INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Simon Raulot

Nonexistence of DEC spin fill-ins

Volume 360 (2022), p. 1049-1054

https://doi.org/10.5802/crmath.366

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2022, Vol. 360, p. 1049-1054
https://doi.org/10.5802/crmath.366

Mathematical physics / *Physique mathématique*

Nonexistence of DEC spin fill-ins

Non-existence de remplissages spinoriels satisfaisant la condition d'énergie dominante

Simon Raulot [a]

[a] Laboratoire de Mathématiques R. Salem UMR 6085 CNRS-Université de Rouen
Avenue de l'Université, BP.12 Technopôle du Madrillet 76801
Saint-Étienne-du-Rouvray, France.
E-mail: simon.raulot@univ-rouen.fr

Abstract. In this note, we show that a closed spin Riemannian manifold does not admit a spin fill-in satisfying the dominant energy condition (DEC) if a certain generalized mean curvature function is point-wise large.

Résumé. Dans cette note, on montre qu'une variété riemannienne fermée munie d'une structure spin n'admet pas de remplissage spinoriel satisfaisant la condition d'énergie dominante (DEC) si une certaine fonction, généralisant la courbure moyenne, est suffisamment grande.

Manuscript received 5 April 2022, accepted 15 April 2022.

Given a closed n-dimensional Riemannian manifold (Σ,γ), it is a very interesting question to know whether there exists an $(n+1)$-dimensional compact Riemannian manifold (Ω,g) with nonnegative scalar curvature (NNSC) whose boundary is isometric to (Σ,γ). If so, the Riemannian manifold is called a fill-in of (Σ,γ) and the set of all such fill-ins, referred to as NNSC fill-ins of (Σ,γ), is denoted by $\mathscr{F}(\Sigma,\gamma)$. The problem of the existence of such fill-ins has recently been solved by Shi, Wang and Wei [13] where it is shown that if Σ is the boundary of an $(n+1)$-dimensional compact manifold Ω then, any metric γ on Σ can be extended to a Riemannian metric g on Ω with positive scalar curvature.

One can also try to find a fill-in whose mean curvature is prescribed by a smooth function H on Σ. This problem is tightly related to the Bartnik quasi-local mass [2] and a triplet (Σ,γ,H) is then usually called a Bartnik data.

In [10], Miao proved that if Σ is the boundary of some compact $(n+1)$-dimensional manifold Ω, then given any Riemannian metric γ on Σ, there exists a constant H_0, depending on γ and Ω, such that, if $\min_\Sigma H\geq H_0$, there does not exist NNSC fill-ins of (Σ,γ,H). The proof makes use of the work of Shi, Wang and Wei [13] and of Schoen and Yau's results on closed manifolds [11, 12]. Such a result is also obtained for fill-ins with a negative scalar curvature lower bound.

This fact was previously demonstrated by Gromov [7] for spin manifolds. More precisely, he showed that if (Ω,g) is a NNSC spin fill-in of (Σ,γ,H), then

$$\min_\Sigma H\leq\frac{n}{\mathrm{Rad}(\Sigma,\gamma)}$$

where $\mathrm{Rad}(\Sigma,\gamma)$ is a constant only depending on (Σ,γ) and known as the hyperspherical radius of (Σ,γ). The main remark here, which is the key point of our argument, is that the nonexistence of NNSC spin fill-ins can be obtained by taking another route. Indeed, it can be deduced from an eigenvalue estimate on the first eigenvalue $\lambda_1(\Sigma,\gamma)$ of the Dirac operator of (Σ,γ) proved by Hijazi, Montiel and Zhang [9] and which states that

$$\min_{\Sigma} H \leq 2|\lambda_1(\Sigma,\gamma)|. \tag{1}$$

Since $\lambda_1(\Sigma,\gamma)$ depends only on Σ, γ and the involved spin structure, the nonexistence of NNSC spin fill-ins is a direct consequence of (1). This inequality is sharp since round balls in the Euclidean space satisfy the equality case. Note that a similar result can be deduced for spin fill-ins with a negative scalar curvature lower bound from [8] as in [7].

