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# Shinbrot's energy conservation criterion for the 3D Navier–Stokes–Maxwell system

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**Abstract.** This paper concerns the energy conservation for the weak solutions to the Navier–Stokes–Maxwell system. Although the Maxwell equation with hyperbolic nature, we still establish a $L^q L^p$ type condition guarantee validity of the energy equality for the weak solutions. We mention that there no regularity assumption on the electric field $E$.

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## 1. Introduction

In this paper, we study the incompressible Navier–Stoke–Maxwell system

$$\begin{cases} \partial_t u + u\cdot\nabla u - \mu\Delta u = -\nabla p + j\times B, & \operatorname{div} u = 0, \\ \frac{1}{c}\partial_t E - \nabla\times B = -j, & j = \sigma(cE + u\times B), \\ \frac{1}{c}\partial_t B + \nabla\times E = 0, & \operatorname{div} B = 0, \end{cases} \tag{1}$$

with the initial data

$$u(0,x) = u_0, E(0,x) = E_0, B(0,x) = B_0 \tag{2}$$

for $(t,x)\in\mathbb{R}^+\times\Omega$, where $\Omega = \mathbb{T}^3$ is a periodic domain in $\mathbb{R}^3$. Here $c > 0$ denotes the speed of light, $\mu > 0$ is the viscosity of the fluid, and $\sigma > 0$ is the electrical conductivity. In the above system (1), $u = (u_1,u_2,u_3) = u(t,x)$ stands for the velocity field of the (incompressible) fluid, while $E = (E_1,E_2,E_3) = E(t,x)$ and $B = (B_1,B_2,B_3) = B(t,x)$ are the electric and magnetic fields, respectively. The scalar function $p = p(t,x)$ is the pressure and is also an unknown. Observe, though, that the electric current $j = j(t,x)$ is not an unknown, for it is fully determined by $(u,E,B)$ through Ohm's law. For simplicity, we will take $\mu = c = \sigma = 1$.

The Navier–Stokes–Maxwell system (1) describes the evolution of a plasma (i.e., a charged fluid) subject to a self-induced electromagnetic Lorentz force $j\times B$. Mathematically, (1) is a

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coupled system, constituted by the parabolic nature of the Navier–Stokes equations from fluid dynamics and the hyperbolic of the Maxwell equation from electromagnetism. Moreover, it can be derived from the Vlasov–Maxwell–Boltzmann system [2]. In the 2D case, Masmoudi [8] proved global existence of regular solutions to the Maxwell–Navier–Stokes system (1), and also provide an exponential growth estimate for the $H^s$ norm of the solution when the time goes to infinity. In the 3D case, Ibrahim–Keraani [10] showed the existence of global small mild solutions of system (1). Recently, Arsenio–Gallagher [1] have made important and new progress in this direction, they established that global existence of solutions to the 3D system (1) holds whenever the initial datum $(u_0, E_0, B_0)$ is chosen in the natural energy space $L^2$, while the electromagnetic field $(E_0, B_0)$ alone lies in $\dot{H}^s$, for some given $s \in [\frac{1}{2}, \frac{3}{2})$, and is sufficiently small when compared to some non-linear function of the initial energy

$$\mathscr{E}_0 := \frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right).$$

Before discussing the contents of this paper let us recall some well-established facts regarding the incompressible Navier–Stokes equations (corresponding to the case when $(E, B) = 0$ in (1)), in relation with this work. Pioneering works of Leray [5] and Hopf [3] showed the global existence of a weak solution called Leray–Hopf solution with initial data $u_0 \in L^2(\Omega)$. As usual, a weak solution $u$ satisfies the energy inequality

$$\|u(t)\|_{L^2}^2 + 2\int_0^T \int_\Omega |\nabla u|^2\, dx dt \le \|u_0\|_{L^2}^2 \tag{3}$$

for any $t \in (0, T)$. On the other hand, regular solutions to the 3D Navier–Stokes equations satisfy the energy equality:

$$\|u(t)\|_{L^2}^2 + 2\int_0^T \int_\Omega |\nabla u|^2\, dx dt = \|u_0\|_{L^2}^2 . \tag{4}$$

A natural question that still remains open is whether energy equality, which should be expected from a physical point of view, is valid for weak solutions. Lions [6] and Ladyzhenskaya [4] proved independently that such solutions satisfy the (global) energy equality (4) under the additional assumption $u \in L^4L^4$. Shinbrot [9] generalized the Lions–Ladyzhenskaya condition to $u \in L^r(0, T; L^s(\Omega))$ with $\frac{2}{r} + \frac{2}{s} \le 1$, $s \ge 4$. Recently, Tan and Yin [12] proved that $u \in L^{2,\infty}(0, T; BMO)$ implies energy equality (4).

