INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus Mathématique

Pratyoosh Kumar and Sumit Kumar Rano

Dynamics of semigroups generated by analytic functions of the Laplacian on Homogeneous Trees

Volume 361 (2023), p. 1-13

https://doi.org/10.5802/crmath.382

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 1-13
https://doi.org/10.5802/crmath.382

Harmonic analysis, Dynamical systems / *Analyse harmonique, Systèmes dynamiques*

Dynamics of semigroups generated by analytic functions of the Laplacian on Homogeneous Trees

Pratyoosh Kumar [a] **and Sumit Kumar Rano** [*, a]

[a] Department of Mathematics, Indian Institute of Technology Guwahati, Guwahati, 781039, India

E-mails: pratyoosh@iitg.ac.in (P. Kumar), sumitrano1992@gmail.com (S. K. Rano)

Abstract. Let Ψ be a non-constant complex-valued analytic function defined on a connected, open set containing the L^p-spectrum of the Laplacian $\mathscr{L}$ on a homogeneous tree. In this paper we give a necessary and sufficient condition for the semigroup $T(t) = e^{t\Psi(\mathscr{L})}$ to be chaotic on L^p-spaces. We also study the chaotic dynamics of the semigroup $T(t) = e^{t(a\mathscr{L}+b)}$ separately and obtain a sharp range of b for which $T(t)$ is chaotic on L^p-spaces. It includes some of the important semigroups such as the heat semigroup and the Schrödinger semigroup.

2020 Mathematics Subject Classification. 39A12, 43A85, 47D06, 47A16, 39A33.

Funding. Sumit Kumar Rano is supported by Institute fellowships of IIT Guwahati.

Manuscript received 2 February 2022, revised 18 May 2022, accepted 20 May 2022.

1. Introduction

A homogeneous tree $\mathfrak{X}$ of degree $q+1$ is a connected graph with no circuits such that every vertex is connected to $q+1$ other vertices. Henceforth we assume $q \geq 2$. We denote by $d(x,y)$ the natural distance between any two vertices x and y, which is the number of edges joining them. The canonical Laplacian $\mathscr{L}$ on $\mathfrak{X}$ is defined by

$$\mathscr{L} f(x) = f(x) - \frac{1}{q+1} \sum_{y : d(x,y)=1} f\left(y\right).$$

Unlike many other spaces, $\mathscr{L}$ defines a bounded linear operator on the Lebesgue spaces $L^p(\mathfrak{X})$ for every $p \in [1,\infty]$. Let $\sigma_p(\mathscr{L})$ denote the L^p-spectrum of the Laplacian $\mathscr{L}$. Let Ψ be a non-constant complex holomorphic function defined on a connected open set containing $\sigma_p(\mathscr{L})$. Then by the usual Riesz functional calculus (see [14, Definition 10.26 at page 261]), it follows that the semigroup

[*] Corresponding author.

ISSN (electronic) : 1778-3569 https://comptes-rendus.academie-sciences.fr/mathematique/

$$T(t) = e^{t\Psi(\mathscr{L})}, \quad \text{where } t \geq 0, \tag{1}$$

consists of bounded linear operators on $L^p(\mathfrak{X})$ for every $p \in [1,\infty]$. Moreover, the semigroup $T(t)$ is *strongly continuous*, that is, $\|T(t)f - f\|_{L^p(\mathfrak{X})} \to 0$ as $t \to 0$, for every $f \in L^p(\mathfrak{X})$.

In this paper we study the chaotic dynamics of the semigroup $T(t) = e^{t\Psi(\mathscr{L})}$ on $L^p(\mathfrak{X})$. Before stating our main results, we recall some basic definitions. For details, we refer to [5]. Let X be a Banach space and $\mathscr{B}(X)$ be the space of all bounded linear operators from X into itself. A *semigroup* on X is a map $T : [0,\infty) \to \mathscr{B}(X)$ such that $T(0) = I$ and $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$. Furthermore, $T(t)$ is said to be *hypercyclic* if there exists $w \in X$ such that $\{T(t)w : t \geq 0\}$ is dense in X. A point $w \in X$ is said to be *periodic* for $T(t)$ if there exists $t > 0$ such that $T(t)w = w$. The set of all periodic points will henceforth be denoted by X_{per}. The semigroup $T(t)$ is said to be *chaotic* if it is hypercyclic and its set of periodic points is dense in X.

It follows from the definition of hypercyclicity that the existence of a hypercyclic semigroup on a Banach space implies that it is separable. Therefore, it is obvious that $T(t)$ cannot be hypercyclic, hence not chaotic on $L^\infty(\mathfrak{X})$. For other values of p, we shall prove the following results.

Theorem A. *Let $2 < p < \infty$ and $T(t) = e^{t\Psi(\mathscr{L})}$ be a semigroup on $L^p(\mathfrak{X})$ as defined in* (1). *Then the following statements are equivalent.*

(1) *$T(t)$ is chaotic on $L^p(\mathfrak{X})$.*
(2) *$T(t)$ has a non-trivial periodic point, that is $L^p(\mathfrak{X})_{per} \neq \{0\}$.*
(3) *The set of periodic points of $T(t)$ is dense in $L^p(\mathfrak{X})$, that is $\overline{L^p(\mathfrak{X})}_{per} = L^p(\mathfrak{X})$.*

Theorem B. *Let $1 \leq p \leq 2$ and $T(t)$ be as in Theorem A. Then we have the following.*

(1) *$T(t)$ has no non-trivial periodic point in $L^p(\mathfrak{X})$.*
(2) *$T(t)$ is not hypercyclic on $L^p(\mathfrak{X})$.*

In particular $T(t)$ is not chaotic on $L^p(\mathfrak{X})$.

