_i} is a transitive nilpotent permutation group, the above argument follows that $G^{(2),\Omega}$ is nilpotent. □

Next, we discuss the Universal Embedding Theorem. Let G be a finite group, and N a normal subgroup of G. Suppose that N acts faithfully on a set Δ. Let $\Gamma := G/N$, and let T be a right transversal for N in G. Also, for $g = Ny \in \Gamma$, let t_g be the unique element of T such that $Ny = Nt_g$. Then G acts faithfully on the finite set $\Omega := \Delta \times \Gamma$ via the rule

$$(\delta, g)^x := \left(\delta^{t_g x t_{g\psi(x)}^{-1}}, g\psi(x)\right),$$

where $\psi : G \to \Gamma$ is a homomorphism of G onto K with kernel N. Let χ be the permutation character of N acting on Δ. Then the permutation character of G acting on Ω is the induced character $\chi\uparrow_N^G$. Whence, Mackey's Theorem [8, Proposition 6.20] implies that the permutation character of N on Ω is $\chi\uparrow_N^G\downarrow_N = \sum_{i=1}^m \chi_{N^{x_i}\cap N}\uparrow^N$, where $\{x_1, \dots, x_m\}$ is a set of representatives for the (N, N) double cosets in G. Since N is normal in G, it follows that $N^{x_i} \cap N = N$, and in fact that $\{x_1, \dots, x_m\}$ is a set of representatives for the right cosets of N in G. Thus, we conclude that $\chi\uparrow_N^G\downarrow_N = m\chi$. That is, the permutation action of N on Δ is permutation isomorphic to the natural action of N on a disjoint union of $[G:N]$ copies of Δ. In fact, this permutation isomorphism can be viewed as follows: the orbits of N in its action on Ω are the sets $\Delta_g := \{(\delta, g) : \delta \in \Delta\}$, for $g \in \Gamma$. The permutation isomorphism $\theta_g : (N, \Delta_g) \to (N, \Delta)$ is given by $n \to t_g n t_g^{-1}$, $(\delta, g) \to \delta$.

Next, let $\mathscr{P}$ be a group theoretic property which is closed under normal extension. That is, if H and K are normal $\mathscr{P}$-subgroups of a finite group G, then HK is a (necessarily) normal $\mathscr{P}$-subgroup of G. Thus, we can define the largest normal $\mathscr{P}$-subgroup of any finite group G: we denote this subgroup by $O_{\mathscr{P}}(G)$. Examples of group theoretic properties which are closed under normal extension include nilpotency and solubility.

Lemma 20. *Let $\mathscr{P}$ be a group theoretic property which is closed under normal extension with the property that whenever Ω is a finite set with $X \leq \mathrm{Sym}(\Omega)$ a $\mathscr{P}$-subgroup, we have that $X^{(2),\Omega}$ has property $\mathscr{P}$. Suppose that $G \leq \mathrm{Sym}(\Omega)$ is 2-closed. Then $O_{\mathscr{P}}(G)$ is 2-closed.*

Proof. By Lemma 5, the group $O_{\mathscr{P}}(G)^{(2),\Omega}$ is normal in $G = G^{(2),\Omega}$. The hypothesis on $\mathscr{P}$ then guarantees that $O_{\mathscr{P}}(G)^{(2),\Omega}$ is $\mathscr{P}$, and hence that $O_{\mathscr{P}}(G)^{(2),\Omega} \leq O_{\mathscr{P}}(G^{(2),\Omega}) = O_{\mathscr{P}}(G)$. The result follows. □

Proposition 21. *Let G be a finite totally 2-closed group. Then the Fitting subgroup $F(G)$ of G is totally 2-closed.*

Proof. Let $F := F(G)$ and let Δ be a set on which F acts faithfully. Let $\Gamma := G/F$ and $\Omega := \Delta \times \Gamma$. Then G acts faithfully on Ω by the Universal Embedding Theorem. With this embedding, we have that $F(G)^{(2),\Omega} = F(G)$, by Lemma 20 and Corollary 19.

Now, for each $g \in \Gamma$, let $F_g := F^{(2),\Delta_g} \leq \mathrm{Sym}(\Delta_g)$, and let $\mu_g : F_1 \to F_g$ be a permutation isomorphism (see the paragraph above for an explanation of this notation). Then F_1 acts faithfully on Ω by the rule $(\delta, g)^z = (\delta^{\mu_g(z)}, g)$, for $\delta \in \Delta$, $g \in \Gamma$, and $z \in F_1$. Furthermore, the natural copy $\widetilde{F}$ of F in F_1 acts faithfully on Ω via restriction, and by the paragraph above this action is permutation isomorphic to the action of F on Ω coming from the Universal Embedding Theorem. Denote by $\widetilde{F}'$ and F_1' the images of $\widetilde{F}$ and F_1 in $\mathrm{Sym}(\Omega)$ under this embedding. Since $(\widetilde{F}', \Omega)$ is permutation isomorphic to (F, Ω), we have that $\widetilde{F}'^{(2),\Omega} = \widetilde{F}'$.