For system (1), by results of Arsenio–Gallagher [1], it is know that for initial data $(u_0, E_0, B_0)$, with $\operatorname{div} u_0 = \operatorname{div} B_0 = 0$, belongs to

$$\left(L^2 \times \left(H^s\right)^2\right)\left(\mathbb{R}^3\right), \frac{1}{2} \le s < \frac{3}{2},$$

and

$$\|(E_0, B_0)\|_{\dot{H}^s} C_* \mathscr{E}_0^{s-\frac{1}{2}} e^{C_* \mathscr{E}_0} \le 1,$$

then there is a global weak solution $(u, E, B)$ to the 3D Navier–Stokes–Maxwell system (1) such that

$$u \in L^\infty\left(\mathbb{R}^+; L^2\right) \cap L^2\left(\mathbb{R}^+; \dot{H}^1\right), \quad (E, B) \in L^\infty\left(\mathbb{R}^+; L^2\right), \quad j \in L^2\left(\mathbb{R}^+; L^2\right) \tag{5}$$

satisfying the energy inequality, for almost all $t > 0$,

$$\begin{aligned} &\frac{1}{2}\left(\|u(t)\|_{L^2}^2 + \|E(t)\|_{L^2}^2 + \|B(t)\|_{L^2}^2\right) + \int_0^t \left(\|\nabla u(\tau)\|_{L^2}^2 + \|j(\tau)\|_{L^2}^2\right) d\tau \\ \le &\frac{1}{2}\left(\|u_0\|_{L^2}^2 + \|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right). \end{aligned} \tag{6}$$

Moreover, the regular solutions to (1) satisfy the corresponding energy equality

$$\begin{aligned}&\frac{1}{2}\left(\|u(t)\|_{L^2}^2+\|E(t)\|_{L^2}^2+\|B(t)\|_{L^2}^2\right)+\int_0^t\left(\|\nabla u(\tau)\|_{L^2}^2+\|j(\tau)\|_{L^2}^2\right)d\tau\\=&\frac{1}{2}\left(\|u_0\|_{L^2}^2+\|E_0\|_{L^2}^2+\|B_0\|_{L^2}^2\right).\end{aligned}\tag{7}$$

Although we can not prove that the weak solution is regular on time, due to the fact that the Navier–Stoke equations is a sub-system of (1). An interesting question is whether or not the energy equality (7) is holds for such weak solutions. By using the lemma introduced by Lions, which has been used in the Yu's article [13], we shall give an positive answer for this problem if $(u,B)\in L^qL^p$.

We state our main result as follows:

**Theorem 1.** *Let $u\in L^\infty(0,T;L^2(\Omega))\cap L^2(0,T;H^1(\Omega))$, $(E,B)\in L^\infty(0,T;L^2(\Omega))$ and $j\in L^2(0,T;L^2(\Omega))$ be a weak solution to system (1). In addition, if $u\in L^q(0,T;L^p(\Omega))$ for any $\frac{1}{q}+\frac{1}{p}\le\frac{1}{2}$, $p\ge4$, and $B\in L^r(0,T;L^s(\Omega))$ for any $\frac{1}{r}+\frac{1}{s}\le\frac{1}{2}$, $s\ge4$, then*

$$\begin{aligned}&\int_\Omega|u(t,x)|^2+|E(t,x)|^2+|B(t,x)|^2\,dx+2\int_0^T\int_\Omega|\nabla u|^2+|j|^2dxdt\\=&\int_\Omega|u_0|^2+|E_0|^2+|B_0|^2\,dx\end{aligned}$$

*for any $t\in[0,T]$.*

**Remark 2.** This result extends the well-known Shinbrot's energy conservation criterion to the Navier–Stokes–Maxwell system (1).

**Remark 3.** By means of arguement in [11], it is interesting to generalize Theorem 1 from periodic domain to bounded Lipschitz domain.