In this note, we use the aforementioned observation to generalize this result in the context of spin fill-ins satisfying the dominant energy condition (DEC). Following [3], a 5-uple $(\Sigma,\gamma,H,\alpha,\mathfrak{h})$ is called a spacetime Bartnik data set if (Σ,γ) is an oriented, closed Riemannian manifold, H and $\mathfrak{h}$ are smooth functions on Σ and α is a smooth 1-form on Σ. In this situation, a triplet (Ω,g,k) is a fill-in of such a spacetime Bartnik data if

(1) (Ω,g,k) is a compact initial data set, that is (Ω,g) is an $(n+1)$-dimensional compact Riemannian manifold with boundary and k is a smooth symmetric $(0,2)$-tensor field on Ω,
(2) there exists an isometry $f:(\Sigma,\gamma)\to(\partial\Omega,g_{|\partial\Omega})$ such that
    (a) $f^*H_g = H$, where H_g is the mean curvature of $\partial\Omega$ in (Ω,g) with respect to the outward unit normal $\widetilde{\nu}$,
    (b) $f^*\big(k(\widetilde{\nu},\cdot)^T\big)=\alpha$,
    (c) $f^*\big(\mathrm{Tr}_{g_{|\partial\Omega}}\,k\big)=\mathfrak{h}$.

Here $\mathrm{Tr}_{g_{|\partial\Omega}}$ denotes the trace operator on $\partial\Omega$ and ω^T is the tangent part to Σ of a 1-form ω defined along Σ. In the following, we will omit the isometry f in the identification between Σ and $\partial\Omega$. Then, a fill-in (Ω,g,k) of $(\Sigma,\gamma,H,\alpha,\mathfrak{h})$ satisfies the dominant energy condition, or is a DEC fill-in, if

$$\mu \geq |J|_g$$

where μ and J are respectively the mass density and the current density defined by

$$\mu = \frac{1}{2}\big(R_g + (\mathrm{Tr}_g\,k)^2 - |k|_g^2\big)$$

and

$$J = \mathrm{div}_g\big(k-(\mathrm{Tr}_g\,k)g\big).$$

Here R_g and Tr_g denote respectively the scalar curvature and the trace operator of (Ω,g). In this situation, a natural generalization of the mean curvature is given by the function

$$\mathscr{H}_g := H_g - \sqrt{|k(\widetilde{\nu},\cdot)^T|_g^2 + (\mathrm{Tr}_{g_{|\partial\Omega}}\,k)^2}$$

which corresponds, for a spacetime Bartnik data, to

$$\mathscr{H} := f^*\mathscr{H}_g = H - \sqrt{|\alpha|_\gamma^2 + \mathfrak{h}^2}. \tag{2}$$

When Σ is endowed with a spin structure, we will say that (Ω,g,k) is a DEC spin fill-in of the spacetime Bartnik data $(\Sigma,\gamma,H,\alpha,\mathfrak{h})$ if (Ω,g,k) is a DEC fill-in and if Ω is a spin manifold which induces the given spin structure on Σ. We then have the following result.

Theorem 1. *Given any Riemannian metric γ on an n-dimensional spin manifold Σ, there exists a constant $\mathscr{H}_0$, depending only on Σ, γ and the spin structure of Σ, such that, if $\min_\Sigma \mathscr{H} \geq \mathscr{H}_0$, there do not exist DEC spin fill-ins of $(\Sigma,\gamma,H,\alpha,\mathfrak{h})$.*

This question was tackled by Tsang in [14] where several partial results are proved. We remark that if $k = 0$ then $\alpha = 0$ and $\mathfrak{h} = 0$, and the DEC gives the nonnegativity of the scalar curvature so that a DEC fill-in of (Σ, γ, H) corresponds to a NNSC fill-in. In a same way, if $k = cg$, $c \neq 0$, then $\alpha = 0$ and $\mathfrak{h} = nc$, and the DEC condition implies that the scalar curvature of (Ω, g) is bounded from below by a negative constant, namely $R_g \geq -n(n+1)c^2$. These remarks imply that our result covers both of these cases. As mentioned above, the proof is a direct consequence of an eigenvalue estimate for the first eigenvalue of the Dirac operator of a spacetime Bartnik spin data which admits a DEC spin fill-in. This lower bound is stated as follow.