For $2 < p < \infty$, let $\delta_p = 1/p - 1/2$, and $\gamma(z)$ be as in (3). Define

$$\Phi_p(a) = \left(1 - \gamma\left(i\delta_p\right)\right) \cdot \left((\Re a)^2 + \tanh^2\left(\delta_p \log q\right)(\Im a)^2\right)^{1/2}. \tag{2}$$

Then we have the following result regarding the chaoticity of the affine semigroup. As a consequence of this result, we obtain the sharp range of perturbations for which the heat and the Schrödinger semigroups are chaotic.

Theorem C. *Suppose that $T(t) = e^{t(a\mathscr{L}+b)}$, $t \geq 0$ where a is a non-zero complex number and b is real. Let $2 < p < \infty$. Then the following statements are equivalent.*

(1) *$T(t)$ is chaotic on $L^p(\mathfrak{X})$.*
(2) *$T(t)$ is hypercyclic.*
(3) *a and b satisfy $-\Re a - \Phi_p(a) < b < -\Re a + \Phi_p(a)$, where $\Phi_p(a)$ is given by* (2).

Remark 1. If we assume b to be a complex number, then there won't be any significant change in the proof of the Theorem C because $|e^{it\Im b}| = 1$ for all $t \geq 0$. However, there will be a minor modification in the statement of Theorem C (3), where b will be replaced by $\Re b$.

Our inspiration to study this circle of ideas originated from the papers [9, 13]. In [9], Ji and Weber initiated the study of chaotic dynamics of the heat semigroup on Riemannian symmetric spaces of non-compact type. They studied the chaotic behaviour of certain shifts of the heat semigroup corresponding to the Laplace–Beltrami operator on the space of all radial L^p-functions (see [9, Theorem 3.1]). Later, Pramanik and Sarkar extended these results and gave a complete characterization of the chaotic behaviour of the heat semigroup on the whole of L^p-spaces and its related subspaces (see [13, Theorems 1.2-1.4]). In both articles, the results show a significant difference among the settings $1 \le p \le 2$ and $2 < p < \infty$. This difference can also be seen in Theorems A and B of this paper. Similar results concerning the chaotic behaviour of the L^p-heat semigroup and the weighted L^p-Dunkl heat semigroup are also known on harmonic NA groups and Euclidean spaces, respectively (see [1, Theorem 1.3], and [15, Theorem A]). Another motivation to study this class of semigroups is the work of deLaubenfels and Emamirad [11]. In [11], the authors studied the dynamical behaviour of a class of operators generated by non-constant analytic functions of the shift operator on weighted $L^p(\mathbb{N})$ spaces. Apart from the papers cited above, we also mention in particular, a recent paper by Cohen et al. [2] which deals with similar topics about hypercyclic operators on spaces of functions on homogeneous trees. These functions, however, are spanned by polyharmonic functions of $\mathscr{L}$, typically not in $L^p(\mathfrak{X})$.

We end this section by providing a quick outline of the contents of this article. We have collected all relevant notation, definitions, and facts about the homogeneous trees in Section 2. The proofs of Theorem A and Theorem B are given in Section 3. In Section 4, we prove Theorem C and discuss some of its important consequences.

2. Background materials on homogeneous trees

2.1. *General Notations*

The letters $\mathbb{R}$ and $\mathbb{C}$ will denote the fields of real and complex numbers, respectively. For $z \in \mathbb{C}$ we use the notation $\Re z$ and $\Im z$ for real and imaginary parts of z, respectively. We will use the standard practice of using the letter C for constant, whose value may change from one line to another line. For every Lebesgue exponent $p \in (1,\infty)$, we write p' to denote the conjugate exponent $p/(p-1)$. Further, we define $p' = \infty$ when $p = 1$ and vice-versa. For $p \in (1,\infty)$, let

$$\delta_p = \frac{1}{p} - \frac{1}{2} \quad \text{and} \quad S_p = \left\{z \in \mathbb{C} : |\Im z| \le \left|\delta_p\right|\right\}.$$

We assume $\delta_1 = -\delta_\infty = 1/2$ so that $S_1 = \{z \in \mathbb{C} : |\Im z| \le 1/2\}$. We shall henceforth write S_p° and ∂S_p to denote the usual interior and the boundary of S_p, respectively. The notation $\|T\|_{p\to p}$ will denote the operator norm of a bounded linear operator T defined on the Lebesgue space $L^p(\mathfrak{X})$. Moreover, we shall write $\sigma_p(T)$, $P\sigma_p(T)$ to respectively denote the set of spectrum and point spectrum of T in $L^p(\mathfrak{X})$.