## 2. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1. By using interpolation inequalities

$$\|f\|_{L^4(0,T;L^4(\Omega))}\le C\|f\|^\theta_{L^\infty(0,T;L^2(\Omega))}\|f\|^{1-\theta}_{L^n(0,T;L^m(\Omega))}\le C$$

with $\frac{1}{n}+\frac{1}{m}=\frac{1}{2}$, $m>4$, we need only prove the case $p=q=r=s=4$. For the sake of simplicity, we will proceed as if the solution is differentiable in time. The extra arguments needed to mollify in time are straightforward.

Let $\eta:\mathbb{R}^3\to\mathbb{R}$ be a standard mollifier, i.e. $\eta(x)=Ce^{\frac{1}{|x|^2-1}}$ for $|x|<1$ and $\eta(x)=0$ for $|x|\geqslant1$, where constant $C>0$ selected such that $\int_{\mathbb{R}^3}\eta(x)\mathrm{d}x=1$. For any $\varepsilon>0$, we define the rescaled mollifier $\eta_\varepsilon(x)=\varepsilon^{-3}\eta\left(\frac{x}{\varepsilon}\right)$. For any function $f\in L^1_{\text{loc}}(\Omega)$, its mollified version is defined as

$$f^\varepsilon(x)=\left(f*\eta_\varepsilon\right)(x)=\int_\Omega\eta_\varepsilon(x-y)f(y)\mathrm{d}y.$$

If $f\in W^{1,p}(\Omega)$, the following local approximation is well known

$$f^\varepsilon(x)\to f\quad\text{in}\quad W^{1,p}_{loc}(\Omega)\quad\forall\,p\in[1,\infty).$$

The crucial ingredient to prove Theorem 1 is the following important lemma proved by Lions [7]

**Lemma 4.** *Let $\partial$ be a partial derivative in one direction. Let $f,\partial f\in L^p(\mathbb{R}^+\times\Omega)$, $g\in L^q(\mathbb{R}^+\times\Omega)$ with $1\le p,q\le\infty$, and $\frac{1}{p}+\frac{1}{q}\le1$. Then, we have*

$$\left\|\partial(fg)*\eta_\varepsilon-\partial\left(f\left(g*\eta_\varepsilon\right)\right)\right\|_{L^r(\mathbb{R}^+\times\Omega)}\le C\left\|\partial f\right\|_{L^p(\mathbb{R}^+\times\Omega)}\|g\|_{L^q(\mathbb{R}^+\times\Omega)}$$

*for some constant $C>0$ independent of $\varepsilon$, $f$ and $g$, and with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. In addition,*

$$\partial(fg)*\eta_\varepsilon-\partial\left(f\left(g*\eta_\varepsilon\right)\right)\to 0\quad in\quad L^r\left(\mathbb{R}^+\times\Omega\right)$$

*as* $\varepsilon\to 0$, *if* $r<\infty$.

With Lemma 4 in hand, we are ready to prove our main result. First, applying the mollifier to the system (1) and multiplying all equations of the system by $u^\varepsilon$, $E^\varepsilon$ and $B^\varepsilon$, respectively, one has

$$\begin{cases}\int_\Omega u^\varepsilon\left(\partial_t u+u\cdot\nabla u-\Delta u+\nabla p-j\times B\right)^\varepsilon dx=0,\\ \int_\Omega E^\varepsilon\left(\partial_t E-\nabla\times B+j\right)^\varepsilon dx=0,\\ \int_\Omega B^\varepsilon\left(\partial_t B+\nabla\times E\right)^\varepsilon dx=0.\end{cases}\tag{8}$$

This yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega\left(\left|u^\varepsilon\right|^2+\left|E^\varepsilon\right|^2+\left|B^\varepsilon\right|^2\right)dx+\int_\Omega\left|\nabla u^\varepsilon\right|^2dx&\\ =-\int_\Omega \operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon dx+\int_\Omega(j\times B)^\varepsilon\cdot u^\varepsilon dx+\int_\Omega(\nabla\times B)^\varepsilon\cdot E^\varepsilon dx&\\ -\int_\Omega j^\varepsilon\cdot E^\varepsilon dx-\int_\Omega(\nabla\times E)^\varepsilon\cdot B^\varepsilon dx.&\end{aligned}\tag{9}$$