Now, fix $z \in F_1'$. Then by definition of F_1' we have that for all $(\delta_1, \delta_2) \in \Delta \times \Delta$ there exists $f \in \widetilde{F}'$ such that $(\delta_1, \delta_2)^z = (\delta_1, \delta_2)^f$. Hence, for all $((\delta_1, g_1), (\delta_2, g_2)) \in \Omega \times \Omega$, we have

$$\begin{aligned}\left((\delta_1, g_1), (\delta_1, g_2)\right)^z &= \left(\left(\delta_1^{\mu_{g_1}(z)}, g_1\right), \left(\delta_2^{\mu_{g_2}(z)}, g_2\right)\right)\\ &= \left(\left(\delta_1^{\mu_{g_1}(f)}, g_1\right), \left(\delta_2^{\mu_{g_2}(f)}, g_2\right)\right)\\ &= \left((\delta_1, g_1), (\delta_2, g_2)\right)^f.\end{aligned}$$

Hence, $z \in \widetilde{F}'^{(2),\Omega} = \widetilde{F}'$. Whence, $F_1' \leq \widetilde{F}'$. Since $|F_1| = |F_1'| \leq |F| \leq |F_1|$, so we have $|F| = |F_1|$ and hence $F = F^{\Delta}$, as needed. □

The proof of Proposition 21 can be adapted to prove that the centraliser in G of any normal subgroup of a totally 2-closed group is totally 2-closed.

Proposition 22. *Let G be a finite totally 2-closed group, and let N be a normal subgroup of G. Then $C_G(N)$ is totally 2-closed.*

Proof. Let $C := C_G(N)$ and let Δ be a set on which C acts faithfully. Let $\Gamma := G/C$ and $\Omega := \Delta \times \Gamma$. Then G acts faithfully on Ω by the Universal Embedding Theorem. Also, $C^{(2),\Omega} = C$ by Lemma 3. The result now follows as in the second paragraph of the proof of Proposition 21 above. □

Proof of Theorem B. Let G be a totally 2-closed. Then $F(G)$ is a totally 2-closed group by Proposition 21. Since, by [11, Corollary 1.28], $F(G)$ is nilpotent, Theorem 9 implies that $F(G)$ is cylic or a direct product of a generalized quaternion group with a cyclic group of odd order, as desired. □

4. The proof of Theorem C

The purpose of this section is to prove Theorem C.

Proof of Theorem C. Suppose that G is a finite group of minimal order with the property that G is an insoluble, totally 2-closed group, and $F := F(G) = 1$. Write $F^* := F^*(G) = F \circ E(G)$ for the generalized Fitting subgroup of G, where $E(G)$ denotes the layer of G.

Let $F \neq Z(G)$. Then, by Proposition 22, $C_G(F)$ is a totally 2-closed group of order strictly smaller than $|G|$. Further, $C_G(F)$ contains $Z(F)$, which is non-trivial since F is non-trivial. Thus, $C_G(F)$ is a finite totally 2-closed group with non-trivial Fitting subgroup, and $|C_G(F)| < |G|$. Hence, $C_G(F)$ is soluble by hypothesis. It follows from Theorem A that $C_G(F)$ is nilpotent, so $C_G(F) \leq F$. Hence $F = F^*$ by [11, Corollary 9.9]. It follows from Theorem B that F^* is either cyclic or a direct product of a cyclic group of odd order with a generalized quaternion group. Thus, $G/Z(F^*) \leq \mathrm{Aut}(F^*)$ is soluble. This contradicts our choice of G.

So we must have that $F = Z(G)$. It follows that $F^*(G)$ has shape $Z(G) \circ T_1 \circ \ldots \circ T_s$, where $T_i \trianglelefteq G$ is a central product of (say) t_i copies of a quasisimple group, and these t_i copies are permuted transitively by G (see [4, Chapter 11]). If $i > 1$, then $C_G(T_i)$ is a finite insoluble totally 2-closed group with non-trivial Fitting subgroup by Proposition 22. However, since $|C_G(T_i)| < |G|$, this contradicts our choice of G. Thus, we must have $i = 1$, whence $F^*(G)$ has shape $Z(G) \circ T$, where $T \trianglelefteq G$ is a central product of (say) t copies of a finite quasimple group S permuted transitively by G, and $Z(T) \leq Z(G)$. In particular, by Theorem 8, $Z(T)$ is cyclic.

We claim that $T/Z(T) \cong TZ(G)/Z(G) \cong (S/Z(S))^t$ is the unique minimal normal subgroup of $G/Z(G)$. Indeed, if $M/Z(G)$ is a minimal normal subgroup of $G/Z(G)$ with $M \neq TZ(G)$, then $M/Z(G)$ must be nonabelian (otherwise, $M/Z(G)$, and hence M, is nilpotent, so $M \leq F(G) = Z(G)$). Write $M/Z(G) = S_1/Z(G) \times S_2/Z(G) \times \ldots \times S_e/Z(G)$, where the groups $S_i/Z(G)$ are nonabelian simple. Then choose $R_i \leq S_i$ minimal with the property that $R_iZ(G) = S_i$. Then $Z(G) \cap R_i \leq \Phi(R_i)$, where $\Phi(R_i)$ is the Frattini subgroup of R_i, and $R_i/(R_i \cap Z(G))$ is simple. Thus, $Z(R_i) = R_i \cap Z(G)$, so $R_i/Z(R_i)$ is a nonabelian simple group, and it follows that R_i is quasisimple. Since R_i is subnormal in G, it follows that the group $R = \langle R_1, \ldots, R_e \rangle$ is contained in the layer $E(G)$ of G. Since $E(G) = T$, we then have $R \leq T$, so $M/Z(G) = RZ(G)/Z(G) = TZ(G)/Z(G)$. Thus, $M = TZ(G)$, so $TZ(G)/Z(G)$ is the unique minimal normal subgroup of $G/Z(G)$, as claimed. This completes the proof. □

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