2.2. *Basics*

Here we review some general facts about the homogeneous trees, most of which are already available in [6, 7]. However, it is worth mentioning that we use a different parametrization from that of the above-mentioned references while introducing the terms such as the Poisson transform, the elementary spherical function, and many more, which appear later in this article.

Let o be a fixed reference point in $\mathfrak{X}$, and Ω be the boundary of $\mathfrak{X}$, that is, the set of all infinite geodesic rays starting at o. For $z \in \mathbb{C}$ and a suitable function F defined on Ω, its *Poisson transformation* $\mathscr{P}_z F$ (see [7, Chapter 4, formula (1) at page 53]) is given by

$$\mathscr{P}_z F(x) = \int_\Omega p^{1/2+iz}(x,\omega) F(\omega)\, d\nu(\omega),$$

where $p(x,\omega)$, defined on $\mathfrak{X} \times \Omega$, is the Poisson kernel. For details regarding the Poisson kernel, we refer to [6, page 285] and [7, Chapter 3, Section 2]. It follows from the definition that $\mathscr{P}_z = \mathscr{P}_{z+\tau}$, where $\tau = 2\pi/\log q$. It is also a well-known fact that $\mathscr{L}\mathscr{P}_z F(x) = \gamma(z)\mathscr{P}_z F(x)$ where γ is an analytic function defined by the formula

$$\gamma(z) = 1 - \frac{q^{1/2+iz} + q^{1/2-iz}}{q+1}. \tag{3}$$

For $z \in \mathbb{C}$, the *elementary spherical function* ϕ_z is defined as the Poisson transform of the constant function $\mathbf{1}$. A function f on $\mathfrak{X}$ is said to be radial if $f(x) = f(y)$ whenever $d(o,x) = d(o,y)$. It is known that ϕ_z is a radial eigenfunction of $\mathscr{L}$ with eigenvalue $\gamma(z)$ and every radial eigenfunction of $\mathscr{L}$ with eigenvalue $\gamma(z)$ is a constant multiple of ϕ_z (see [6, Theorem 1]). Below we enlist some properties of ϕ_z which can be easily derived from its explicit formula given in [6, Theorem 2 (ii)-(iii)] and [7, Chapter 3, Theorem 2.2].

Lemma 2. *For $z \in \mathbb{C}$, let $\phi_z = \mathscr{P}_z\mathbf{1}$. Then the following assertions hold:*

(i) *For every $x \in \mathfrak{X}$, the map $z \mapsto \phi_z(x)$ is an entire function.*
(ii) *$\phi_z \in L^\infty(\mathfrak{X})$ if and only if $z \in S_1$.*
(iii) *For $1 < p < 2$, $\phi_z \in L^{p'}(\mathfrak{X})$ if and only if $z \in S_p^\circ$.*
(iv) *For $1 \le p \le 2$, $\phi_z \notin L^p(\mathfrak{X})$ for any $z \in S_p$.*

For the proof of Lemma 2 (ii) and (iii), we refer to [6, Corollary (i)-(ii) at page 288] and [7, Chapter 3, Corollary 2.3]. We also need the following estimates of $\mathscr{P}_zF$, which can be considered as a generalisation of the size estimates of ϕ_z given above. These estimates follow from the first two inequalities in [10, page 735], the analytic interpolation [8, Theorem 1.3.7] and the fact that $\|\mathscr{P}_zF\|_{L^{p'}(\mathfrak{X})} \le C\|\mathscr{P}_zF\|_{L^{p',1}(\mathfrak{X})}$ (see [8, Proposition 1.4.10]).

Proposition 3. *Let $1 < p < 2$ and $z \in \mathbb{C}$ be such that $\Im z = \delta_{r'}$ where $p < r < p'$. Then for all $F \in L^{r'}(\Omega)$,*

$$\|\mathscr{P}_zF\|_{L^{p'}(\mathfrak{X})} \le C\|F\|_{L^{r'}(\Omega)}. \tag{4}$$

The *Helgason–Fourier transform* $\widetilde{f}$ of a finitely supported function f is a function on $\mathbb{C} \times \Omega$ defined by the formula

$$\widetilde{f}(z,\omega) = \sum_{x \in \mathfrak{X}} f(x) p^{1/2+iz}(x,\omega).$$

For a finitely supported function f on $\mathfrak{X}$ and a continuous function F on Ω, we have

$$\int_\Omega \widetilde{f}(z,\omega)F(\omega)\,d\nu(\omega) = \sum_{x \in \mathfrak{X}} f(x)\left(\int_\Omega p^{1/2+iz}(x,\omega)F(\omega)\,d\nu(\omega)\right) = \sum_{x \in \mathfrak{X}} f(x)\mathscr{P}_zF(x). \tag{5}$$

The estimates of the Poisson transform from Proposition 3 and the duality relation (5) together gives us the following 'restriction type' inequality for the Helgason–Fourier transform (see [10, Theorem 4.2] for a more general version):

Theorem 4. *Let $1 < p < 2$ and $f \in L^p(\mathfrak{X})$. For $p < r < p'$ and $z \in \mathbb{C}$ with $\Im z = \delta_{r'}$, there exists a constant $C_{p,r} > 0$ such that*

$$\left\|\widetilde{f}(z,\cdot)\right\|_{L^r(\Omega)} \le C_{p,r}\left\|f\right\|_{L^p(\mathfrak{X})}.$$