Clearly,

$$\begin{aligned}&\int_\Omega\left(\left|u^\varepsilon\right|^2+\left|E^\varepsilon\right|^2+\left|B^\varepsilon\right|^2\right)dx-\int_\Omega\left(\left|u_0^\varepsilon\right|^2+\left|E_0^\varepsilon\right|^2+\left|B_0^\varepsilon\right|^2\right)dxdt+2\int_0^T\int_\Omega\left|\nabla u^\varepsilon\right|^2dxdt\\ =&-2\int_0^T\int_\Omega \operatorname{div}(u\otimes u)^\varepsilon\cdot u^\varepsilon dxdt+2\int_0^T\int_\Omega(j\times B)^\varepsilon\cdot u^\varepsilon dxdt+2\int_0^T\int_\Omega(\nabla\times B)^\varepsilon\cdot E^\varepsilon dxdt\\ &-2\int_0^T\int_\Omega j^\varepsilon\cdot E^\varepsilon dxdt-2\int_0^T\int_\Omega(\nabla\times E)^\varepsilon\cdot B^\varepsilon dxdt\\ =&I_1+I_2+I_3+I_4+I_5.\end{aligned}\tag{10}$$

Notice that

$$\begin{aligned}\operatorname{div}(u\otimes u)^\varepsilon&=\left[\operatorname{div}(u\otimes u)^\varepsilon-\operatorname{div}\left(u\otimes u^\varepsilon\right)\right]+\left[\operatorname{div}\left(u\otimes u^\varepsilon\right)-\operatorname{div}\left(u^\varepsilon\otimes u^\varepsilon\right)\right]+\operatorname{div}\left(u^\varepsilon\otimes u^\varepsilon\right)\\ &=I_{11}+I_{12}+I_{13},\end{aligned}$$

thus

$$I_1=-2\int_0^T\int_\Omega(I_{11}+I_{12}+I_{13})\cdot u^\varepsilon dxdt.\tag{11}$$

From Lemma 4, one obtains

$$\|I_{11}\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)}\le C\|u\|_{L^4(0,T;L^4(\Omega))}\|\nabla u\|_{L^2(0,T;L^2(\Omega))}$$

and it converges to zero in $L^{\frac{4}{3}}(0,T;L^{\frac{4}{3}}(\Omega))$ as $\varepsilon$ tends to zero. Thus, as $\varepsilon$ goes to zero, it follows that

$$\begin{aligned}\left|-2\int_0^T\int_\Omega I_{11}u^\varepsilon dxdt\right|&=\left|\int_0^T\int_\Omega\left[\operatorname{div}(u\otimes u)^\varepsilon-\operatorname{div}\left(u\otimes u^\varepsilon\right)\right]\cdot u^\varepsilon dxdt\right|\\ &\le\|I_{11}\|_{L^{\frac{4}{3}}\left(0,T;L^{\frac{4}{3}}(\Omega)\right)}\left\|u^\varepsilon\right\|_{L^4(0,T;L^4(\Omega))}\\ &\to 0.\end{aligned}\tag{12}$$

Moreover, we have

$$\begin{aligned}\left|-2\int_0^T\int_\Omega I_{12}u^\varepsilon dxdt\right| &= \left|\int_0^T\int_\Omega \left[\operatorname{div}\left(u\otimes u^\varepsilon\right)-\operatorname{div}\left(u^\varepsilon\otimes u^\varepsilon\right)\right]\cdot u^\varepsilon dxdt\right| \\ &= \left|\int_0^T\int_\Omega \left(u\otimes u^\varepsilon-u^\varepsilon\otimes u^\varepsilon\right)\cdot\nabla u^\varepsilon dxdt\right| \\ &\le \left\|u-u^\varepsilon\right\|_{L^4(0,T;L^4(\Omega))}\left\|u^\varepsilon\right\|_{L^4(0,T;L^4(\Omega))}\left\|\nabla u^\varepsilon\right\|_{L^2(0,T;L^2(\Omega))} \\ &\to 0.\end{aligned} \tag{13}$$

