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On the 2D "viscous incompressible fluid + rigid body" system with Navier conditions and unbounded energy

Sur le mouvement d'un corps rigide dans un écoulement bidimensionel d'un fluide visqueux incompressible avec conditions au bord de Navier et énergie infinie

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Abstract. In this paper we consider the motion of a rigid body in a viscous incompressible fluid when some Navier slip conditions are prescribed on the body's boundary. The whole "viscous incompressible fluid + rigid body" system is assumed to occupy the full plane $\mathbb{R}^2$. We prove the existence of global-in-time weak solutions with constant non-zero circulation at infinity.

Résumé. Dans cet article, nous considérons le mouvement d'un corps rigide dans un fluide visqueux incompressible avec des conditions de glissement avec friction de Navier à l'interface. Le système "fluide+corps rigide" est supposé occuper le plan tout entier. Nous prouvons l'existence de solutions globales en temps avec une circulation constante non nulle à l'infini.

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Introduction

The problem of well-posedness of Navier–Stokes equations with infinite energy in dimension two has been studied a lot in the past years. We recall the work [8], where the authors prove existence for initial data which have measure vorticity and the corresponding uniqueness result is available in [5]. Other interesting works are [17] and [18], where the authors prove existence of weak solutions in loc-uniform Lebesque spaces. The first result deals with solutions defined in $\mathbb{R}^3$, the second one defined in the half space $\mathbb{R}^3_+$. For exterior domains, where no slip boundary condition is prescribed on the boundary, it was proved in [15] an existence result for initial data in the weak-L^2 space with some restriction on the concentration of the initial energy. These solutions will remain uniformly bounded in weak-L^2 norm for almost every time and bounded in the K_4 norm which is the Kato norm for $p = 4$.

In this paper we study weak solutions for viscous incompressible fluid + rigid body system where Navier-type boundary condition are prescribed on the boundary of the solid and the energy is allowed to be infinity for lack of integrability at infinity, more precisely the solutions behaves like $x^\perp/2\pi|x|^2$ at infinity.

In the case of finite energy, a wide literature is present for example [2], [6], [20], in particular in [20] existence of weak solutions is proved. The goals of this work are to extend the definition of weak solutions presented in [20] in our setting and prove existence. The main contributions are the extension of the definition of weak solutions and the density argument presented in Lemma 10, this last result is also essential to make the proof of Theorem 1 of [20] correct.

1. The 2D "viscous incompressible fluid + rigid body" system with Navier conditions

We study the Cauchy problem for a system describing the motion of a rigid body immersed in a viscous incompressible fluid when some Navier slip conditions are prescribed on the body's boundary. In [20], the existence of global in time weak solutions with finite energy to the Cauchy problem were established in the case where the whole system occupies the full space $\mathbb{R}^3$. Moreover, several properties of these solutions were exhibited. We consider here the 2D case, for which our analysis can be carried out for initial data corresponding to unbounded fluid kinetic energy.

Let us therefore consider $\mathcal{S}_0$ a closed, bounded, connected and simply connected subset of the plane with smooth boundary. We assume that the body initially occupies the domain $\mathcal{S}_0$ and we denote $\mathcal{F}_0 = \mathbb{R}^2 \setminus \mathcal{S}_0$ the domain occupied by the fluid.

The equations, in the unknown (u, l, r, p), that model the dynamics of the system in the body frame read then

$$\frac{\partial u}{\partial t} + \left[(u - l - r x^\perp)\cdot\nabla\right] u + r u^\perp + \nabla p = \nu\Delta u \qquad x \in \mathcal{F}_0, \tag{1}$$

$$\operatorname{div} u = 0 \qquad x \in \mathcal{F}_0, \tag{2}$$

$$u\cdot n = \left(l + r x^\perp\right)\cdot n \qquad x \in \partial\mathcal{S}_0, \tag{3}$$

$$(D(u)n)\cdot\tau = -\alpha(u - l - r x^\perp)\cdot\tau \qquad x \in \partial\mathcal{S}_0, \tag{4}$$

$$m l'(t) = -\int_{\partial\mathcal{S}_0} \sigma n \,\mathrm{d}s - m r l^\perp, \tag{5}$$

$$\mathcal{J} r'(t) = -\int_{\partial\mathcal{S}_0} x^\perp\cdot\sigma n \,\mathrm{d}s, \tag{6}$$

$$u(0,x) = u_0(x) \qquad x \in \mathcal{F}_0, \tag{7}$$

$$l(0) = l_0,\ r(0) = r_0, \tag{8}$$

where $u_0\cdot n = (l_0 + r_0 x^\perp)\cdot n$ for any $x \in \partial\mathcal{S}_0$. Here $u = (u_1, u_2)$ and p denote the velocity and pressure fields, $\nu > 0$ is the viscosity, n and τ are the unit outwards normal and counterclockwise tangent vectors to the boundary of the fluid domain, $\alpha \geqslant 0$ is a material constant (the friction coefficient). m and $\mathcal{J}$ denote respectively the mass and the moment of inertia of the body while the fluid is supposed to be homogeneous of density 1, to simplify the notations. The Cauchy stress tensor is defined by $\sigma = -p\,\mathrm{Id}_2 + 2\nu D(u)$, where $D(u) = (\frac{1}{2}(\partial_j u_i + \partial_i u_j))_{1\leqslant i,j\leqslant 2}$ is the deformation tensor.

When $x = (x_1, x_2)$ the notation $x^\perp$ stands for $x^\perp = (-x_2, x_1)$, $l(t)$ is the velocity of the center of mass of the body and $r(t)$ denotes the angular velocity of the rigid body. Finally to shorter the notation we will write $u_S = l + r x^\perp$.

Remark 1. In general the friction coefficient α can depend on the position on the boundary of the solid. In the case of the fluid alone some results are already available, see for instance [1]. At this moment it is not clear if the analysis from [1] can be adapted in the fluid-structure interaction problem (1)-(8).

2. Leray-type solutions with infinite energy

We are interested in solutions with initial datum (u_0, l_0, r_0) with fluid velocity of the form

$$u_0 = \tilde{u}_0 + \beta H_{\mathscr{S}_0} \in L^2(\mathscr{F}_0) \oplus \mathbb{R} H_{\mathscr{S}_0},$$

where u_0 is divergence free in a distributional sense and $H_{\mathscr{S}_0}$ is the unique solution vanishing at infinity of

$$\begin{aligned} \operatorname{div} H_{\mathscr{S}_0} &= 0 \quad \text{for } x \in \mathscr{F}_0, \\ \operatorname{curl} H_{\mathscr{S}_0} &= 0 \quad \text{for } x \in \mathscr{F}_0, \\ H_{\mathscr{S}_0} \cdot n &= 0 \quad \text{for } x \in \partial \mathscr{S}_0, \\ \int_{\partial \mathscr{S}_0} H_{\mathscr{S}_0} \cdot \tau \, \mathrm{d}s &= 1. \end{aligned}$$

See, for instance, [14]. This solution is smooth and decays like $1/|x|$ at infinity. For any x_0 in the interior of $\mathscr{S}_0$, we also have

$$H_{\mathscr{S}_0}, \nabla H_{\mathscr{S}_0} \in L^\infty(\mathscr{F}_0) \text{ and } H_{\mathscr{S}_0} - \frac{(x-x_0)^\perp}{2\pi |x-x_0|^2}, \nabla H_{\mathscr{S}_0}, H_{\mathscr{S}_0}^\perp - (x-x_0)^\perp \cdot \nabla H_{\mathscr{S}_0} \in L^2(\mathscr{F}_0), \tag{9}$$

but $H_{\mathscr{S}_0}$ is not a L^2 function. In the case of regular solutions to the Euler equations this vector field is useful to take into account the velocity associated with the circulation around the body, which is a conserved quantity according to Kelvin's theorem.

First of all we note that the quadruple $(u, l, r, p) = (H_{\mathscr{S}_0}, 0, 0, -|H_{\mathscr{S}_0}|^2/2)$ satisfies the equations (1)-(3) unless the boundary condition (4). This leads us to expect that a solution (u, l, r, p) of (1)-(8) with initial data $(\tilde{u}_0 + \beta H_{\mathscr{S}_0}, l_0, r_0)$ is of the form

$$(u, l, r, p) = (\tilde{u}, l, r, p + \beta^2 |H_{\mathscr{S}_0}|^2/2) + (\beta H_{\mathscr{S}_0}, 0, 0, -\beta^2 |H_{\mathscr{S}_0}|^2/2), \quad \text{with } \tilde{u} \in L^2(\mathscr{F}_0)$$

and β is independent of time.

We now introduce a definition of Leray-type solutions for these initial data. First of all in the literature, for example in [20], there is already a definition of weak solutions of Leray-type with finite energy, i.e. with $\beta = 0$, so we want to be coherent with this definition. In the next subsection we recall the definition of weak solution with finite energy coming from [20] and then we notice that we can extend this definition in a straight-forward way to our setting.

2.1. *A weak formulation with finite energy*

Let Ω an open subset of $\mathbb{R}^2$, we use the classical notation $L^2_\sigma(\Omega)$ to indicate the closure in $L^2(\Omega)$-norm of $C^\infty_{c,\sigma}(\Omega)$, which is the space of infinitely differentiable function divergence free and with compact support. Moreover we denote by $\mathscr{H}$ the following space

$$\mathscr{H} = \left\{ \phi \in L^2(\mathbb{R}^2) \,\middle|\, \operatorname{div} \phi = 0 \text{ in } \mathbb{R}^2 \text{ and } D(\phi) = 0 \text{ in } \mathscr{S}_0 \right\}.$$

For all $\phi \in \mathscr{H}$, there exist $\ell_\phi \in \mathbb{R}^2$ and $r_\phi \in \mathbb{R}$ such that for any $x \in \mathscr{S}_0$, $\phi(x) = \ell_\phi + r_\phi x^\perp$. Conversely if $(u, l, r) \in L^2(\mathscr{F}_0) \times \mathbb{R}^2 \times \mathbb{R}$ is a triple such that the extension u_e of u by $\ell + x^\perp r$ in $\mathscr{S}_0$ is divergence free in a distributional sense in $\mathbb{R}^2$, then $u_e \in \mathscr{H}$. Note that the extension of the initial data u_0 by setting $u_0 = \ell_0 + r_0 x^\perp$ for $x \in \mathscr{S}_0$ is an element of $\mathscr{H}$. Finally, when $\phi \in \mathscr{H}$, we denote $\phi_{\mathscr{S}} = l_\phi + r_\phi x^\perp$. Now we endow the space $\mathscr{H}$ with the following inner product

$$(\phi,\psi)_{\mathscr{H}} = \int_{\mathscr{F}_0} \phi\cdot\psi \,\mathrm{d}x + m\ell_\phi\cdot\ell_\psi + \mathscr{J} r_\phi r_\psi,$$

which is equivalent to the restriction of the $L^2(\mathbb{R}^2)$ inner product to the subspace $\mathscr{H}$. Let us also denote

$$\underline{\mathscr{V}} = \left\{\phi\in\mathscr{H} \,\middle|\, \int_{\mathscr{F}_0} |\nabla\phi(y)|^2 \mathrm{d}y < +\infty\right\} \quad \text{with norm} \quad \|\phi\|_{\underline{\mathscr{V}}} = \|\phi\|_{\mathscr{H}} + \|\nabla\phi\|_{L^2(\mathscr{F}_0,\mathrm{d}y)},$$

$$\mathscr{V} = \left\{\phi\in\mathscr{H} \,\middle|\, \int_{\mathscr{F}_0} |\nabla\phi(y)|^2(1+|y|^2) \mathrm{d}y < +\infty\right\} \quad \text{with norm} \quad \|\phi\|_{\mathscr{V}} = \|\phi\|_{\mathscr{H}} + \|\nabla\phi\|_{L^2(\mathscr{F}_0,(1+|y|^2)^{\frac12}\mathrm{d}y)},$$

$$\widehat{\mathscr{V}} = \left\{\phi\in\mathscr{V} \,\middle|\, \phi|_{\mathscr{F}_0} \in \mathrm{Lip}(\overline{\mathscr{F}_0})\right\} \quad \text{with norm} \ \|\phi\|_{\widehat{\mathscr{V}}} = \|\phi\|_{\mathscr{V}} + \|\phi\|_{\mathrm{Lip}(\overline{\mathscr{F}_0})}.$$

Let us emphasize that $\widehat{\mathscr{V}} \subset \mathscr{V} \subset \underline{\mathscr{V}}$. We define formally for appropriate u and v,

$$a(u,v) = -\alpha\int_{\partial\mathscr{S}_0} (u-u_{\mathscr{S}})\cdot(v-v_{\mathscr{S}}) - \int_{\mathscr{F}_0} D(u):D(v)$$

$$b(u,v,w) = \int_{\mathscr{F}_0} \Big([(u-u_{\mathscr{S}})\cdot\nabla w]\cdot v - r_u v^\perp\cdot w)\Big) - m r_u \ell_v^\perp\cdot\ell_w,$$

we recall that $u_{\mathscr{S}} = \ell_u - r_u x^\perp$. The next straight-forward proposition clarify in which spaces a and b are defined.