We now recall some important facts related to the L^p-point spectrum of $\mathscr{L}$. It follows from Lemma 2 above that for $p \in (2,\infty)$ and $z \in S_p^\circ$, ϕ_z is an L^p -eigenfunction of $\mathscr{L}$ and hence $\gamma(S_p^\circ)$ lies inside the set $P\sigma_p(\mathscr{L})$ of L^p-point spectrum of $\mathscr{L}$. In fact, $\gamma(S_p^\circ)$ is exactly the set $P\sigma_p(\mathscr{L})$. To prove this, let us assume that there exists a non-zero function u in $L^p(\mathfrak{X})$ such that $\mathscr{L}u = \gamma(z)u$ for some $z \notin S_p^\circ$. Suppose that $u(x_o) \neq 0$ for some $x_o \in \mathfrak{X}$. Then $f(x) = \int_K u(x_o kx)\,dk$ is a radial eigenfunction of $\mathscr{L}$ with eigenvalue $\gamma(z)$, hence f is a constant multiple ϕ_z. Since $u \in L^p(\mathfrak{X})$, hence so is f, which is clearly not possible as $\phi_z \notin L^p(\mathfrak{X})$ for $z \notin S_p^\circ$ (by Lemma 2 (iii)). In fact, by using Lemma 2 and a similar technique as above, one can completely summarize the L^p-point spectrum of $\mathscr{L}$ as follows:

Proposition 5. *Regarding the L^p-point spectrum of $\mathscr{L}$, we have the following results.*

(i) *For $1 \le p \le 2$, the point spectrum of $\mathscr{L}$ on $L^p(\mathfrak{X})$ is empty.*
(ii) *For $2 < p < \infty$, the point spectrum of $\mathscr{L}$ on $L^p(\mathfrak{X})$ is the set $\gamma(S_p^\circ)$.*

The complete description of the L^p-spectrum of $\mathscr{L}$ is given in the following proposition. For details we refer to [7, Chapter 3, Proposition 3.1 and Theorem 3.3].

Proposition 6. *For every $p \in [1,\infty]$, the L^p-spectrum $\sigma_p(\mathscr{L})$ of $\mathscr{L}$ is the image of S_p under the map γ, which is precisely the set of all w in $\mathbb{C}$ which satisfies*

$$\left[\frac{1-\Re w}{b\cosh\left(\delta_p \log q\right)}\right]^2 + \left[\frac{\Im w}{b\sinh\left(\delta_p \log q\right)}\right]^2 \leq 1, \quad \textit{where } b = \frac{2\sqrt{q}}{q+1}.$$

In particular, $\sigma_2(\mathscr{L})$ degenerates into the line segment $[1-b, 1+b]$.

The elliptic region in Figure 1 represents the L^p-spectrum of $\mathscr{L}$, when $1 < p < \infty$. Moreover, for $2 < p < \infty$, the shaded open elliptic region in this figure also represents the L^p-point spectrum of $\mathscr{L}$, that is, $\gamma(S_p^\circ)$.

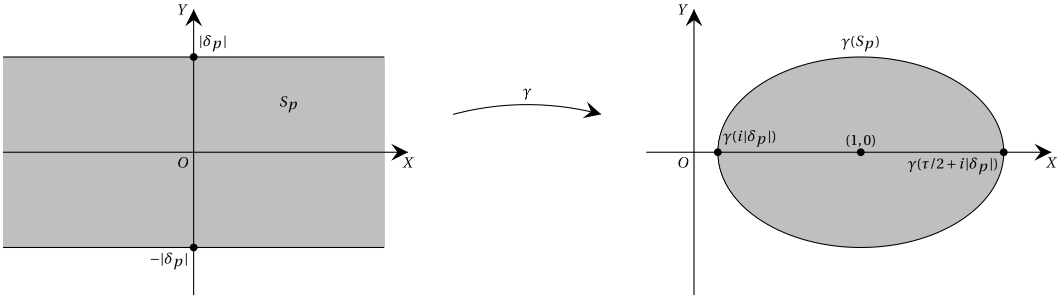

Figure 1. L^p-spectrum of $\mathscr{L}$, $1 < p < \infty$.

3. Proof of Theorem A and Theorem B

To prove Theorem A and Theorem B, we collect some key results which will be used very frequently. We begin with the spectral mapping theorem (see [14, Theorem 10.28, Theorem 10.33]).

Theorem 7. *Suppose T is a bounded linear operator on $L^p(\mathfrak{X})$ and g is a non-constant complex holomorphic function defined on a connected open set containing $\sigma_p(T)$. Then we have the following.*

(a) $\sigma_p(g(T)) = g(\sigma_p(T))$.
(b) $P\sigma_p(g(T)) = g(P\sigma_p(T))$.

In comparison to [14, Theorem 10.28, Theorem 10.33], there is a slight abuse of notation in the statement of Theorem 7, namely, the operator-valued version of g, which is denoted by $g(T)$ in Theorem 7, corresponds to $\widetilde{g}(T)$ in [14, Theorem 10.28, Theorem 10.33]. We have skipped over this notational inconvenience to avoid confusion with the Helgason–Fourier transform, which we have also denoted by $\widetilde{f}$.