Theorem 2. *If a spacetime Bartnik spin data $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ admits a DEC spin fill-in (Ω, g, k) with $\mathscr{H} > 0$, then the first eigenvalue $\lambda_1(\Sigma, \gamma)$ of the Dirac operator of (Σ, γ) satisfies*

$$\min_{\Sigma} \mathscr{H} \leq 2|\lambda_1(\Sigma, \gamma)|.$$

One can consider $\mathscr{F}_{spin}(\Sigma, \gamma)$, the set of the DEC spin fill-ins of the Riemannian spin manifold (Σ, γ) without specifying the data H, α and $\mathfrak{h}$. Then Theorem 2 implies that if $\mathscr{F}_{spin}(\Sigma, \gamma) \neq \emptyset$, it holds that

$$\sup_{(\Omega, g, k) \in \mathscr{F}_{spin}(\Sigma, \gamma)} \min_{\Sigma} \mathscr{H} \leq 2|\lambda_1(\Sigma, \gamma)| < \infty.$$

The proof of Theorem 2 relies on spin geometry and we refer especially to [4–6] for more details on this subject. Let us briefly recall what we need here. Since (Ω, g) is a Riemannian spin manifold, there exists a smooth Hermitian vector bundle over Ω, the spinor bundle, denoted by $\mathbf{S}\Omega$, whose sections are called spinor fields. The Hermitian scalar product is denoted by $\langle\, , \rangle$. Moreover, the tangent bundle $T\Omega$ acts on $\mathbf{S}\Omega$ by Clifford multiplication $X \otimes \psi \mapsto c(X)\psi$ for any tangent vector fields X and any spinor fields ψ. On the other hand, the Riemannian Levi-Civita connection ∇ lifts to the so-called spin Levi-Civita connection, also denoted by ∇, and defines a metric covariant derivative on $\mathbf{S}\Omega$ that preserves the Clifford multiplication. A quadruplet $(\mathbf{S}\Omega, c, \langle\, , \rangle, \nabla)$ which satisfies the previous assumptions is usually referred to as a Dirac bundle. The Dirac operator is then the first order elliptic differential operator acting on $\mathbf{S}\Omega$ defined by $D := c \circ \nabla$. The spin structure on Ω induces, via a choice of an unit normal field to $\partial\Omega \simeq \Sigma$, a spin structure on Σ. This allows to define the *extrinsic* spinor bundle $\not{\mathbf{S}} := \mathbf{S}\Omega_{|\Sigma}$ over Σ on which there exists a Clifford multiplication $\not{c}$ and a metric covariant derivative $\not{\nabla}$. The quadruplet $(\not{\mathbf{S}}, \not{c}, \langle\, , \rangle, \not{\nabla})$ is thus endowed with a Dirac bundle structure. Similarly, the extrinsic Dirac operator is defined by taking the Clifford trace of the covariant derivative $\not{\nabla}$ that is $\not{D} := \not{c} \circ \not{\nabla}$. It is by now well-known that this operator can be expressed using the Dirac operator D_Σ of (Σ, γ) endowed with the induced spin structure. What is important to us here is that the first nonnegative eigenvalue of the extrinsic Dirac operator $\not{D}$ corresponds to $|\lambda_1(\Sigma, \gamma)|$, the absolute value of the first eigenvalue of D_Σ and so it only depends on (Σ, γ) and the spin structure on Σ.

Proof of Theorem 2. Let (Ω, g, k) be a DEC spin fill-in of the spacetime Bartnik spin data $(\Sigma, \gamma, H, \alpha, \mathfrak{h})$ and consider the modified spin covariant derivatives defined by

$$\nabla^{\pm}_X \psi := \nabla_X \psi \pm \frac{i}{2} c\big(k(X)\big)\psi \tag{3}$$

for $X \in \Gamma(T\Omega)$ and $\psi \in \Gamma(\mathbf{S}\Omega)$. The associated Dirac-type operators given by $D^{\pm} := c \circ \nabla^{\pm}$ are easily seen to satisfy

$$D^{\pm}\psi = D\psi \mp \frac{i}{2}(\mathrm{Tr}_g\, k)\psi. \tag{4}$$

These are first order elliptic differential operators whose formal adjoints, with respect to the L^2 scalar product on $\mathbf{S}\Omega$, are $D^{\mp}$ as deduced from the following integration by parts formulae

$$\int_\Omega \langle D^{\pm}\psi, \varphi\rangle \mathrm{d}\mu = \int_\Omega \langle \psi, D^{\mp}\varphi\rangle \mathrm{d}\mu - \int_\Sigma \langle c(\nu)\psi, \varphi\rangle \mathrm{d}\sigma \tag{5}$$