Proposition 2. *The following holds true:*

(i) *b is a trilinear continuous map from $\underline{\mathscr{V}}\times\underline{\mathscr{V}}\times\mathscr{V}$ to $\mathbb{R}$, i.e. there exists a constant $C>0$ such that for any $(u,v,w)\in\underline{\mathscr{V}}\times\underline{\mathscr{V}}\times\mathscr{V}$,*

$$|b(u,v,w)| \leqslant C\|u\|_{\underline{\mathscr{V}}}\|v\|_{\underline{\mathscr{V}}}\|w\|_{\mathscr{V}}.$$

Moreover if $v\in\underline{\mathscr{V}}$ it holds $b(u,v,v)=0$ and if $v,w\in\mathscr{V}$, it holds $b(u,v,w) = -b(u,w,v)$.

(ii) *b can be extended to a continuous map from $\mathscr{H}\times\mathscr{H}\times\widehat{\mathscr{V}}$ to $\mathbb{R}$, i.e. there exists a constant $C>0$ such that for any $(u,v,w)\in\mathscr{H}\times\mathscr{H}\times\widehat{\mathscr{V}}$,*

$$|b(u,v,w)| \leqslant C\|u\|_{\mathscr{H}}\|v\|_{\mathscr{H}}\|w\|_{\widehat{\mathscr{V}}}.$$

(iii) *$a(\cdot,\cdot)$ is a continuous map from $\underline{\mathscr{V}}\times\underline{\mathscr{V}}$ to $\mathbb{R}$, i.e. for any u, v in $\underline{\mathscr{V}}$,*

$$|a(u,v)| \leqslant C\|u\|_{\underline{\mathscr{V}}}\|v\|_{\underline{\mathscr{V}}}.$$

Proof. We present here only the proof of point i..

$$|b(u,v,w)| = \left|\int_{\mathscr{F}_0}\Big([(u-u_{\mathscr{S}})\cdot\nabla w]\cdot v - r_u v^\perp\cdot w)\Big) - m r_u\ell_v^\perp\cdot\ell_w,\right|$$

$$\leqslant \left|\int_{\mathscr{F}_0}\Big([(u\cdot\nabla)w]\cdot v - r_u v^\perp\cdot w)\Big) - m r_u\ell_v^\perp\cdot\ell_w,\right| \tag{10}$$

$$+\left|\int_{\mathscr{F}_0}[(u_{\mathscr{S}}\cdot\nabla)w]\cdot v\right|. \tag{11}$$

We separately estimates (10) and (11). By Hölder and interpolation estimates, it holds

$$(10) \leqslant C\|u\|_{\underline{\mathscr{V}}}\|v\|_{\underline{\mathscr{V}}}\|w\|_{\underline{\mathscr{V}}} \leqslant C\|u\|_{\underline{\mathscr{V}}}\|v\|_{\underline{\mathscr{V}}}\|w\|_{\mathscr{V}},$$

and by the fact that $u_{\mathscr{S}} = \ell_u + r_u x^\perp$, we have

$$\begin{aligned}(11) &= \left|\int_{\mathscr{F}_0} [((\ell_u + r_u x^\perp)\cdot\nabla)w]\cdot v\right| \\ &\leqslant C|\ell_u|\|\nabla w\|_{L^2(\mathscr{F}_0)}\|v\|_{L^2(\mathscr{F}_0)} + C|r_u|\||x|\nabla w\|_{L^2(\mathscr{F}_0)}\|v\|_{L^2(\mathscr{F}_0)} \\ &\leqslant C\|u\|_{\underline{\mathscr{V}}}\|v\|_{\underline{\mathscr{V}}}\|w\|_{\underline{\mathscr{V}}} + C\|u\|_{\underline{\mathscr{V}}}\|v\|_{\underline{\mathscr{V}}}\|w\|_{\mathscr{V}}.\end{aligned}$$ □

We are now able to state the definition of weak solution defined in [20].

Definition 3. *Let $v_0 \in \mathscr{H}$, we say that $v \in C(0,T;\mathscr{H}) \cap L^2(0,T;\underline{\mathscr{V}})$ is a solution of* (1)-(8) *with finite energy if and only if for all $\varphi \in C^\infty([0,T];\mathscr{H})$ such that $\varphi|_{\mathscr{F}_0} \in C^\infty(0,T;C_c^\infty(\overline{\mathscr{F}_0}))$ and for a.e. $t \in [0,T]$ it holds*

$$(v,\varphi)_{\mathscr{H}}(t) - (v_0,\varphi|_{t=0})_{\mathscr{H}} = \int_0^t \Big[(v,\partial_t\varphi)_{\mathscr{H}} + 2\nu a(v,\varphi) - b(v,\varphi,v)\Big].$$

2.2. *A weak formulation with infinite energy*

To extend the definition of weak solution in the case of unbounded energy we start with noticing that we can continuously extend the map a and b in our new setting. First of all for X one of the spaces $\mathscr{H}$, $\underline{\mathscr{V}}$, $\mathscr{V}$ or $\widehat{\mathscr{V}}$, the space $X \oplus \mathbb{R}H$ is endowed with the norm $\|u\|_{X\oplus\mathbb{R}H} = \|\tilde{u} + \beta H\|_{X\oplus\mathbb{R}H} = \|\tilde{u}\|_X + |\beta|$, moreover we use the convention that $u_{\mathscr{S}} = \tilde{u}_{\mathscr{S}}$, $l_u = l_{\tilde{u}}$ and $r_u = r_{\tilde{u}}$, i.e. we extend the function H by 0 inside the solid $\mathscr{S}_0$.

Proposition 4. *The map a and b can be linearly extended as follow:*

(i) *the map b can be continuously extended to a trilinear map on $(\underline{\mathscr{V}} \oplus \mathbb{R}H) \times \underline{\mathscr{V}} \times (\mathscr{V} \oplus \mathbb{R}H)$ by*
$$b(u,\tilde{v},w) = \int_{\mathscr{F}_0}\Big([(u - u_{\mathscr{S}})\cdot\nabla w]\cdot\tilde{v} - r_u\tilde{v}^\perp\cdot w)\Big) - m r_u \ell_{\tilde{v}}^\perp\cdot\ell_w.$$
The continuity assumption is equivalent to the following inequality : there exists a constant $C > 0$ such that for any $(u = \tilde{u} + \beta_1 H, \tilde{v}, w = \tilde{w} + \beta_3 H) \in (\underline{\mathscr{V}} \oplus \mathbb{R}H) \times \underline{\mathscr{V}} \times (\mathscr{V} \oplus \mathbb{R}H)$,
$$|b(u,\tilde{v},w)| \leqslant C(\|\tilde{u}\|_{\underline{\mathscr{V}}} + |\beta_1|)\|\tilde{v}\|_{\underline{\mathscr{V}}}(\|\tilde{w}\|_{\mathscr{V}} + |\beta_3|).$$

(ii) *The map $b(H,\cdot,\cdot)$, $b(\cdot,\cdot,H)$ are continuous bilinear map from $\underline{\mathscr{V}} \times \underline{\mathscr{V}}$ to $\mathbb{R}$ and $b(H,\tilde{v},H) = 0$ for any $\tilde{v} \in \underline{\mathscr{V}}$.*

(iii) *For $u \in \underline{\mathscr{V}} \oplus \mathbb{R}H$ and $\tilde{v} \in \underline{\mathscr{V}}$, we have $b(u,\tilde{v},\tilde{v}) = 0$. Moreover if $\tilde{v},\tilde{w} \in \mathscr{V}$, it holds $b(u,\tilde{v},\tilde{w}) = -b(u,\tilde{w},\tilde{v})$.*

(iv) *The trilinear map b can be extended in a unique way on $(\mathscr{H} \oplus \mathbb{R}H) \times \mathscr{H} \times (\widehat{\mathscr{V}} \oplus \mathbb{R}H)$ in a continuous way, i.e. there exists a constant $C > 0$ such that for any $(u,\tilde{v},w) = (\tilde{u} + \beta_1 H, \tilde{v}, \tilde{w} + \beta_3 H)$,*
$$|b(u,\tilde{v},w)| \leqslant C(\|\tilde{u}\|_{\mathscr{H}} + |\beta_1|)\|\tilde{v}\|_{\mathscr{H}}(\|\tilde{w}\|_{\widehat{\mathscr{V}}} + |\beta_3|).$$

(v) *$a(\cdot,\cdot)$ can be extended to a continuous bilinear map from $(\underline{\mathscr{V}} \oplus \mathbb{R}H) \times \underline{\mathscr{V}}$ to $\mathbb{R}$, where for any $(u,\tilde{v})$*
$$a(u,\tilde{v}) = -\alpha\int_{\partial\mathscr{S}_0}(u - u_{\mathscr{S}})\cdot(\tilde{v} - \tilde{v}_{\mathscr{S}}) - \int_{\mathscr{F}_0} D(u) : D(\tilde{v}).$$