For $1 \leq p < \infty$ define

$$X_0 = \left\{f \in L^p(\mathfrak{X}) : \left\|T(t)f\right\|_{L^p(\mathfrak{X})} \to 0 \text{ as } t \to \infty\right\}, \tag{6}$$

$$X_\infty = \Big\{f \in L^p(\mathfrak{X}) : \forall\, \epsilon > 0 \;\exists\; g \in L^p(\mathfrak{X}) \text{ and } t_0 > 0 \text{ s.t. } \left\|g\right\|_{L^p(\mathfrak{X})} < \epsilon, \left\|T(t_0)g - f\right\|_{L^p(\mathfrak{X})} < \epsilon\Big\}. \tag{7}$$

The following sufficient condition for hypercyclicity which was proved by Desch, Schappacher and Webb is useful in the sequel.

Proposition 8 ([4, Theorem 2.3]). *Let $T(t)$, $t \geq 0$ be a strongly continuous semigroup on $L^p(\mathfrak{X})$ for $1 \leq p < \infty$. If both the sets X_0 and X_∞ are dense in $L^p(\mathfrak{X})$, then $T(t)$ is hypercyclic.*

We also need the following lemma about the existence of the Helgason–Fourier transform of L^p-functions.

Lemma 9. *Suppose $f \in L^p(\mathfrak{X})$ for $1 < p < 2$. Then there exists a subset Ω_p of Ω of full Haar measure in K, such that $\widetilde{f}(z,\omega)$ exists for all $\omega \in \Omega_p$ and $z \in S_p^\circ$. Moreover for every $\omega \in \Omega_p$, the map $z \mapsto \widetilde{f}(z,\omega)$ is analytic on S_p°.*

Proof. Fix $1 < p < 2$ and suppose that $f \in L^p(\mathfrak{X})$. Consider a sequence $\{r_n\}$ of real numbers satisfying the following conditions:

(a) For all $n \in \mathbb{N}$,

$$p < r_n < 2 \quad \text{and hence} \quad 2 < r_n' < p'. \tag{8}$$

(b) The following convergences hold.

$$\frac{1}{r_n} \uparrow \frac{1}{p} \quad \text{and hence} \quad \frac{1}{r_n'} \downarrow \frac{1}{p'} \quad \text{as } n \to \infty. \tag{9}$$

By using (8) and Theorem 4 it follows that for each n, there exist subsets Ω_{r_n} and $\Omega_{r_n'}$ of Ω, of full Haar measure in K such that $\widetilde{|f|}(i\delta_{r_n},\omega)$ and $\widetilde{|f|}(i\delta_{r_n'},\omega)$ exist for all $\omega \in \Omega_{r_n}$ and $\omega \in \Omega_{r_n'}$ respectively. Now define the set

$$\Omega_p = \bigcap_{k=r_n,r_n'} \Omega_k.$$

Then $\widetilde{|f|}(\pm i\delta_{r_n},\omega)$ exists for every $n \in \mathbb{N}$ and $\omega \in \Omega_p$. Furthermore, Ω_p is a subset of Ω with full Haar measure in K. We claim that $\widetilde{f}(z,\omega)$ exists for all $\omega \in \Omega_p$ and $z \in S_p^\circ$. To prove this, let us first assume that $z \in S_p^\circ$. Then $z = \alpha + i\delta_{r'}$ for some r satisfying $p < r < p'$. By using this fact and (9), we can find a natural number k such that

$$\frac{1}{p'} < \frac{1}{r_k'} < \frac{1}{r} < \frac{1}{r_k} < \frac{1}{p}. \tag{10}$$

Hence for each $\omega \in \Omega_p$,

$$\left|\widetilde{f}(z,\omega)\right| = \left|\sum_{x\in\mathfrak{X}} p^{1/r+i\alpha}(x,\omega) f(x)\right| \leq \sum_{x\in E_{\omega,1}} p^{1/r}(x,\omega)\left|f(x)\right| + \sum_{x\in E_{\omega,2}} p^{1/r}(x,\omega)\left|f(x)\right|,$$

where the sets $E_{\omega,1}$ and $E_{\omega,2}$ are defined as follows:

$$E_{\omega,1} = \left\{x \in \mathfrak{X} : p(x,\omega) < 1\right\} \quad \text{and} \quad E_{\omega,2} = \left\{x \in \mathfrak{X} : p(x,\omega) \geq 1\right\}.$$

Using (10) in the last inequality, we finally get

$$\begin{aligned}\left|\widetilde{f}(z,\omega)\right| &\leq \sum_{x\in E_{\omega,1}} p^{1/r_k'}(x,\omega)\left|f(x)\right| + \sum_{x\in E_{\omega,2}} p^{1/r_k}(x,\omega)\left|f(x)\right| \\ &\leq \sum_{x\in\mathfrak{X}} p^{1/r_k'}(x,\omega)\left|f(x)\right| + \sum_{x\in\mathfrak{X}} p^{1/r_k}(x,\omega)\left|f(x)\right| = \widetilde{|f|}\left(i\delta_{r_k},\omega\right) + \widetilde{|f|}\left(i\delta_{r_k'},\omega\right) < \infty.\end{aligned}$$

This completes the proof of the existence while the analyticity of the Helgason–Fourier transform follows from the standard use of Fubini's theorem, Morera's theorem and the fact that for each x and ω, the map $z \mapsto p^{1/2+iz}(x,\omega)$ is analytic on S_p°. $\square$

3.1. *Proof of Theorem A*

Proof. Fix $p \in (2,\infty)$. It follows from the definition that (1) implies (2) and (3). To complete the proof we only need to show that (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (1).
We first prove (2) $\Longrightarrow$ (3): For a clear understanding, we have divided this proof into the following steps.