for all smooth spinor fields ψ, φ on Ω. Here $\mathrm{d}\mu$ (resp. $\mathrm{d}\sigma$) denotes the Riemannian volume (resp. area) element of (Ω,g) (resp. (Σ,γ)) and ν is the inner unit normal to Σ in (Ω,g). In a same way, a straightforward computation implies that

$$\big(\nabla^{\pm}\big)^*\nabla^{\pm}\psi = -\sum_{j=1}^{n+1}\nabla^{\mp}_{e_j}\nabla^{\pm}_{e_j}\psi \tag{6}$$

where $\big(\nabla^{\pm}\big)^*$ denote the formal adjoints of the modified connection $\nabla^{\pm}$ and $\{e_1,\cdots,e_{n+1}\}$ is a local g-orthonormal frame of $T\Omega$. In particular, the Stokes formula leads to

$$\int_\Omega\langle\big(\nabla^{\pm}\big)^*\nabla^{\pm}\psi,\psi\rangle\mathrm{d}\mu = \int_\Omega|\nabla^{\pm}\psi|^2\mathrm{d}\mu + \int_\Sigma\langle\nabla^{\pm}_\nu\psi,\psi\rangle\mathrm{d}\sigma. \tag{7}$$

Then, it follows from the fact that $\big(D^{\pm}\big)^* = D^{\mp}$, from (6) and the classical Schrödinger–Lichnerowicz formula

$$D^2\psi = \nabla^*\nabla\psi + \frac{R_g}{4}\psi$$

that

$$\big(D^{\pm}\big)^*D^{\pm}\psi = \big(\nabla^{\pm}\big)^*\nabla^{\pm}\psi + \frac{1}{2}\big(\mu\psi \pm ic(J)\psi\big) \tag{8}$$

for all $\psi\in\Gamma(\mathbf{S}\Omega)$. This is the $(n+1)$-dimensional Riemannian counterpart of the formula obtained by Witten [15] in his proof of the positive energy theorem. Now observe that since the point-wise symmetric endomorphism $J^{\pm} := \pm ic(J)$ satisfies $\big(J^{\pm}\big)^2\psi = |J|^2_g\psi$ it holds that

$$\langle J^{\pm}\psi,\psi\rangle \geq -|J|_g|\psi|^2$$

so that the DEC and (8) imply the point-wise inequalities

$$\langle\big(D^{\pm}\big)^*D^{\pm}\psi,\psi\rangle \geq \langle\big(\nabla^{\pm}\big)^*\nabla^{\pm}\psi,\psi\rangle$$

for all $\psi\in\Gamma(\mathbf{S}\Omega)$. Now integrating by parts these estimates on Ω using (5) and (7) leads to the following important integral inequalities

$$\int_\Omega\Big(|\nabla^{\pm}\psi|^2 - |D^{\pm}\psi|^2\Big)\mathrm{d}\mu \leq -\int_\Sigma\langle\nabla^{\pm}_\nu\psi + c(\nu)D^{\pm}\psi,\psi\rangle\mathrm{d}\sigma. \tag{9}$$

From the very definitions (3) and (4) of the modified covariant derivatives and the associated Dirac operators, we compute that

$$-\nabla^{\pm}_\nu\psi - c(\nu)D^{\pm}\psi = \mathbb{D}\psi - \frac{1}{2}\big(H\psi \mp ic(\mathcal{V})\psi\big) \tag{10}$$

where

$$\mathcal{V} := \alpha^\sharp + \mathfrak{h}\nu \in \Gamma(T\Omega_{|\Sigma})$$

since (Ω,g,k) is a fill-in of the data $(\Sigma,\gamma,H,\alpha,\mathfrak{h})$. Here $\sharp : T^*\Omega \to T\Omega$ denotes the classical musical isomorphism between the cotangent bundle and the tangent bundle. Observe that the endomorphisms $\mathcal{V}^{\pm} := \pm ic(\mathcal{V})$ of $\mathbb{S}$ is point-wise symmetric with respect to the Hermitian structure and satisfies $(\mathcal{V}^{\pm})^2\psi = |\mathcal{V}|^2\psi$ in such a way that

$$\langle\mathcal{V}^{\pm}\psi,\psi\rangle \geq -|\mathcal{V}|_g|\psi|^2 = -\sqrt{\mathfrak{h}^2 + |\alpha|^2_\gamma}\,|\psi|^2 \tag{11}$$

for all $\psi\in\Gamma(\mathbb{S})$. Combining (9), (10) and (11) yields the following integral inequalities

$$\int_\Omega\Big(|\nabla^{\pm}\psi|^2 - |D^{\pm}\psi|^2\Big)\mathrm{d}\mu \leq \int_\Sigma\langle\mathbb{D}\psi - \frac{1}{2}\mathscr{H}\psi,\psi\rangle\mathrm{d}\sigma. \tag{12}$$

which hold for all $\psi\in\Gamma(\mathbf{S}\Omega)$ and where $\mathscr{H}$ is the generalized mean curvature function defined in (2).