Proof. Point (i) is direct consequence of point (ii) so we begin by (ii). For $(\tilde{u},\tilde{v}) \in \underline{\mathscr{V}} \times \underline{\mathscr{V}}$,

$$b(\tilde{u},\tilde{v},H) = \int_{\mathscr{F}_0}[\tilde{u}\cdot\nabla H]\cdot\tilde{v} - \int_{\mathscr{F}_0}[\ell_{\tilde{u}}\cdot\nabla H]\cdot\tilde{v} - r_{\tilde{u}}\int_{\mathscr{F}_0}(x^\perp\cdot\nabla H - H^\perp)\cdot\tilde{v}$$

is well defined thanks to (9), moreover there exists $C > 0$ such that $|b(\tilde{u},\tilde{v},H)| \leqslant C\|\tilde{u}\|_{\underline{\mathscr{V}}}\|\tilde{v}\|_{\underline{\mathscr{V}}}$. For $(\tilde{v},\tilde{w}) \in \underline{\mathscr{V}} \times \underline{\mathscr{V}}$, $b(H,\tilde{v},\tilde{w}) = \int_{\mathscr{F}_0}[H\cdot\nabla]\tilde{w}\cdot\tilde{v}$. Thanks to (9), it is clear that $|b(H,\tilde{v},\tilde{w})| \leqslant C\|\tilde{v}\|_{\underline{\mathscr{V}}}\|\tilde{w}\|_{\underline{\mathscr{V}}}$. Moreover

$$b(H,\tilde{v},H) = \int_{\mathscr{F}} H\cdot\nabla H\cdot\tilde{v} = \int_{\partial\mathscr{F}}\frac{|H|^2}{2}\tilde{v}\cdot n = \frac{l_{\tilde{v}}}{2}\cdot\int_{\partial\mathscr{F}}|H|^2 n + \frac{r_{\tilde{v}}}{2}\int_{\partial\mathscr{F}}|H|^2(x^\perp\cdot n)$$

where we use the fact that $\operatorname{curl} H = 0$ and an integration by parts. By Blasius' lemma applied to $f = g = H$ (see Lemma 12 in the Appendix), it holds

$$\int_{\partial\mathscr{F}} |H|^2 n \,\mathrm{d}s = i\left(\int_{\partial\mathscr{F}} H^2\right)^* = 0 \quad \text{and} \quad \int_{\partial\mathscr{F}} |H|^2 (x^\perp \cdot n)\,\mathrm{d}s = \operatorname{Re}\left(\int_{\partial\mathscr{F}} z\overline{H}^2\right) = \operatorname{Re}\left(-\frac{i}{2\pi}\right) = 0,$$

where we use the residue theorem due to the fact that $\overline{H}$ is holomorphic and the behaviour at infinity from Lemma 11 to compute the residue. This concludes the proof of point (i) and (ii).

For (iii), we use an integration by parts to see that for any $\widetilde{v} \in \underline{\mathscr{V}}$ we have $b(H, \widetilde{v}, \widetilde{v}) = 0$, which implies, together with point (i) of Proposition 2, that it holds $b(u, \widetilde{v}, \widetilde{v}) = 0$ for any $u = \widetilde{u} + \beta H \in \underline{\mathscr{V}} \oplus \mathbb{R}H$. Integrating by part we have also that for any $u \in \underline{\mathscr{V}} \oplus \mathbb{R}H$, for any $\widetilde{v}, \widetilde{w} \in \mathscr{V}$, $b(u, \widetilde{v}, \widetilde{w}) = -b(u, \widetilde{w}, \widetilde{v})$.

Point (iv) is trivial after notice that $\nabla H \in L^\infty$ and recall (ii) of Proposition 2.

Finally to prove (v) we use the same procedure of point (iii) of Proposition 2. □

We now introduce the definition of weak solution, with possibly unbounded energy, of the system (1)-(8).

Definition 5 (Weak solution with β circulation at infinity). *Let $u_0 = \widetilde{u}_0 + \beta H \in \mathscr{H} \oplus \mathbb{R}H$ and $T > 0$. We say that $u = \widetilde{u} + \beta H$ where*

$$\widetilde{u} \in C_w([0,T]; \mathscr{H}) \cap L^2(0,T; \underline{\mathscr{V}})$$

is a weak solution for 2D Navier–Stokes with β circulation at infinity if for any test function $\varphi \in C^1([0,T]; \mathscr{H})$ such that $\varphi|_{\overline{\mathscr{F}_0}} \in C^1([0,T]; C_c^\infty(\overline{\mathscr{F}_0}))$, it holds

$$(\widetilde{u}, \varphi)_{\mathscr{H}}(t) - (\widetilde{u}_0, \varphi|_{t=0})_{\mathscr{H}} = \int_0^t \Big[(\widetilde{u}, \partial_t \varphi)_{\mathscr{H}} + 2\nu a(u, \varphi) - b(u, \varphi, u)\Big].$$

Observe that we took into account here that H and β are time independent and the fact that for any function $\varphi \in \mathscr{H}$ such that $\varphi|_{\mathscr{F}_0} \in C_c^\infty(\overline{\mathscr{F}_0})$, it holds $b(u, u, \varphi) = -b(u, \varphi, u)$. For our convenience we give an equivalent but more explicit definition of weak formulation of the system (1)-(8).

Definition 6 (Weak solution with β circulation at infinity). *Let $\widetilde{u}_0 \in \mathscr{H}$ and $T > 0$ given. We say that*

$$\widetilde{u} \in C_w([0,T]; \mathscr{H}) \cap L^2((0,T); \underline{\mathscr{V}})$$

is a weak solution for 2D Navier–Stokes with β circulation at infinity if for every test function $\varphi \in C^1([0,T]; \mathscr{H})$ with $\varphi|_{\overline{\mathscr{F}_0}} \in C^1([0,T]; C_c^\infty(\overline{\mathscr{F}_0}))$, it holds

$$(\widetilde{u}(t), \varphi(t))_{\mathscr{H}} - (\widetilde{u}_0, \varphi(0))_{\mathscr{H}} = \int_0^t \Big[(\widetilde{u}, \partial_t \varphi)_{\mathscr{H}} + 2\nu a(\widetilde{u}, \varphi) + 2\beta\nu a(H, \varphi) - b(\widetilde{u}, \varphi, \widetilde{u}) - \beta b(H, \varphi, \widetilde{u}) - \beta b(\widetilde{u}, \varphi, H)\Big]\mathrm{d}t. \tag{12}$$

To conclude this section, we observe that any smooth solution of (1)-(8) with infinite energy is also a weak solution.

Proposition 7. *Let $u = \widetilde{u} + \beta H$ a smooth solution of* (1)-(8) *with initial data $u_0 = \widetilde{u}_0 + \beta H$, then $\widetilde{u}$ is a weak solution for 2D Navier–Stokes with β circulation at infinity.*

Proof. Multiply the equation (1) by the test function φ, integrate in all $\mathscr{F}_0$, integrate by parts and use the boundary condition. □

3. Result

The following result establishes the existence of global weak solutions of the system (1)-(8).

Theorem 8. *Let $\tilde{u}_0 \in \mathscr{H}$ and let $T > 0$. Then there exists a weak solution $\tilde{u} \in \mathscr{H}$ of 2D Navier–Stokes with β circulation at infinity in $C_w([0,T];\mathscr{H}) \cap L^2(0,T;\underline{\mathcal{V}})$ such that it satisfies the following energy inequality: for almost every $t \in [0,T]$ we have*

$$\frac{1}{2}\|\tilde{u}(t,\cdot)\|^2_{\mathscr{H}} + 2\nu \int_{(0,t)\times\mathscr{F}_0} |D(\tilde{u})|^2 + 2\alpha\nu \int_0^t \int_{\partial\mathscr{S}_0} |\tilde{u} - \tilde{u}_{\mathscr{S}}|^2 \leqslant C(1 + \|\tilde{u}_0\|^2_{\mathscr{H}}),$$

where C depends on $T, \mathscr{S}_0, \beta$ and ν. Moreover $(l,r) \in H^1(0,T;\mathbb{R}^2 \times \mathbb{R})$.

The motivation that drives us to study this special infinite-energy solutions is to study the "inviscid+shrinking-body" limit. First of all in the case of a fixed obstacle with no-slip boundary condition, existence and uniqueness of solutions with initial datum a perturbation of βH is a consequence of [15], where they proved global-in-time well-posedness for initial datum in weak-L^2 space with some restrictions on the concentration of the initial energy. In particular they do not decompose the vector fields in a part with finite energy and the other with infinite energy as in this work. Later in [13] the authors showed that for initial data a perturbation of βH, the solutions remain of the form βH plus a perturbation where β is time independent. Moreover they proved that if the initial datum has small enough circulation around the obstacle, then as the size of the obstacle converges to zero, the solutions tend to the one of the Navier–Stokes equations in the full plane with a perturbation of the point vortex as initial datum. For the inviscid limit we recall the result from [20], where the authors proved that as ν goes to zero, the solutions u_ν of the viscous "fluid+rigid body" problem converge to the solution of the corresponding inviscid "fluid+rigid body" coupled equations. In [20], the "rigid+body" system occupies all the space $\mathbb{R}^3$, in the case of $\mathbb{R}^2$ the situation is a bit more tricky and the argument of [20] holds at least in the case the solid is a disk. Moreover by the work [10], we know that as the size of the object goes to zero (and the mass remains constant) the system converges to a variant of the vortex-wave system where the vortex, placed in the point occupied by the shrunk body is accelerated by a Kutta–Joukowski-type lift force. In the massless case, i.e. the density of the object is constant respect to the scale of the object, similar results are available when the fluid satisfies incompressible Navier–Stokes equation and no-slip boundary conditions are prescribe on the boundary of the solid, for example in [16] it is proven that for a fixed viscosity the "fluid+disk" system converges to the Navier–Stokes system in all $\mathbb{R}^2$ when the object shrinks to a point. The goal of further studies is to understand the limiting equations when both the viscosity and the size of the object go to zero at the same time (in both mass and massless cases) and to find in the limit a similar system of the one in [10]. We expect that the appearance of a Kutta–Joukowski-type lift force in the limiting system is strictly related to the presence of the circulation due to βH, i.e. in the absence of this term we do not expect to see any force on the point mass in the limit. Indeed in the case where the vorticity is integrable, βH denotes the circulation at infinity.

Before moving to the proof of the theorem we present two density results. For the first one we do not claim originality but we were not able to find a reference in the literature. The second result is one of the main contribution of the paper. Lemma 10 is also essential in [20], where we propose to change the set $\mathscr{T} = \{\varphi \in C^\infty_{c,\sigma}(\mathbb{R}^2) | D(\varphi) = 0 \text{ in } \mathscr{S}_0\}$ with the set defined in (13) in the proof of Theorem 1. The set $\mathscr{T}$ is not dense in $\underline{\mathcal{V}}$ neither in $\mathcal{V}$. On the other hand we will introduce below, cf. (13), a set $\mathscr{Y}$ which is dense and has all the property to make the proof of Theorem 1 of [20] working. To see that $\mathscr{T}$ is not dense in $\mathcal{V}$, it is enough to consider $\mathscr{S}_0 = B_1(0)$ and the function

$$f(x) = \begin{cases} 0 & \text{in } B_1(0), \\ \nabla^\perp(x^2\chi) & \text{elsewhere,} \end{cases}$$

where χ is a smooth cut off such that $\chi \equiv 1$ in $B_2(0)$ and $\chi \equiv 0$ outside $B_4(0)$. It is clear that $f \in \mathcal{V} \subset \underline{\mathcal{V}}$. Suppose by contradiction that there exist approximations $f_\varepsilon \in \mathcal{T}$ such that $f_\varepsilon \to f$ in $\mathcal{V}$, then $l_{f_\varepsilon} \to l_f = 0$ and $r_{f_\varepsilon} \to r_f = 0$. By continuity, $f_\varepsilon = l_{f_\varepsilon} + r_{f_\varepsilon} x^\perp$ in $\overline{B_1(0)}$ which implies $f_\varepsilon|_{\partial B_1(0)} = l_{f_\varepsilon} + r_{f_\varepsilon} x^\perp \to l_f + r_f x^\perp = 0$ in $L^2(\partial B_1(0))$. Moreover $f_\varepsilon|_{\mathbb{R}^2 \setminus B_1(0)} \to f|_{\mathbb{R}^2 \setminus B_1(0)}$ in $H^1(\mathbb{R}^2 \setminus B_1(0))$, then by trace theorem $f_\varepsilon|_{\partial B_1(0)} \to f|_{\partial B_1(0)}$, but $f|_{\partial B_1(0)} = 2x^\perp \neq 0$ which is a contradiction.