Step 1. In this step, we prove that $P\sigma_p(\Psi(\mathscr{L})) \cap i\mathbb{R}$ is an infinite set. Condition (2) implies that there exists a nonzero function $h \in L^p(\mathfrak{X})$ such that $T(t_o)h = h$ for some $t_o > 0$, that is $1 \in P\sigma_p(e^{t_o\Psi(\mathscr{L})})$. On the other hand, Proposition 5 (ii) and Theorem 7 (b) together imply that $P\sigma_p(e^{t_0\Psi(\mathscr{L})}) = e^{t_0\Psi(\gamma(S_p^\circ))}$. Therefore there exists $z_0 \in S_p^\circ$ such that $\Psi(\gamma(z_0)) = 2n\pi i/t_o$ for some $n \in \mathbb{Z}$.

Let $\Gamma : S_p^\circ \to \mathbb{C}$ be defined by

$$\Gamma(z) = \left(\Psi \circ \gamma\right)(z) = \Psi\left(\gamma(z)\right). \tag{11}$$

It follows from the assumption on Ψ that Γ is a non-constant holomorphic function on S_p°. Since $\Gamma(z_0) = 2n\pi i/t_o$ for some $z_0 \in S_p^\circ$ and $\Gamma(S_p^\circ) = P\sigma_p(\Psi(\mathscr{L}))$, by the open mapping theorem it follows that $P\sigma_p(\Psi(\mathscr{L}))$ must contains some open ball centered at $2n\pi i/t_o$. Consequently $P\sigma_p(\Psi(\mathscr{L})) \cap i\mathbb{R}$ is an infinite set. In particular the set $V_1 = \{z \in S_p^\circ : \Gamma(z) \in i\mathbb{Q}\}$ is an infinite set which contains a cluster point in S_p°.

Step 2. Let $z \in V_1$. Since $V_1 \subseteq S_p^\circ$, $z = \alpha + i\delta_{r'}$ for some $r \in (p', p)$ and $\alpha \in \mathbb{R}$. Hence the set

$$\mathscr{V}_1 = \bigcup_{z \in V_1} \left\{\mathscr{P}_z F : F \in L^{r'}(\Omega) \quad \text{whenever} \quad \Im z = \delta_{r'} \quad \text{with} \quad p' < r < p\right\}$$

is well defined. It follows from inequality (4) that $\mathscr{V}_1 \subseteq L^p(\mathfrak{X})$. We now show that $\text{span}(\mathscr{V}_1) \subseteq L^p(\mathfrak{X})_{per}$. Since $\Psi(\mathscr{L})\mathscr{P}_z F = \Gamma(z)\mathscr{P}_z F$ for every $z \in S_p^\circ$, $T(t)\mathscr{P}_z F = e^{t\Gamma(z)}\mathscr{P}_z F$. Now if $g \in \text{span}(\mathscr{V}_1)$ then

$$g = \sum_{j=1}^{k} \beta_j \mathscr{P}_{z_j} F_j, \quad \text{where} \quad z_j \in V_1 \quad \text{and} \quad \beta_j \in \mathbb{C}, 1 \le j \le k.$$

For every $j \in \{1,2,\ldots,k\}$, $z_j \in V_1$ and hence, $\Gamma(z_j) = ip_j/q_j$ such that $p_j, q_j \in \mathbb{Z}$ with $q_j \neq 0$. If we choose $s = 2\pi q_1 \cdots q_k$, then $T(s)g = g$. Hence $\text{span}(\mathscr{V}_1) \subseteq L^p(\mathfrak{X})_{per}$.

Step 3. To prove that $\overline{L^p(\mathfrak{X})_{per}} = L^p(\mathfrak{X})$, it is enough to prove that $\text{span}(\mathscr{V}_1)$ is dense in $L^p(\mathfrak{X})$. Let $f \in L^{p'}(\mathfrak{X})$ annihilate $\mathscr{V}_1$, that is, $\sum_{x \in \mathfrak{X}} f(x)\mathscr{P}_z F(x) = 0$ for all $\mathscr{P}_z F \in \mathscr{V}_1$. By the duality relation (5) we have,

$$\int_\Omega \widetilde{f}(z,\omega)F(\omega)\,d\nu(\omega) = \sum_{x \in \mathfrak{X}} f(x)\mathscr{P}_z F(x) = 0, \quad \text{for all } \mathscr{P}_z F \in \mathscr{V}_1.$$