Now we are going to show that one can extend any spinor fields on Σ in a suitable way. For this, we recall that the map $\chi := ic(\nu)$ is a boundary chirality operator (in the sense of [1, Example 7.26]) and so it is an orthogonal involution of $\mathbb{S}$ which induces an orthogonal splitting $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ into the eigenbundles of χ for the eigenvalues ±1. The associated projection maps $P^{\pm} : \mathbb{S} \to \mathbb{S}^{\pm}$

define elliptic local boundary conditions for the Dirac-type operators $D^{\pm}$ (see [1, Corollary 7.23] for example). This implies that the operators

$$D^{+}_{\pm}:\left\{\psi\in H^{1}(\mathbf{S}\Omega)\,/\,P^{\pm}\psi_{|\Sigma}=0\right\}\longrightarrow L^{2}(\mathbf{S}\Omega)$$

are of Fredholm type and that if $\Phi\in\Gamma(\mathbf{S}\Omega)$ and $\Psi\in\Gamma(\mathbb{S})$ are smooth spinor fields, any solutions $\psi\in\Gamma(\mathbf{S}\Omega)$ of the boundary problem

$$\begin{cases} D^{+}\psi=\Phi & \text{on } \Omega\\ P^{\pm}\psi_{|\Sigma}=P^{\pm}\Psi & \text{on } \Sigma\end{cases}\tag{13}$$

are smooth. The same holds for the operators $D^{-}_{\pm}$. It turns out that these operators are isomorphisms. To prove this fact, we notice that it is enough to show that $D^{+}_{\pm}$ and $D^{-}_{\pm}$ are one-to-one since it follows from the integration by parts formulae (5) that the adjoint of $D^{+}_{\pm}$ is $D^{-}_{\mp}$. So take $\psi_{\pm}$, non trivial, in the kernel of $D^{+}_{\pm}$, that is $\psi_{\pm}\in\Gamma(\mathbf{S}\Omega)$ satisfies (13) with $\Phi=0$ and $\Psi=0$. In particular, $\psi_{\pm}$ is smooth on Ω. On the other hand, from the self-adjointness of the Dirac operator $\mathbb{D}$ and the fact that $\mathbb{D}\chi=-\chi\mathbb{D}$, we get that

$$\int_{\Sigma}\langle\mathbb{D}\psi,\psi\rangle d\sigma=2\int_{\Sigma}\mathrm{Re}\langle\mathbb{D}(P^{\pm}\psi),P^{\mp}\psi\rangle d\sigma\tag{14}$$

for all $\psi\in\Gamma(\mathbf{S}\Omega)$. Using this formula for $\psi=\psi_{\pm}$, we deduce from (12) that

$$0\le\int_{\Omega}|\nabla^{\pm}\psi_{\pm}|^{2}d\mu\le-\frac{1}{2}\min_{\Sigma}\mathscr{H}\int_{\Sigma}|\psi_{\pm}|^{2}d\sigma.$$

Since we assumed that $\mathscr{H}$ is positive on Σ, we conclude that $\psi_{\pm}$ is zero on Σ and $\nabla^{\pm}\psi_{\pm}=0$. This leads to a contradiction since this last property implies that $\psi_{\pm}$ is nowhere vanishing on Ω. The same holds for $D^{-}_{\pm}$.