We start by presenting the first density result.

Lemma 9. *Let Ω an open, bounded subset of $\mathbb{R}^2$ with smooth boundary such that $\partial\Omega = \cup \Gamma_i$ where Γ_i for $i = 0, \dots, n$ are open connected components of the boundary with $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, then the set $C^\infty_\sigma(\Omega) \cap L^2_\sigma(\Omega)$ of smooth divergence-free functions with 0 normal component on the boundary $\partial\Omega$ is dense in $H^1(\Omega) \cap L^2_\sigma(\Omega)$.*

Proof. Let $v \in H^1(\Omega) \cap L^2_\sigma(\Omega)$, then by [9, Corollary 3.3] there exists a stream function ψ such that $\nabla^\perp \psi = v$ and $\psi \in H^2(\Omega)$. Using the condition $v \cdot n = 0$ on $\partial\Omega$, ψ satisfies w.l.o.g.

$$\begin{cases} -\Delta\psi = -\operatorname{curl} v & \text{in } \Omega, \\ \psi = 0 & \text{on } \Gamma_0, \\ \psi = c_i & \text{on } \Gamma_i, \end{cases}$$

for some constant c_i. Consider η_ε a symmetric convolution kernel of mass 1 with support in $B_\varepsilon(0)$ and consider χ_ε the characteristic function such that $\chi_\varepsilon(x) = 1$ if $\operatorname{dist}(x, \partial\Omega) > \varepsilon$ and 0 else. We define

$$\begin{cases} -\Delta\psi_\varepsilon = -\left(\chi_{3\varepsilon} \operatorname{curl} v\right) * \eta_\varepsilon & \text{in } \Omega, \\ \psi_\varepsilon = 0 & \text{on } \Gamma_0, \\ \psi_\varepsilon = c_i & \text{on } \Gamma_i. \end{cases}$$

The functions $v_\varepsilon = \nabla^\perp \psi_\varepsilon$ are the desired approximations of v. First of all we prove that $v_\varepsilon \in C^\infty_c(\Omega) \cap L^2_\sigma(\Omega)$. This is clear by elliptic regularity and $v_\varepsilon \cdot n = \nabla^\perp \psi_\varepsilon \cdot n = \nabla\psi \cdot \tau = 0$ on $\partial\Omega$ (ψ_ε is constant in any Γ_i). To prove the convergence we use the elliptic regularity from [3, Theorem 4, Chapter 6] (in particular the remark that follow Theorem 4), we have

$$\| v_\varepsilon - v \|_{H^1(\Omega)} \leqslant \| \psi_\varepsilon - \psi \|_{H^2(\Omega)} \leqslant C \| \left(\chi_{3\varepsilon} \operatorname{curl} v\right) * \eta_\varepsilon - \operatorname{curl} v \|_{L^2(\Omega)} \to 0.$$

□

Lemma 10. *The set*

$$\mathscr{Y} = \left\{ v \in \mathscr{H} \,\middle|\, v|_{\mathscr{F}_0} \in C^\infty_c(\overline{\mathscr{F}_0}) \right\}, \tag{13}$$

is dense in $\mathcal{V}$, $\underline{\mathcal{V}}$ and $\mathscr{H}$.

Proof. The proof in the case of $\mathscr{H}$ is easy. We turn to the case of $\mathcal{V}$ and $\underline{\mathcal{V}}$. The difference between the two spaces is the integrability at $+\infty$ but this will not change much the proof so we will do it only for $\underline{\mathcal{V}}$.

Let $v \in \underline{\mathcal{V}}$ and let l and r such that $v_S = l + x^\perp r$. For $\rho > 0$ such that $\rho > \operatorname{diam}(\mathscr{S}_0)$, we define χ_ρ to be a smooth cut off function such that $0 \leqslant \chi_\rho \leqslant 1$, $\chi_\rho = 1$ in $B_\rho(0)$, $\chi_\rho = 0$ outside $B_{2\rho}(0)$ and $|\nabla\chi_\rho| \leqslant C/\rho$. Fix $R > 0$ such that $R/4 > \operatorname{diam}(\mathscr{S}_0)$, we decompose $v = u + v_1$, where $u = \nabla^\perp(\chi_{R/4}(-l^\perp \cdot x + r/2|x|^2))$. The function $u \in C^\infty_{\sigma,c}(\mathbb{R}^2)$ and $v_1|_{\mathscr{F}_0} \in H^1(\mathbb{R}^2) \cap L^2_\sigma(\mathscr{F}_0)$ and $v_1|_{\mathscr{S}_0} = 0$. By Theorem 3.3 of [9] there exists $\varphi \in H^2(B_{2R}(0) \setminus \mathscr{S}_0)$ such that $v_1 = \nabla^\perp\varphi$. We decompose $v_1 = w + z$ where $w = \nabla^\perp(\chi_R\varphi)$. The function z is such that $z|_{B_R(0)} = 0$ and $z|_{\mathbb{R}^2 \setminus B_R(0)} \in H^1_0(\mathbb{R}^2 \setminus \overline{B_R(0)}) \cap L^2_\sigma(\mathbb{R}^2 \setminus B_R(0)) = E$, where

$$E = \overline{\{C^\infty_{\sigma,c}(\mathbb{R}^2 \setminus \overline{B_R(0)})\}}^{\|.\|_{H^1}},$$

see for example [4, Section III.4.2]. This provides the existence of a sequence $\tilde{z}_\varepsilon \in C^\infty_{\sigma,c}(\mathbb{R}^2 \setminus \overline{B_R(0)})$ such that $\tilde{z}_\varepsilon \to z|_{\mathbb{R}^2 \setminus B_R(0)}$ in $H^1(\mathbb{R}^2 \setminus B_R(0))$. Let z_ε to be the extension by 0 of $\tilde{z}_\varepsilon$ inside $B_R(0)$, then $z_\varepsilon \to z$ in $\underline{\mathcal{V}}$. We now study w. The function $w \in H^1(B_{4R}(0) \setminus \overline{\mathscr{S}_0}) \cap L^2_\sigma(B_{4R}(0) \setminus \overline{\mathscr{S}_0})$. By Lemma 9 there exist $\widetilde{w}_\varepsilon \in C^\infty(B_{4R}(0)\setminus \mathscr{S}_0)\cap L^2_\sigma((B_{4R}(0)\setminus\overline{\mathscr{S}_0})$ such that $\widetilde{w}_\varepsilon \to w|_{B_{4R}(0)\setminus\overline{\mathscr{S}_0}}$ in $H^1(B_{4R}(0)\setminus\overline{\mathscr{S}_0})$. Let $\psi_\varepsilon \in H^2(B_{4R}(0)\setminus\overline{\mathscr{S}_0})$ such that $\widetilde{w}_\varepsilon = \nabla^\perp\psi_\varepsilon$. The function ψ_ε is unique up to a constant, so we choose the unique ψ_ε such that $\int_{B_{4R}(0)\setminus\overline{B_{2R}(0)}}\psi_\varepsilon = 0$. Define $\overline{w}_\varepsilon = \nabla^\perp(\chi_{2R}\psi_\varepsilon)$ and denote by $\overline{w} = w|_{B_{4R}(0)\setminus\overline{\mathscr{S}_0}}$. We have

$$\begin{aligned}\|\overline{w}-\overline{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus\overline{\mathscr{S}_0})} &\leqslant \|\overline{w}-\widetilde{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus\overline{\mathscr{S}_0})} + C\|\widetilde{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus B_{2R}(0))} + \|(\nabla^\perp\chi_{2R})\psi_\varepsilon\|_{H^1(B_{4R}(0)\setminus\overline{\mathscr{S}_0})}\\ &\leqslant o(\varepsilon) + C\|\widetilde{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus B_{2R}(0))} + C\|\psi_\varepsilon\|_{L^2(B_{4R}(0)\setminus B_{2R}(0)))}\\ &\leqslant o(\varepsilon) + C\|\widetilde{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus B_{2R}(0))} = o(\varepsilon),\end{aligned}$$

in fact we can use the Poincaré inequality on the ψ_ε and $\|\widetilde{w}_\varepsilon\|_{H^1(B_{4R}(0)\setminus B_{2R}(0))} = o(\varepsilon)$ because $w = 0$ in $B_{4R}(0)\setminus B_{2R}(0)$ (C is a constant that change from line to line). Let w_ε be the extension by 0 of $\overline{w}_\varepsilon$. The functions

$$v_\varepsilon = u + w_\varepsilon + z_\varepsilon \to v \quad \text{in } \underline{\mathcal{V}},$$

Moreover v_ε, u, w_ε and z_ε are element of $\mathscr{Y}$ (where we extend w_ε by 0 in the interior of $\mathscr{S}_0$). □

Finally we move to the proof of Theorem 8.

Proof. The proof of this theorem follows the proof of Theorem 1 in [20]. The main difficulty is to deal with the fact that the function H is not an $L^2(\mathscr{F}_0)$ function. In this work we emphasize only the changes in the proof in [20], for this reason we divide the proof in several steps as in the paper mentioned above.

The idea of the proof is to use an energy estimate to prove that the Galerkin approximation converges. To get the energy estimate at a formal level is enough to test the equation with $\widetilde{u}$, but this does not work because b is not bounded in $\underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\underline{\mathcal{V}}$ but only in $\underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\mathcal{V}$. The idea is to use a truncation of the solid velocity far from the solid. This procedure was introduced by [19] in a slightly different setting.

For simplicity in the proof we consider the case $\beta = 1$. Dealing with $\beta \neq 1$ is not an issue.

Truncation. As said in the beginning we refer to [20] for more details. Let R_0 such that $\mathscr{S}_0 \subset B(0,R_0/2)$. For $R > R_0$, let $\chi_R : \mathbb{R}^2 \to \mathbb{R}^2$ the map such that

$$\chi_R(x) = \begin{cases}\chi_R(x) = x^\perp & \text{for } x \text{ in } B(0,R),\\ \chi_R(x) = \frac{R}{|x|}x^\perp & \text{for } x \text{ in } \mathbb{R}^2\setminus B(0,R).\end{cases}$$

Note that for $w \in \mathcal{V}$ we have that

$$\chi_R\cdot\nabla w \to x^\perp\cdot\nabla w \quad \text{in } L^2(\mathbb{R}^2) \text{ as } R\to+\infty.$$

We can use the functions χ_R to truncate the solid velocity in the following way: we define

$$u_{\mathscr{S},R}(t,x) = l(t) + r(t)\chi_R(x),$$

and the forms

$$b_R(u,v,w) = mr_u l_u^\perp\cdot l_v^\perp + \mathscr{J}_0 r_u r_v r_w + \int_{\mathscr{F}_0}\left[((u-u_{\mathscr{S},R})\cdot\nabla)w\right]\cdot v - r_u v^\perp\cdot w\,\mathrm{d}x.$$

The advantage of b_R is that it is a continuous form from $\underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\underline{\mathcal{V}}$ to $\mathbb{R}$. Moreover there exists a constant C independent from R such that for any $(u,v,w)\in\underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\mathcal{V}$, $|b_R(u,v,w)| \leqslant C\|u\|_{\underline{\mathcal{V}}}\|v\|_{\underline{\mathcal{V}}}\|w\|_{\mathcal{V}}$ and for any $(u,v)\in\underline{\mathcal{V}}\times\mathcal{V}$, $|b_R(u,u,v)| \leqslant C(\|u\|^2_{L^4(\mathscr{F}_0)} + \|u\|^2_{\mathscr{H}})\|v\|_{\mathcal{V}}$. The cancellation property still hold, in fact for any $(u,v)\in\underline{\mathcal{V}}\times\underline{\mathcal{V}}$, $b_R(u,v,v) = 0$. Finally we note that for any $(u,v,w)\in\underline{\mathcal{V}}\times\underline{\mathcal{V}}\times\mathcal{V}$ $b_R(u,v,w)\to b(u,v,w)$ when R goes to $+\infty$.