Fix $z \in V_1$ and suppose that $z = \alpha + i\delta_{r'}$ for some $r \in (p', p)$. Then for every $F \in L^{r'}(\mathfrak{X})$, we have

$$\int_\Omega \widetilde{f}(\alpha + i\delta_{r'}, \omega)\, F(\omega)\,d\nu(\omega) = 0.$$

Since $F \in L^{r'}(\mathfrak{X})$ is arbitrary, from Theorem 4 and the above expression, we have $\widetilde{f}(\alpha + i\delta_{r'}, \omega) = 0$ for almost every $\omega \in \Omega$. Thus for every $z \in V_1$, $\widetilde{f}(z,\omega) = 0$ for almost every $\omega \in \Omega$. By Lemma 9, for almost every $\omega \in \Omega$, the function $z \mapsto \widetilde{f}(z,\omega)$ is analytic on S_p°. So we conclude that for almost every ω, the set of zeros of $\widetilde{f}$ has a cluster point in S_p°, and hence $\widetilde{f}(z,\omega) = 0$ for every $z \in S_p^\circ$ and for almost every ω. Since $f \in L^{p'}(\mathfrak{X}) \subseteq L^2(\mathfrak{X})$ (as $\mathfrak{X}$ is a discrete space) whenever $2 < p < \infty$, by Plancherel Theorem [7, Chapter 3, Theorem 4.1] we conclude that $f \equiv 0$. This proves that $\text{span}(\mathscr{V}_1)$ is dense in $L^p(\mathfrak{X})$, hence (2) $\Longrightarrow$ (3).

Now we will prove(3) $\Longrightarrow$ (1): Since the density of the periodic points is already assumed, to prove our assertion it is enough to show that $T(t)$ is hypercyclic. In view of Proposition 8, we only need to show that the sets X_0 and X_∞ (given by (6) and (7)) are dense in $L^p(\mathfrak{X})$. We define the sets

$$V_2 = \left\{z \in S_p^\circ : \Re(\Gamma(z)) < 0\right\} \quad \text{and} \quad V_3 = \left\{z \in S_p^\circ : \Re(\Gamma(z)) > 0\right\}.$$

By repeating the arguments of Step 1 and Step 2 in the previous proof, one may prove that V_2 and V_3 are non-empty open sets and both contain a cluster point in S_p°. Corresponding to the sets V_i, we define (for $i = 2,3$)

$$\mathcal{V}_i = \bigcup_{z \in V_i} \left\{\mathscr{P}_z F : F \in L^{r'}(\Omega) \quad \text{whenever } \Im z = \delta_{r'} \text{ with } p' < r < p\right\}.$$

Adopting a similar approach as in Step 3, we may easily show that both span$(\mathcal{V}_2)$ and span$(\mathcal{V}_3)$ are dense in $L^p(\mathfrak{X})$. The proof will be complete once we show that span$(\mathcal{V}_2) \subseteq X_0$ and span$(\mathcal{V}_3) \subseteq X_\infty$. For every $z \in V_2$,

$$\lim_{t\to\infty} \|T(t)\mathscr{P}_z F\|_{L^p(\mathfrak{X})} = \lim_{t\to\infty} e^{t\Re(\Gamma(z))} \|\mathscr{P}_z F\|_{L^p(\mathfrak{X})} = 0.$$

This shows that $\mathcal{V}_2$ is a subset of X_0, hence so is span$(\mathcal{V}_2)$.

Next we prove that span$(\mathcal{V}_3) \subseteq X_\infty$. Let $g \in$ span$(\mathcal{V}_3)$ be of the form

$$g = \sum_{j=1}^{k} \alpha_j \mathscr{P}_{z_j} F_j, \quad \text{where } z_j \in V_3 \text{ and } \alpha_j \in \mathbb{C}, 1 \le j \le k.$$

If we choose

$$g_t = \sum_{j=1}^{k} e^{-t\Gamma(z_j)} \alpha_j \mathscr{P}_{z_j} F_j,$$

then $T(t)g_t = g$ for all $t \ge 0$. Since $\Re(\Gamma(z_j)) > 0$ for each j, the limit $\|g_t\|_{L^p(\mathfrak{X})} \to 0$ as $t \to \infty$. Hence it follows from definition (7) that span$(\mathcal{V}_3) \subseteq X_\infty$. This completes the proof of Theorem A. □

3.2. *Proof of Theorem B*

Proof.

Part 1. Let $p \in [1,2]$. We know from Proposition 5 (i) that $P\sigma_p(\mathscr{L}) = \emptyset$. Hence by Theorem 7 (b) it follows that $P\sigma_p(T(t)) = \emptyset$ for all $t > 0$. Seeking a contradiction, suppose that $T(t)$ has a non-trivial periodic point in $L^p(\mathfrak{X})$. Then there exist $t_0 > 0$ and a non-zero $h \in L^p(\mathfrak{X})$ such that $T(t_0)h = h$, consequently $1 \in P\sigma_p(T(t_0))$, a contradiction to the fact that $P\sigma_p(T(t_0)) = \emptyset$. This shows that only the zero function is a periodic point, that is, $T(t)$ has no non-trivial periodic point in $L^p(\mathfrak{X})$ for any $p \in [1,2]$.