Now take $\Psi_1\in\Gamma(\mathbb{S})$ an eigenspinor for the operator $\mathbb{D}$ associated with the eigenvalue $|\lambda_1(\Sigma,\gamma)|$. From the previous discussion, there exists an unique smooth solution $\varphi\in\Gamma(\mathbf{S}\Omega)$ satisfying

$$\begin{cases} D^{+}\varphi=0 & \text{on } \Omega\\ P^{+}\varphi_{|\Sigma}=P^{+}\Psi_1 & \text{on } \Sigma.\end{cases}$$

Taking $\psi=\varphi$ in (12) and using the fact that

$$2\,\mathrm{Re}\langle P^{-}\Psi_1,P^{-}\varphi\rangle\le|P^{-}\Psi_1|^{2}+|P^{-}\varphi|^{2},$$

finally lead to

$$0\le\Big(|\lambda_1(\Sigma,\gamma)|-\frac{1}{2}\min_{\Sigma}\mathscr{H}\Big)\int_{\Sigma}|\varphi|^{2}d\sigma$$

which implies the estimate of Theorem 2. To get this last inequality, we implicitly used the fact that

$$\begin{aligned}|\lambda_1(\Sigma,\gamma)|\int_{\Sigma}|P^{-}\Psi_1|^{2}d\sigma&=\int_{\Sigma}\langle\mathbb{D}(P^{+}\varphi),P^{-}\Psi_1\rangle d\sigma\\&=\int_{\Sigma}\langle P^{+}\varphi,\mathbb{D}(P^{-}\Psi_1)\rangle d\sigma\\&=\int_{\Sigma}\langle P^{+}\varphi,P^{+}(\mathbb{D}\Psi_1)\rangle d\sigma\\&=|\lambda_1(\Sigma,\gamma)|\int_{\Sigma}|P^{+}\varphi|^{2}d\sigma\end{aligned}$$

which follows from the identity (14) and the self-adjointness of $\mathbb{D}$. □

We conclude this note by noticing that this method can be generalized to other situations (like the Einstein–Maxwell equations with nonpositive cosmological constant).

References

[1] C. Bär, W. Ballmann, "Boundary value problems for elliptic differential operators of first order", in *Surveys in differential geometry. Vol. XVII*, Surveys in Differential Geometry, vol. 17, International Press, 2012, p. 1-78.

[2] R. Bartnik, "New definition of quasilocal mass", *Phys. Rev. Lett.* **62** (1989), no. 20, p. 2346-2348.

[3] ———, "Energy in general relativity", in *Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991)*, International Press, 1997, p. 5-27.

[4] J.-P. Bourguignon, O. Hijazi, J.-L. Milhorat, A. Moroianu, S. Moroianu, *A spinorial approach to Riemannian and conformal geometry*, EMS Monographs in Mathematics, European Mathematical Society, 2015, ix+452 pages.

[5] T. Friedrich, "On the spinor representation of surfaces in Euclidean 3-space", *J. Geom. Phys.* **28** (1998), no. 1-2, p. 143-157.

[6] N. Ginoux, *The Dirac spectrum*, Lecture Notes in Mathematics, vol. 1976, Springer, 2009, xvi+156 pages.

[7] M. Gromov, "Scalar curvature of manifolds with boundaries: natural questions and artificial constructions", https://arxiv.org/abs/1811.04311, 2018.

[8] O. Hijazi, S. Montiel, A. Roldán, "Dirac operators on hypersurfaces of manifolds with negative scalar curvature", *Ann. Global Anal. Geom.* **23** (2003), no. 3, p. 247-264.

[9] O. Hijazi, S. Montiel, X. Zhang, "Dirac operator on embedded hypersurfaces", *Math. Res. Lett.* **8** (2001), no. 1-2, p. 195-208.

[10] P. Miao, "Nonexistence of NNSC fill-ins with large mean curvature", *Proc. Am. Math. Soc.* **149** (2021), no. 6, p. 2705-2709.

[11] R. Schoen, S.-T. Yau, "On the structure of manifolds with positive scalar curvature", *Manuscr. Math.* **28** (1979), no. 1-3, p. 159-183.

[12] ———, "Positive scalar curvature and minimal hypersurface singularities", in *Surveys in differential geometry 2019. Differential geometry, Calabi-Yau theory, and general relativity*, Surveys in Differential Geometry, vol. 24, International Press, 2019, p. 441-480.

[13] Y. Shi, W. Wang, G. Wei, "Total mean curvature of the boundary and nonnegative scalar curvature fill-ins", *J. Reine Angew. Math.* **784** (2022), p. 215-250.

[14] T.-Y. Tsang, "On a spacetime positive mass theorem with corners", https://arxiv.org/abs/2109.11070, 2022.

[15] E. Witten, "A new proof of the positive energy theorem", *Commun. Math. Phys.* **80** (1981), no. 3, p. 381-402.