Existence for the truncated system. In this step we present the existence of a solution for the truncated system. We claim that for any $\widetilde{u}_0 \in \mathscr{H}$ and $T > 0$, there exists $\widetilde{u}_R \in C([0,T];\mathscr{H}) \cap$

$L^2([0,T];\underline{V})$ such that for all $\varphi \in C^\infty([0,T];\mathscr{H})$ and $\varphi|_{\overline{\mathscr{F}_0}} \in C^1([0,T];C_c^\infty(\overline{\mathscr{F}_0}))$, and for all $t \in [0,T]$, it holds

$$(\widetilde{u}_R(t),\varphi(t))_{\mathscr{H}} - (\widetilde{u}_{R,0},\varphi(0))_{\mathscr{H}} = \int_0^t \Big[(\widetilde{u}_R,\partial_t\varphi)_{\mathscr{H}} + 2\nu a(\widetilde{u}_R,\varphi) + 2\nu a(H,\varphi) - b_R(\widetilde{u}_R,\varphi,\widetilde{u}_R) - b(H,\varphi,\widetilde{u}_R) - b(\widetilde{u}_R,\varphi,H)\Big]\mathrm{d}t.$$

Moreover $\widetilde{u}_R$ satisfies for almost every $t \in [0,T]$ the energy inequality

$$\frac{1}{2}\|\widetilde{u}_R(t)\|^2_{\mathscr{H}} + \int_0^T\int_{\partial\mathscr{S}_0}|\widetilde{u}_R - \widetilde{u}_{R,\mathscr{S}}|^2\mathrm{d}s\mathrm{d}t + \int_0^t\int_{\mathscr{F}_0}|D(\widetilde{u}_R)|^2\mathrm{d}x\mathrm{d}t \leqslant C\int_0^T\left(\|\widetilde{u}_R\|^2_{\mathscr{H}}+1\right)\mathrm{d}t.$$

The idea of the proof is based on the Galerkin method. We consider the set

$$\mathscr{Y} = \left\{ v \in \mathscr{H} \;\middle|\; v|_{\mathscr{F}_0} \in C_c^\infty(\overline{\mathscr{F}_0})\right\},$$

which is dense in $\underline{V}$. Therefore there exists a base $\{w_i\}_{i\in\mathbb{N}}$ of the Hilbert space $\underline{V}$ such that $w_i \in \mathscr{Y}$ for all i. We consider the approximate solution

$$\widetilde{u}_N(t,x) = \widetilde{u}_{N,R}(t,x) = \sum_{i=1}^N g_{i,N}(t)w_i(x),$$

where we forgot R for simplicity. The function $\widetilde{u}_N$ satisfies

$$(\partial_t\widetilde{u}_N, w_j)_{\mathscr{H}} = 2\nu a(\widetilde{u}_N,w_j) + 2\nu a(H,w_j) + b_R(\widetilde{u}_N,\widetilde{u}_N,w_j) - b(H,w_j,\widetilde{u}_N) - b(\widetilde{u}_N,w_j,H),$$
$$\widetilde{u}_N|_{t=0} = \widetilde{u}_{N0}, \quad (14)$$

where $\widetilde{u}_{N0}$ is the orthogonal projection in $\mathscr{H}$ of $\widetilde{u}_0$ onto the space spanned by $w_1,\dots,w_N$. The existence of such $g_{i,N}$ is due to the Cauchy–Lipschitz theorem applied to the system of ODE:

$$\mathscr{G}_N' = \mathscr{M}_N^{-1}\left[2\nu\mathscr{A}_N\mathscr{G}_N + 2\nu\mathscr{A}_{N,H} - \mathscr{B}_{N,H_1}(\mathscr{G}_N) - \mathscr{B}_{N,H_3}(\mathscr{G}_N) + \mathscr{B}_N(\mathscr{G}_N,\mathscr{G}_N)\right], \quad \mathscr{G}_N(0) = \mathscr{G}_{N,0},$$

where

$$\mathscr{M}_N = [(w_i,w_j)_{\mathscr{H}}]_{1\leqslant i,j\leqslant N}, \quad \mathscr{G}_N = [g_{1,N}\dots g_{N,N}]^T, \quad \mathscr{A}_N = [a(w_i,w_j)]_{1\leqslant i,j\leqslant N},$$
$$\left[\mathscr{B}_{N,H_1}(u)\right]_j = \sum_{k=1}^N u_k b(H,w_j,w_k), \quad \left[\mathscr{B}_{N,H_3}(u)\right]_j = \sum_{i=1}^N u_i b(w_i,w_j,H),$$
$$[\mathscr{B}_N(u,v)]_j = \sum_{i,k=1}^N u_i v_k b_R(w_i,w_j,w_k), \quad \left[\mathscr{A}_{N,H}\right]_j = a(H,w_j).$$

Note that $\mathscr{M}_N$ is invertible because $\{w_i\}_{i\in\mathbb{N}}$ are linear independent in $\mathscr{H}$.

The Cauchy–Lipschitz theorem ensures a local in time existence for the functions $g_{i,N}$. To prove that the existence is in all the interval $[0,T]$ we need an estimate that leads us to conclude that the function $g_{i,N}$ are defined in all $[0,T]$. To do that we multiply (14) by $g_{j,N}$ and we sum over j to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{u}_N\|_{\mathscr{H}} + 2\nu\int_{\mathscr{F}_0}|D(\widetilde{u}_N)|^2\mathrm{d}x + 2\nu\alpha\int_{\partial\mathscr{S}_0}|\widetilde{u}_N - \widetilde{u}_{N,\mathscr{S}}|^2\mathrm{d}s = 2\nu a(H,\widetilde{u}_N) - b(\widetilde{u}_N,\widetilde{u}_N,H). \quad (15)$$

We now estimate the right hand side of the last equality. Note that for any ε there exists C_ε such that

$$|a(H,\widetilde{u}_N)| \leqslant C_\varepsilon + \varepsilon\left(\int_{\mathscr{F}_0}|D(\widetilde{u}_N)|^2\mathrm{d}x + \int_{\partial\mathscr{S}_0}|\widetilde{u}_N - \widetilde{u}_{N,\mathscr{S}}|^2\mathrm{d}s\right) \quad \text{and that} \quad |b(\widetilde{u}_N,\widetilde{u}_N,H)| \leqslant C\|\widetilde{u}_N\|^2_{\mathscr{H}},$$

where C and C_ε do not depend on N and R. If we integrate (15) in $(0,t)$, we use the two inequality above and we bring on the left the terms multiplied by ε we get

$$\|\widetilde{u}_N\|^2_{\mathscr{H}} + \int_0^t\int_{\mathscr{F}_0}|D(\widetilde{u}_N)|^2\mathrm{d}x + \int_0^t\int_{\partial\mathscr{S}_0}|\widetilde{u}_N - \widetilde{u}_{N,\mathscr{S}}|^2\mathrm{d}s \leqslant \int_0^t\left(C + \|\widetilde{u}_N\|^2_{\mathscr{H}}\right)\mathrm{d}t + \|\widetilde{u}_{N0}\|^2_{\mathscr{H}}.$$

Using the Grönwall lemma we obtain the estimate

$$\|\widetilde{u}_N\|^2_{\mathscr{H}} \leqslant te^{tC}\left(C\frac{t}{2} + \|\widetilde{u}_{N0}\|^2_{\mathscr{H}}\right) + Ct + \|\widetilde{u}_{N0}\|^2_{\mathscr{H}},$$

which leads us to conclude that the function $g_{i,N}$ can be extended in all $[0,T]$.

Moreover, by the fact that $\|\widetilde{u}_{N0}\|_{\mathscr{H}} \leqslant \|\widetilde{u}_0\|_{\mathscr{H}}$ and by the Korn inequality, we conclude that

$$\widetilde{u}_N \in L^\infty((0,T);\mathscr{H})$$
$$\widetilde{u}_N \in L^2((0,T);\underline{\mathcal{V}})$$

are uniformly bounded in both the spaces. This leads us to conclude that there exists $\widetilde{u} \in L^\infty((0,T);\mathscr{H}) \cap L^2((0,T);\underline{\mathcal{V}})$ such that $\widetilde{u}_N$ converges to $\widetilde{u}$ weakly in $L^2((0,T);\underline{\mathcal{V}})$ and *-weakly in $L^\infty((0,T);\mathscr{H})$ as N goes to $+\infty$.

We pass to the limit in (14). The only not triviality is to prove the convergence of the non-linear term, i.e. $b_R(\widetilde{u}_N,\widetilde{u}_N,w_j)$ converges to $b_R(\widetilde{u},\widetilde{u},w_j)$. The idea is to notice that $\widetilde{u}_N$ is relatively compact in $L^2((0,T);L^2_{loc}(\mathbb{R}^2))$, in fact this follows from the proof of Theorem 1 in [20], where the only difference is the estimate

$$\|f_N\|_{\underline{\mathcal{V}}'} \leqslant C(1 + \|\widetilde{u}_N\|_{\underline{\mathcal{V}}} + \|\widetilde{u}_N\|^2_{\underline{\mathcal{V}}}),$$

with f_N defined by

$$\langle f_N, w\rangle = 2\nu a(\widetilde{u}_N, w) + 2\nu a(H,w) + b_R(\widetilde{u}_N,\widetilde{u}_N,w) - b(H,w,\widetilde{u}_N) - b(\widetilde{u}_N,w,H).$$

At this point we are able to pass to the limit in

$$(\widetilde{u}_N(t),\varphi(t))_{\mathscr{H}} - (\widetilde{u}_{N0},\varphi(0))_{\mathscr{H}} = \int_0^t \Big[(\widetilde{u}_N,\partial_t\varphi)_{\mathscr{H}} + 2\nu a(\widetilde{u}_N,\varphi) + 2\nu a(H,\varphi) - b_R(\widetilde{u}_N,\varphi,\widetilde{u}_N) - b(H,\varphi,\widetilde{u}_N) - b(\widetilde{u}_N,\varphi,H)\Big]\mathrm{d}t.$$

which means that $\widetilde{u} = \widetilde{u}_R$ satisfies

$$(\widetilde{u}_R(t),\varphi(t))_{\mathscr{H}} - (\widetilde{u}_{R0},\varphi(0))_{\mathscr{H}} = \int_0^t \Big[(\widetilde{u}_R,\partial_t\varphi)_{\mathscr{H}} + 2\nu a(\widetilde{u}_R,\varphi) + 2\nu a(H,\varphi) - b_R(\widetilde{u}_R,\varphi,\widetilde{u}_R) - b(H,\varphi,\widetilde{u}_R) - b(\widetilde{u}_R,\varphi,H)\Big]\mathrm{d}t. \quad (16)$$

Limit of the solutions of the truncated problems. We note that the energy estimate do not depend on R, so there exists sequence $\widetilde{u}_{k,R_k}$ converging to $\widetilde{u} \in L^\infty((0,T);\mathscr{H}) \cap L^2((0,T);\underline{\mathcal{V}})$ *-weakly in $L^\infty((0,T);\mathscr{H})$ and weakly in $L^2((0,T);\underline{\mathcal{V}})$ as k goes to $+\infty$.