Since $\operatorname{div} u^\varepsilon = 0$, one has

$$-2\int_0^T\int_\Omega I_{13}\cdot u^\varepsilon dx = 0. \tag{14}$$

Combining (11)-(14), we know that

$$I_1 \to 0,\ as\ \varepsilon\to 0. \tag{15}$$

For the term $I_2$, we claim that

$$I_2 = 2\int_0^T\int_\Omega (j\times B)^\varepsilon\cdot u^\varepsilon dxdt \to 2\int_0^T\int_\Omega (j\times B)\cdot udxdt,\ as\ \varepsilon\to 0. \tag{16}$$

Indeed,

$$\begin{aligned}&2\int_0^T\int_\Omega (j\times B)^\varepsilon\cdot u^\varepsilon-(j\times B)\cdot udxdt \\ &=2\int_0^T\int_\Omega (j\times B)^\varepsilon\cdot u^\varepsilon-(j\times B)\cdot u^\varepsilon+(j\times B)\cdot u^\varepsilon-(j\times B)\cdot udxdt \\ &=2\int_0^T\int_\Omega \left[(j\times B)^\varepsilon-(j\times B)\right]\cdot u^\varepsilon+(j\times B)\cdot\left(u^\varepsilon-u\right)dxdt \\ &\le 2\left\|(j\times B)^\varepsilon-(j\times B)\right\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)}\|u^\varepsilon\|_{L^4(0,T;L^4(\Omega))} \\ &\quad+2\left\|j\times B\right\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)}\|u^\varepsilon-u\|_{L^4(0,T;L^4(\Omega))} \\ &\quad\to 0,\ as\ \varepsilon\to 0,\end{aligned} \tag{17}$$

where used the fact that

$$\left\|j\times B\right\|_{L^{\frac43}\left(0,T;L^{\frac43}(\Omega)\right)} \le \left\|j\right\|_{L^2(0,T;L^2(\Omega))}\|B\|_{L^4(0,T;L^4(\Omega))}\le C.$$

By using the same trick, we find that

$$I_4 = -2\int_0^T\int_\Omega j^\varepsilon\cdot E^\varepsilon dxdt \to -2\int_0^T\int_\Omega j\cdot Edxdt,\ as\ \varepsilon\to 0. \tag{18}$$

After integration by parts, the term $I_3$ can be dominated as

$$\begin{aligned}I_3 &=2\int_0^T\int_\Omega (\nabla\times B)^\varepsilon\cdot E^\varepsilon dxdt \\ &=2\int_0^T\int_\Omega \left(\epsilon_{ijk}\partial_j B_k\right)^\varepsilon\cdot E_i^\varepsilon dxdt \\ &=-2\int_0^T\int_\Omega \epsilon_{ijk}B_k^\varepsilon\cdot\partial_j E_i^\varepsilon dxdt \\ &=2\int_0^T\int_\Omega \epsilon_{kji}B_k^\varepsilon\cdot\partial_j E_i^\varepsilon dxdt \\ &=2\int_0^T\int_\Omega B^\varepsilon\cdot(\nabla\times E)^\varepsilon dxdt,\end{aligned} \tag{19}$$

So, it follows that

$$I_3 + I_5 = 0. \tag{20}$$

Letting $\varepsilon$ go to zero in (10), and using (15)-(20), what we have proved is that in the limit

$$\begin{aligned}\int_\Omega \left(|u|^2 + |E|^2 + |B|^2\right) dx + 2\int_0^T \int_\Omega |\nabla u|^2 dxdt \\ = \int_\Omega \left(|u_0|^2 + |E_0|^2 + |B_0|^2\right) dxdt + 2\int_0^T \int_\Omega (j \times B) \cdot u dxdt - 2\int_0^T \int_\Omega j \cdot E dxdt.\end{aligned} \tag{21}$$

Thus, it holds that

$$\int_\Omega \left(|u|^2 + |E|^2 + |B|^2\right) dx + 2\int_0^T \int_\Omega |\nabla u|^2 + |j|^2 dxdt = \int_\Omega \left(|u_0|^2 + |E_0|^2 + |B_0|^2\right) dxdt, \tag{22}$$

where we have used the fact that

$$2\int_0^T \int_\Omega (j \times B) \cdot u dxdt - 2\int_0^T \int_\Omega j \cdot E dxdt = -2\int_\Omega (u \times B) \cdot j \mathrm{d}x - 2\int_\Omega E \cdot j \mathrm{d}x$$

and

$$j = E + u \times B.$$

The proof of Theorem 1 is completed.

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