These weak convergences do not lead us to pass directly to the limit because of the non-linearity of b_R, in other words we have to find an argument to prove that

$$\int_0^t b_{R_k}(\widetilde{u}_{k,R_k},\widetilde{u}_{k,R_k},\varphi) \to \int_0^t b(\widetilde{u},\widetilde{u},\varphi)\mathrm{d}x, \quad \text{as } k \text{ goes to } +\infty.$$

As presented in the paper [20], it is enough to prove compactness of $\widetilde{u}_{k,R_k}$ in a space of the type $L^2((0,T);L^2_{loc}(\mathbb{R}^2))$. We have already presented this compactness property for $\widetilde{u}_{N,R}$, but the estimates are R depending so we cannot directly conclude.

The idea is to apply the Aubin–Lions lemma to get the compactness result. First of all we note that $\widetilde{u}_{k,R_k}$ are uniformly bounded in $L^4(0,T;L^4(\mathscr{F}_0))$, in fact

$$\begin{aligned}\|\widetilde{u}_{k,R_k}\|^4_{L^4(0,T;L^4(\mathscr{F}_0))} &= \int_0^t \|\widetilde{u}_{k,R_k}\|^4_{L^4(\mathscr{F}_0)}\mathrm{d}t \\ &\leqslant C\int_0^t \|\widetilde{u}_{k,R_k}\|^2_{L^2(\mathscr{F}_0)}\|\nabla\widetilde{u}_{k,R_k}\|^2_{L^2(\mathscr{F}_0)}\mathrm{d}t \\ &\leqslant C\|\widetilde{u}_{k,R_k}\|_{L^\infty(0,T;L^2(\mathscr{F}_0))}\|\nabla\widetilde{u}_{k,R_k}\|_{L^2(0,T;L^2(\mathscr{F}_0))},\end{aligned}$$

where we use a Ladyzhenskaya's inequality, see Lemma 13 in the Appendix. This leads us to prove that $\partial_t \widetilde{u}_{k,R_k}$ is uniformly bounded in $L^2((0,T);\mathcal{V}')$, in fact the only non-linear term that can be an issue is

$$\int_{\mathscr{F}_0} [(\widetilde{u}_{k,R_k}\cdot\nabla)g]\cdot\widetilde{u}_{k,R_k}\mathrm{d}x,$$

where $g\in L^2(0,T;\mathcal{V})$, but it can be bound by

$$\begin{aligned}\left|\int_0^T\int_{\mathscr{F}_0}[(\widetilde{u}_{k,R_k}\cdot\nabla)g]\cdot\widetilde{u}_{k,R_k}\mathrm{d}x\mathrm{d}t\right| &\leqslant C\int_0^T\|\widetilde{u}_{k,R_k}\|^2_{L^4(\mathscr{F}_0)}\|\nabla g\|_{L^2(\mathscr{F}_0)}\mathrm{d}t\\ &\leqslant C\|\widetilde{u}_{k,R_k}\|^2_{L^4(0,T;L^4(\mathscr{F}_0))}\|g\|_{L^2(0,T;\mathcal{V})}.\end{aligned}$$

We apply Aubin–Lions lemma to pass to the limit in the non-linear term. To do that we decompose the velocity field $\widetilde{u}_{k,R_k}=\widehat{u}_k+\nabla^\perp(\chi(l_k\cdot x^\perp+1/2|x|^2r_k))$, with χ a smooth cut-off which is identical 1 in an open neighbourhood of $\overline{\mathscr{S}}$ and it is identically zero outside a big enough ball centred in zero. It holds that $\widehat{u}_k\in L^\infty(0,T;L^2_\sigma(\mathscr{F}))\cap L^2(0,T;H^1_\sigma(\mathscr{F}))$. First of all note that l_k and r_k are uniformly bounded in H^1. The proof of this result is postponed to the next step "Improved regularity for (l,r)". We deduce that l_k and r_k converge strongly in $L^2(0,T)$ to l and r, in particular $\nabla^\perp(\chi(l_k\cdot x^\perp+1/2|x|^2r_k))$ converges to $\nabla^\perp(\chi(l\cdot x^\perp+1/2|x|^2r))$ in $L^2(0,T;H^k(\mathscr{F}))$ for any $k\in\mathbb{N}$. It remains to prove some strong convergence of $\widehat{u}_k$ in L^2_{loc}. The idea is to restrict the functions $\widehat{u}_k$ in a ball of radius b big enough, to prove that the Leray projection of $\widehat{u}_k$ on $L^2_\sigma(B_b(0)\setminus\overline{\mathscr{S}})$ is strongly converge in L^2 up to subsequence and to show that the remainder passes to the limit in the equation. Let $\widehat{L}_b=\{v\in L^2(B_b(0)\setminus\overline{\mathscr{S}})$ such that $\operatorname{div}v=0$ and $v\cdot n=0$ on $\partial\mathscr{S}\}$ and define the projector $\mathbb{P}_b:\widehat{L}_b\to L^2_\sigma(B_b(0)\setminus\overline{\mathscr{S}})$ where $v\mapsto v-\nabla q$ with q solution of $-\Delta q=0$ in $B_b(0)\setminus\overline{\mathscr{S}}$, $\nabla q\cdot n=v\cdot n$ on $\partial B_b(0)$ and $\nabla q\cdot n=0$ on $\partial\mathscr{S}$. It holds that $\mathbb{P}_b\widehat{u}_k\in L^\infty(0,T;L^2_\sigma(B_b(0)\setminus\overline{\mathscr{S}}))\cap L^2(0,T;H^1_\sigma(B_b(0)\setminus\overline{\mathscr{S}}))$, in fact $\mathbb{P}$ is an orthogonal projection in $L^2_\sigma(B_b(0)\setminus\overline{\mathscr{S}})$ and

$$\|\mathbb{P}_b\widehat{u}_k\|_{H^1}\leqslant\|\widehat{u}_k\|_{H^1}+\|\nabla q\|_{H^1}\leqslant C\left(\|u_k\|_{H^1}+\|u_k\cdot n\|_{H^{1/2}}\right)\leqslant\widetilde{C}\|u_k\|_{H^1}.$$

Consider the triple $H^1_\sigma(B_b(0)\setminus\overline{\mathscr{S}}))\hookrightarrow L^2_\sigma(B_b(0)\setminus\overline{\mathscr{S}}))\hookrightarrow(H^1_{\sigma,0}(B_b(0)\setminus\overline{\mathscr{S}})))'$ which satisfies the hypothesis of Aubin–Lions lemma. We already showed that $\mathbb{P}_b\widehat{u}_k\in L^\infty(0,T;L^2_\sigma(B_b(0)\setminus\overline{\mathscr{S}}))\cap L^2(0,T;H^1_\sigma(B_b(0)\setminus\overline{\mathscr{S}}))$, it remains to prove that $\partial_t\mathbb{P}_b\widehat{u}_k\in L^2(0,T;(H^1_{\sigma,0}(B_b(0)\setminus\overline{\mathscr{S}})))')$. For $\varphi\in C^\infty_c((0,T)\times B_b(0)\setminus\overline{\mathscr{S}})$ such that $\operatorname{div}\varphi=0$ in $B_b(0)\setminus\overline{\mathscr{S}}$, it holds

$$\int_{\mathscr{F}}\mathbb{P}_b\widehat{u}_k\cdot\partial_t\varphi=\int_{\mathscr{F}}\widehat{u}_k\cdot\partial_t\mathbb{P}_b\varphi=\int_{\mathscr{F}}\widehat{u}_k\cdot\partial_t\varphi=\int_{\mathscr{F}}\widetilde{u}_{k,R_k}\cdot\partial_t\varphi-\int_{\mathscr{F}}\nabla^\perp\left(\chi\left(l_k\cdot x^\perp+\frac{1}{2}|x|^2r_k\right)\right)\cdot\partial_t\varphi.$$

After an integration by parts in time we deduce that

$$\begin{aligned}\left|\int_0^T\int_{\mathscr{F}}\mathbb{P}_b\widehat{u}_k\cdot\partial_t\varphi\right| &\leqslant\left|\int_0^T\int_{\mathscr{F}}\widetilde{u}_{k,R_k}\cdot\partial_t\varphi\right|+\left|\int_0^T\int_{\mathscr{F}}\nabla^\perp\left(\chi\left(l'_k\cdot x^\perp+\frac{1}{2}|x|^2r'_k\right)\right)\cdot\varphi\right|\\ &\leqslant C\left(\|\partial_t\widetilde{u}_{k,R_k}\|_{L^2(0,T;\mathcal{V}')}+\|(l_k,r_k)\|_{H^1(0,T;\mathbb{R}^3)}\right)\|\varphi\|_{L^2(0,T;H^1_{\sigma,0}(B_b(0)\setminus\bar{\mathscr{S}}))},\end{aligned}$$

which shows that $\partial_t\mathbb{P}_b\widehat{u}_k\in L^2(0,T;(H^1_{\sigma,0}(B_b(0)\setminus\overline{\mathscr{S}})))')$. The Aubin–Lions lemma ensures the strong convergence of $\mathbb{P}_b\widehat{u}_k$ in $L^2(B_b(0)\setminus\overline{\mathscr{S}})$ and by uniqueness of the limit, it converges to $\mathbb{P}_b\widetilde{u}$.

The difficult term to pass to the limit is

$$\int_{\mathscr{F}}(\widetilde{u}_{k,R_k}-\widetilde{u}_{k,R_k,S})\cdot\nabla\varphi\cdot\widetilde{u}_{k,R_k}.$$

Let $b>0$ such that the support of φ and χ are contained in $B_{b/2}(0)$. To short the notation, define $\widetilde{u}_{k,S}=\nabla^\perp(\chi(l_k\cdot x^\perp+1/2|x|^2r_k))$ and $\nabla q_{k,b}=\widehat{u}_k-\mathbb{P}_b(\widehat{u}_k)$. Then we have the identity $\widetilde{u}_{k,R_k}=\widetilde{u}_{k,S}+\mathbb{P}_b(\widehat{u}_k)+\nabla q_{k,b}$ in $B_b(0)$. Recall that $\widetilde{u}_{k,S}$ and $\mathbb{P}_b(\widehat{u}_k)$ converge strongly in $L^2((0,T)\times B_b(0))$. Moreover $\|\nabla q_{k,b}\|_{L^2(B_b(0)\setminus\bar{\mathscr{S}})}\leqslant C_b\|\widetilde{u}_{k,R_k}\|_{L^2(\partial B_b(0))}\leqslant\widetilde{C}_b$, which implies that $\nabla q_{k,b}$ weakly converges in $L^2((0,T)\times B_b(0)\setminus\overline{\mathscr{S}})$ to ∇q_b, where $\nabla q_b=\widehat{u}-\mathbb{P}_b\widehat{u}$ by uniqueness.

Using the above decomposition of $\widetilde{u}_{k,R_k}$, the only term for which the convergence is not justify is the one involving two times $\nabla q_{k,b}$, in fact for the others we can use the strong convergence of one of the sequences. In the case the test functions φ have 0 normal component on the boundary the term involving two times $\nabla q_{k,b}$ is zero, as shown in [7] Section 4. In our setting, it reads

$$\begin{aligned}
&\int_{\mathscr{F}} (\nabla q_{k,b}\cdot\nabla)\varphi\cdot\nabla q_{k,b}\\
&\quad= -\int_{\mathscr{F}\cap B_b(0)} \Delta q_{k,b}\varphi\cdot\nabla q_{k,b} - \int_{\mathscr{F}\cap B_b(0)} (\nabla q_{k,b}\cdot\nabla)\nabla q_{k,b}\cdot\varphi + \int_{\partial\mathscr{F}\cup\partial B_b(0)} \nabla q_{k,b}\cdot n\nabla q_{k,b}\cdot\varphi\\
&\quad= -\int_{\mathscr{F}\cap B_b(0)} \nabla\frac{|\nabla q_{k,b}|^2}{2}\cdot\varphi\\
&\quad= -\int_{\partial\mathscr{F}} \frac{|\nabla q_{k,b}|^2}{2}\varphi\cdot n\\
&\quad= -l_\varphi\cdot\int_{\partial\mathscr{F}} \frac{|\nabla q_{k,b}|^2}{2} n + r_\varphi\int_{\partial\mathscr{F}} \frac{|\nabla q_{k,b}|^2}{2} x^\perp\cdot n.
\end{aligned}$$

where we use that $\varphi = 0$ on $\partial B_b(0)$, $\nabla q_{k,b}\cdot n = 0$ on $\partial\mathscr{F}$ and $\operatorname{div}\varphi = 0$. Note that $\int_{\partial\mathscr{F}}|\nabla q_{k,b}|^2$ is bounded by $\int_{\partial B_b(0)}|\widetilde{u}_{k,R_k}|^2$, in particular we deduce that up to subsequence the terms

$$\int_{\partial\mathscr{F}} \frac{|\nabla q_{k,b}|^2}{2} n \quad\text{and}\quad \int_{\partial\mathscr{F}} \frac{|\nabla q_{k,b}|^2}{2} x^\perp\cdot n$$

weakly converge in $L^2(0,T)$. We can now pass to the limit in (16) and if we sum and subtract the term $1/2\int_{\partial\mathscr{F}}|\nabla q_b|^2\varphi\cdot n$, we are left with the desired result plus the error term

$$\int_0^T\int_{\partial\mathscr{F}} \frac{|\nabla q_b|^2}{2}\varphi\cdot n - \lim_{k\to+\infty}\int_0^T\int_{\partial\mathscr{F}} \frac{|\nabla q_{k,b}|^2}{2}\varphi\cdot n. \tag{17}$$

Note that in the equality obtained after passing to the limit, the only term which still depends on b is only the above one. We deduce that (17) is constant in b. Recall that $q_{k,b}$ and q_b are solution of a Laplace problem and Lemma 14 applies. It holds

$$\int_0^T\int_{\partial\mathscr{F}}|\nabla q_{k,b}|^2 \leqslant \frac{C}{b}\int_0^T\int_{\partial B_b(0)}|\widetilde{u}_{k,R_k}\cdot n|^2 \leqslant \frac{\widetilde{C}}{b}\|u_{k,R_k}\|_{L^\infty(0,T;L^2(\mathscr{F}))}\|u_{k,R_k}\|_{L^2(0,T;H^1(\mathscr{F}))},$$

with C and $\widetilde{C}$ independent of b and k. For b big enough the absolute value of the expression (17) is smaller the 2ε, which implies that (17) is zero.

Improved regularity for (l,r). From the energy inequality we deduce that $(l_k,r_k)\in L^\infty(0,T;\mathbb{R}^2\times\mathbb{R})$. What we will show is that actually (l_k,r_k) satisfy a uniform estimates in a more regular space. In particular we will prove that (l_k,r_k) are uniformly bounded in $H^1(0,T;\mathbb{R}^2\times\mathbb{R})$, that leads us to conclude that up to subsequence (l_k,r_k) converges strongly in L^2 to (l,r), moreover (l,r) is $H^1(0,T)$. In two dimensions the Kirchhoff potentials are the solutions $\Phi=(\Phi_i)_{i=1,2,3}$ of the following problems:

$$\begin{aligned}
-\Delta\Phi_i &= 0 \quad \text{for } x\in\mathscr{F}_0,\\
\Phi_i &\longrightarrow 0 \quad \text{for } |x|\to\infty,\\
\frac{\partial\Phi_i}{\partial n} &= K_i \quad \text{for } x\in\partial\mathscr{F}_0,
\end{aligned}$$

where

$$(K_1,K_2,K_3) = (n_1,n_2,x^\perp\cdot n).$$

These functions are smooth and decay at infinity as follows:

$$\nabla\Phi_i = \mathscr{O}\left(\frac{1}{|x|^2}\right) \quad\text{and}\quad \nabla^2\Phi_i = \mathscr{O}\left(\frac{1}{|x|^3}\right) \quad \text{as } x\to\infty.$$

For $i = 1,2,3$ we define the three functions v_i as

$$v_i = \nabla\Phi_i \text{ in } \mathscr{F}_0 \quad \text{and} \quad v_i = \begin{cases} e_i & \text{if } i = 1,2, \\ x^\perp & \text{if } i = 3, \end{cases} \quad \text{in } \mathscr{S}_0.$$

Note that $v_i \in \widehat{\mathscr{V}}$.

If we test the weak formulation (16) with v_i and we do some integrations by parts in a similar way to [20] Section 3.4, we can rephrase the body's equations as follow.

$$\mathscr{M}\begin{bmatrix} l_k \\ r_k \end{bmatrix}' = (2\nu a(\tilde{u}_{k,R_k}, v_i) + 2\nu a(H, v_i) + b_{R_k}(\tilde{u}_{k,R_k}, \tilde{u}_{k,R_k}, v_i) \\ + b(H, \tilde{u}_{k,R_k}, v_i) - b(\tilde{u}_{k,R_k}, v_i, H))_{i\in\{1,\dots,3\}},$$

where

$$\mathscr{M} = \begin{bmatrix} m\,\mathrm{Id}_2 & 0 \\ 0 & \mathscr{J} \end{bmatrix} + \left[\int_{\mathscr{F}_0} v_a \cdot v_b \, dx\right]_{a,b\in\{1,2,3\}}.$$

Since the matrix $\mathscr{M}$ is symmetric and positive definite, the norm $\|x\|_M = \sqrt{\langle \mathscr{M}x, \mathscr{M}x\rangle}$ is equivalent to the Euclidean norm on $\mathbb{R}^3$, $\langle .,.\rangle$ is the Euclidean scalar product. In particular the square of L^2 norm of (l'_k, r'_k) is equivalent to

$$\int_0^T \left\langle \mathscr{M}\begin{bmatrix} l'_k \\ r'_k \end{bmatrix}, \mathscr{M}\begin{bmatrix} l'_k \\ r'_k \end{bmatrix} \right\rangle \leqslant C\sum_i \int_0^T \Big(|a(\tilde{u}_{k,R_k}, v_i)| + |a(H, v_i)| \\ + |b_{R_k}(\tilde{u}_{k,R_k}, \tilde{u}_{k,R_k}, v_i)| + |b(H, \tilde{u}_{k,R_k}, v_i)| + |b(\tilde{u}, v_i, H)|\Big)^2 \tag{18}$$

by Young inequality and Proposition 4 we estimates the right hand side as follow:

$$\int_0^T |a(\tilde{u}_{k,R_k}, v_i)|^2 \leqslant \int_0^T \|\tilde{u}_{k,R_k}\|^2_{\underline{\mathscr{V}}} \|v_i\|^2_{\underline{\mathscr{V}}} \leqslant C\|v_i\|^2_{\underline{\mathscr{V}}} \int_0^T \|\tilde{u}_{k,R_k}\|^2_{\underline{\mathscr{V}}}, \qquad \int_0^T |a(H, v_i)|^2 \leqslant C\|v_i\|^2_{\underline{\mathscr{V}}} T,$$

$$\int_0^T |b_{R_k}(\tilde{u}_{k,R_k}, \tilde{u}_{k,R_k}, v_i)|^2 \leqslant C\big(\|v_i\|_{Lip(\mathscr{F})} + \|v_i\|_{\mathscr{V}}\big) \int_0^T \|\tilde{u}_{k,R_k}\|^4_{\mathscr{H}},$$

$$\int_0^T |b(H, \tilde{u}, v_i)|^2 \leqslant C\|H\|^2_{L^\infty(\mathscr{F})} \|v_i\|^2_{\underline{\mathscr{V}}} \int_0^T \|\tilde{u}_{k,R_k}\|^2_{\mathscr{H}},$$

$$\int_0^T |b(\tilde{u}_{k,R_k}, v_i, H)|^2 \leqslant C\|v_i\|^2_{\mathscr{H}} \|v_i\|^2_{\mathscr{H}} \int_0^T \|\tilde{u}_{k,R_k}\|^2_{\mathscr{H}}.$$

From the energy inequality we deduce that the right hand side of (18) is uniformly bounded, which implies that the (l_k, r_k) are uniformly bounded in $H^1(0,T)$. ☐

Appendix

In this appendix we recall four results for lack of completeness. The first one describes some regularity properties of H. The second one is the Blasius' Lemma. The third one is the Ladyzhenskaya's inequality in exterior domain. The last one is a scaling estimate.

Lemma 11. *Let identify $\mathbb{R}^2$ with $\mathbb{C}$. Then $\overline{H}$, the complex conjugate of H is an holomorphic function in $\mathscr{F}$. Moreover*

$$\overline{H}(z) = \frac{1}{2i\pi z} + O\left(\frac{1}{z^2}\right) \quad \text{as} \quad |z| \longrightarrow +\infty.$$

Proof. The function H is smooth and satisfies $\operatorname{div} H = 0$ and $\operatorname{curl} H = 0$, which implies that $\overline{H}$ satisfies the Cauchy–Riemann equations. This shows that $\overline{H}$ is holomorphic. To prove that

$$\overline{H}(z) = \frac{1}{2i\pi z} + O\left(\frac{1}{z^2}\right) \quad \text{as} \quad |z| \longrightarrow +\infty,$$

we recall that $\overline{H}_{\mathbb{R}^2\setminus B_1(0)} = 1/(2i\pi z)$ and that $\overline{H}(z) = b\overline{H}_{\mathbb{R}^2\setminus B_1(0)}(T(z))$ with T the biholomorphic map from Lemma 2.1 of [12]. In particular $T(z) = bz + h(t)$ with h a bounded holomorphic function. This last information concludes the proof. □

Lemma 12 (Blasius' Lemma). *Let Γ a smooth Jordan curves and let $f := (f_1, f_2)$ and $g := (g_1, g_2)$ two smooth tangent vector fields. Then it holds*

$$\int_\Gamma (f\cdot g)n\,\mathrm{d}s = i\left(\int_\Gamma \overline{f}\overline{g}\right)^*,$$

$$\int_\Gamma (f\cdot g)(x^\perp\cdot n)\,\mathrm{d}s = \mathrm{Re}\left(\int_\Gamma z\overline{f}\overline{g}\right).$$

Proof. See for instance [10]. □

Lemma 13 (Ladyzhenskaya's inequality). *Let $\mathscr{F}\subset\mathbb{R}^2$ an exterior domain with smooth boundary. Then the following inequality holds true*

$$\int_{\mathscr{F}} u^4 \leqslant C\int_{\mathscr{F}} u^2 \int_{\mathscr{F}} |\nabla u|^2,$$

for any function $u : \mathscr{F} \longrightarrow \mathbb{R}$ in $H^1(\mathscr{F})$.

Proof. The proof is a corollary of the well-known Ladyzhenskaya's inequality in $\mathbb{R}^2$. The idea of the proof is to extend u in $H^1(\mathbb{R}^2)$ in a proper way. Fix R such that $\mathbb{R}^2\setminus\mathscr{F}\subset B_{R/2}(0)$. Then there exists a continuous operator

$$\widetilde{E} : L^2(B_R(0)\cap\mathscr{F}) \longrightarrow L^2(B_R(0))$$

such that $\widetilde{E}(f)|_{B_R(0)\setminus\mathscr{F}} = f$, such that the operator restricted to the H^1 functions is continuous as

$$\widetilde{E}|_{H^1(B_R(0)\setminus\mathscr{F})} : H^1(B_R(0)\setminus\mathscr{F}) \longrightarrow H^1(B_R(0)).$$

We choose $\widetilde{E}$ to be defined as in the proof of Theorem 1 of Section 5.4 of [3] (see (3)). Consider now the extension operator defined as follow

$$E(f)(x) = \begin{cases} f & \text{if } x\in\mathscr{F},\\ \widetilde{E}\left(f|_{B_R(0)\cap\mathscr{F}} - \fint_{B_R(0)\cap\mathscr{F}} f\right) + \fint_{B_R(0)\cap\mathscr{F}} f & \text{if } x\in\mathbb{R}^2\setminus\mathscr{F}.\end{cases}$$

It holds

$$\int_{\mathscr{F}} u^4 \leqslant \int_{\mathbb{R}^2} E(u)^4 \leqslant \int_{\mathbb{R}^2} |E(u)|^2 \int_{\mathbb{R}^2} |\nabla E(u)|^2 \leqslant C\int_{\mathscr{F}} u^2\,\mathrm{d}x \int_{\mathscr{F}} |\nabla u|^2,$$

where the last inequality follows from the estimates

$$\begin{aligned}\int_{\mathbb{R}^2} |E(u)|^2 &= \int_{\mathscr{F}} u^2 + \int_{\mathbb{R}^2\setminus\mathscr{F}} |E(u)|^2\\ &\leqslant \|u\|^2_{L^2(\mathscr{F})} + C\left\|u - \fint_{B_R(0)\cap\mathscr{F}} u\right\|^2_{L^2(B_R(0)\cap\mathscr{F})} + \left\|\fint_{B_R(0)\cap\mathscr{F}} u\right\|^2_{L^2(B_R(0)\cap\mathscr{F})}\\ &\leqslant C\|u\|^2_{L^2(\mathscr{F})},\end{aligned}$$

where we use the continuity of $\widetilde{E}$ and C is a constant that may change line by line. In a similar way

$$\begin{aligned}\int_{\mathbb{R}^2} |\nabla E(u)|^2 &= \int_{\mathscr{F}} |\nabla u|^2 + \int_{\mathbb{R}^2\setminus\mathscr{F}} |\nabla E(u)|^2\\ &\leqslant \|\nabla u\|^2_{L^2(\mathscr{F})} + \int_{\mathbb{R}^2\setminus\mathscr{F}} |\nabla\widetilde{E}(u)|^2\\ &\leqslant \|\nabla u\|^2_{L^2(\mathscr{F})} + C\left\|u - \fint_{B_R(0)\cap\mathscr{F}} u\right\|_{H^1(B_R(0)\cap\mathscr{F})}\\ &\leqslant C\|\nabla u\|^2_{L^2(\mathscr{F})},\end{aligned}$$

where we use the continuity of $\widetilde{E}$ and the Poincaré inequality in the last step. □

We conclude with the scaling estimate.

Lemma 14. *For big enough $R>0$, it holds that solutions w_R of*

$$\begin{aligned} -\Delta w_R &= 0 && \text{in } B_R(0)\setminus\mathscr{S},\\ \nabla w_R\cdot n &= g_R && \text{on } \partial B_R(0),\\ \nabla w_R\cdot n &= 0 && \text{on } \partial\mathscr{S}, \end{aligned}$$

with $\int g_R = 0$, satisfy the estimate

$$\int_{\partial\mathscr{S}} |\nabla w_R|^2 \leqslant \frac{C}{R}\int_{\partial B_R(0)} |g_R|^2 \tag{19}$$

with C independent on R.

Proof. Consider the functions $v(x) = w_R(Rx)$ and $\overline{g}(x) = Rg_R(xR)$, which satisfy

$$\begin{aligned} -\Delta v &= 0 && \text{in } B_1(0)\setminus\mathscr{S}_{1/R},\\ \nabla v\cdot n &= \overline{g} && \text{on } \partial B_1(0),\\ \nabla v\cdot n &= 0 && \text{on } \partial\mathscr{S}_{1/R}, \end{aligned}$$

where $\mathscr{S}_{1/R} = \{x \text{ such that } Rx\in\mathscr{S}\}$. The inequality (19) is equivalent to

$$\int_{\partial\mathscr{S}_{1/R}} |\nabla v|^2 \leqslant \frac{C}{R}\int_{\partial B_1(0)} |\overline{g}|^2.$$

To show the above estimate, we decompose $v = \overline{v} + \check{v} + \hat{v}$ and for any component we prove the inequality. The function $\overline{v}$ satisfies

$$\begin{aligned} -\Delta \overline{v} &= 0 && \text{in } B_1(0),\\ \nabla \overline{v}\cdot n &= \overline{g} && \text{on } \partial B_1(0). \end{aligned}$$

This implies that $\overline{v}$ is smooth in $B_{1/2}(0)$ and the $W^{1,\infty}$ norm on $B_{1/2}(0)$ is bounded by a constant multiply by the L^1 norm of $\overline{g}$. We deduce that

$$\int_{\partial\mathscr{S}_{1/R}} |\nabla \overline{v}|^2 \leqslant \frac{C}{R}\int_{\partial B_1(0)} |\overline{g}|^2.$$

The functions $v-\overline{v}$ satisfy

$$\begin{aligned} -\Delta (v-\overline{v}) &= 0 && \text{in } B_1(0)\setminus\mathscr{S}_{1/R},\\ \nabla (v-\overline{v})\cdot n &= 0 && \text{on } \partial B_1(0),\\ \nabla (v-\overline{v})\cdot n &= -\nabla\overline{v}\cdot n && \text{on } \partial\mathscr{S}_{1/R}. \end{aligned}$$

Note that $\nabla\overline{v}$ is smooth in $\partial\mathscr{S}_{1/R}$ and we can bound any H^k norm by $1/\sqrt{R}\|\overline{g}\|_{L^2}$.

Consider the space $\widehat{H}_{1/R} = \{u\in L^2(\partial\mathscr{S}_{1/R}) \text{ such that } \|u\|_{L^2(\partial\mathscr{S}_{1/R})} + R^{-1/2}\|u\|_{\dot{H}^{1/2}(\partial\mathscr{S}_{1/R})} < \infty\}$. Note that the functions $\nabla\overline{v}\cdot n\in\widehat{H}_{1/R}$. By reflection method (see the Remark 15 after the proof), there exist $\alpha_1\in H^{1/2}$ and $\alpha_2\in\widehat{H}_{1/R}$ such that $\|\alpha_1\|_{H^{1/2}} + \|\alpha_2\|_{\hat{H}_{1/R}} \leqslant C\|\nabla\overline{v}\cdot n\|_{\hat{H}_{1/R}}$ for which $v-\overline{v} = \check{v}+\hat{v}$ solutions of

$$\begin{aligned} -\Delta \hat{v} &= 0 && \text{in } B_1(0), && -\Delta\check{v} = 0 && \text{in } \mathbb{R}^2\setminus B_{1/R}(0),\\ \nabla\hat{v}\cdot n &= \alpha_1 && \text{on } \partial B_1(0), && \nabla\check{v}\cdot n = \alpha_2 && \text{on } \partial B_{1/R}(0), \end{aligned}$$

the estimates of $\nabla\hat{v}$ on $\partial\mathscr{S}_{1/R}$ is exactly the same of $\overline{v}$. Regarding $\check{v}$ we perform scaling estimates. Fix $R=1$ and denote by $\check{v}_1(Rx) = \check{v}(x)$ and $\tilde{\alpha}_2(Rx) = 1/R\alpha_2(x)$, the following estimates holds

$$\|\nabla\check{v}_1\|_{L^2(\partial\mathscr{S})} \leqslant C_1\|\nabla\check{v}_1\|^{1/2}_{L^2(\mathbb{R}^2\setminus\mathscr{S})}\|\nabla^2\check{v}_1\|^{1/2}_{L^2(\mathbb{R}^2\setminus\mathscr{S})} \leqslant C_2\|\tilde{\alpha}_2\|_{\hat{H}_1}$$

by scaling it holds

$$\frac{1}{\sqrt{R}}\|\nabla\check{v}\|_{L^2(\partial\mathscr{S}_{1/R})} \leqslant \frac{C_1}{\sqrt{R}}\|\nabla\check{v}\|^{1/2}_{L^2(\mathbb{R}^2\setminus\mathscr{S}_{1/R})}\|\nabla^2\check{v}\|^{1/2}_{L^2(\mathbb{R}^2\setminus\mathscr{S}_{1/R})} \leqslant \frac{C_2}{\sqrt{R}}\|\alpha_2\|_{\hat{H}_{1/R}} \leqslant \frac{C_3}{R}\|\overline{g}\|_{L^2(\partial B_b(0))},$$

which concludes the proof. □

Remark 15. We refer to Section 3.1.2 of [11] for an introduction to the reflection method. Let us note that for big enough $R>0$ the linear map $\mathscr{T}:\widetilde{H}^{1/2}(\partial B_1(0))\times\widetilde{H}_{1/R}\longrightarrow\widetilde{H}^{1/2}(\partial B_1(0))\times\widetilde{H}_{1/R}$ which sends (α_1,α_2) to the couple $(\nabla\widehat{f}[\alpha_2-\nabla g[\alpha_1]\cdot n]\cdot n,0)$ has norm $\|\mathscr{T}\|_{\tilde{H}^{1/2}(\partial B_1(0))\times\tilde{H}_{1/R}}\leqslant 1/2$, where $\widetilde{H}^{1/2}(\partial B_1(0))$ and $\widetilde{H}_{1/R}$ are the subspaces of respectively $H^{1/2}(\partial B_1(0))$ and $\widehat{H}_{1/R}$ with zero average and for $\beta\in\widetilde{H}_{1/R}$, the function $\widehat{f}[\beta]$ satisfies $-\Delta\widehat{f}[\beta]=0$ in $\mathbb{R}^2\setminus\mathscr{S}_{1/R}$ and $\nabla\widehat{f}[\beta]\cdot n=\beta$ on $\partial\mathscr{S}_{1/R}$. The above observation is the key Lemma 3.6 of [11]. To show that the reflection method applies, we verify that $\|\mathscr{T}\|_{\tilde{H}^{1/2}(\partial B_1(0))\times\tilde{H}_{1/R}}\leqslant 1/2$, which follows from the estimate

$$\|\nabla\widehat{f}[\alpha_2-\nabla g[\alpha_1]\cdot n]\cdot n\|_{\tilde{H}^{1/2}(\partial B_1(0))}\leqslant\frac{C}{\sqrt{R}}\|\alpha_2-\nabla g[\alpha_1]\cdot n\|_{L^2(\partial\mathscr{S}_{1/R})}\leqslant\frac{C}{\sqrt{R}}\|\alpha_2\|_{L^2(\partial\mathscr{S}_{1/R})}+\frac{\widetilde{C}}{R}\|\alpha_1\|_{\partial B_1(0)},$$

and for R big enough.

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