Part 2. Now we will show that $T(t)$ is not hypercyclic on $L^p(\mathfrak{X})$. Let us first assume that $p = 2$. We proof this assertion by contradiction. If possible, assume that there exists a non-zero $h \in L^2(\mathfrak{X})$ such that the set $\{T(t)h : t \ge 0\}$ is dense in $L^2(\mathfrak{X})$. Then for $g = 2h$, there exists a sequence of non-negative real numbers $\{t_n\}$ such that $T(t_n)h \to g$ in $L^2(\mathfrak{X})$ as $n \to \infty$.

If the sequence $\{t_n\}$ is bounded, then there exists a subsequence $\{t_{n_k}\}$ of $\{t_n\}$ and some number $t_0 \ge 0$ such that $t_{n_k} \to t_0$ as $k \to \infty$. Using the strong continuity of $T(t)$, it follows that $T(t_{n_k})h \to T(t_0)h$ as $k \to \infty$. Hence $T(t_0)h = g = 2h$, which is impossible as $P\sigma_2(T(t_0)) = \emptyset$.

If the sequence $\{t_n\}$ is unbounded, then without the loss of generality we may assume that $\{t_n\}$ is strictly increasing to ∞. By using the Plancherel Theorem [7, Chapter 3, Theorem 4.1], we have

$$4\|h\|^2_{L^2(\mathfrak{X})} = \lim_{n\to\infty} \|T(t_n)h\|^2_{L^2(\mathfrak{X})} = \lim_{n\to\infty} \int_{-\tau/2}^{\tau/2} \int_{\Omega} \exp\left\{2t_n \Re\left(\Psi\left(\gamma(s)\right)\right)\right\} \left|\widetilde{h}(s,\omega)\right|^2 d\nu(\omega) d\mu(s).$$

Let

$$S_1 = \left\{ s \in [-\tau/2, \tau/2) : \exp\left\{ 2\Re\left(\Psi\left(\gamma(s) \right) \right) \right\} \leq 1 \right\}$$

and

$$S_2 = \left\{ s \in [-\tau/2, \tau/2) : \exp\left\{ 2\Re\left(\Psi\left(\gamma(s) \right) \right) \right\} > 1 \right\}.$$

If the Plancherel measure of S_2 is zero then $4\|h\|^2_{L^2(\mathfrak{X})} \leq \|h\|^2_{L^2(\mathfrak{X})}$. This implies that $\|h\|_{L^2(\mathfrak{X})} = 0$, which is a contradiction to our assumption that $h \neq 0$. If S_2 has a positive Plancherel measure, then

$$4\|h\|^2_{L^2(\mathfrak{X})} \geq \lim_{n\to\infty} \int_{S_2} \int_{\Omega} \exp\left\{ 2t_n \Re\left(\Psi\left(\gamma(s) \right) \right) \right\} \left| \widetilde{h}(s,\omega) \right|^2 d\nu(\omega) d\mu(s).$$

By the Monotone Convergence Theorem, it follows that the above integral tends to infinity as n tends to infinity. This again leads to a contradiction. Hence we conclude that for any $h \in L^2(\mathfrak{X})$ and $h \neq 0$, the set $\{T(t)h : t \geq 0\}$ can never approximate $2h$. This proves our assertion for $p = 2$.

Now we assume $p \in [1,2)$. Since $\mathfrak{X}$ is a discrete space, hence $L^p(\mathfrak{X}) \subseteq L^2(\mathfrak{X})$ whenever $1 \leq p < 2$ and $\|h\|_{L^2(\mathfrak{X})} \leq \|h\|_{L^p(\mathfrak{X})}$ for every $h \in L^p(\mathfrak{X})$. This implies that for any $h \in L^p(\mathfrak{X})$ and $h \neq 0$, the set $\{T(t)h : t \geq 0\}$ can never approximate $2h$. This completes the proof of Theorem B. □

4. Proof of Theorem C and some of its consequences

Before going into the details, we recall some important facts related to the operators $e^{\xi\mathscr{L}}$, $\xi \in \mathbb{C}$. For a finitely supported function f defined on $\mathfrak{X}$, $e^{\xi\mathscr{L}} f = f * h_{-\xi}$, where

$$h_\xi(x) = e^{-\xi} \sum_{k=0}^{\infty} \frac{\xi^k}{k!} \mu_1^{(*k)}, \tag{12}$$

μ_1 is the normalised radial measure concentrated on the set $\{x \in \mathfrak{X} : d(o,x) = 1\}$, and $\mu_1^{(*k)}$ denotes the k^{th} convolution power of μ_1. For details, we refer to [16, Section 2, page 747]. Note that we use a different parametrization: the operators $e^{\xi\mathscr{L}}$ correspond to $\mathscr{H}_{-\xi}$ in [16, page 747], the measure μ_1 in (12) corresponds to ν in [16, page 746], our $\gamma(z)$ corresponds to $1-\gamma(z)$ in [16, page 744], and $\gamma(0)$ corresponds to b_2 in [16, page 745].

Lemma 10. *Let $e^{\xi\mathscr{L}}$ be the operator defined as above. Then for $p > 2$ the following hold.*

$$\exp\left\{ \Re\xi + \Phi_p(\xi) \right\} \leq \left\| e^{\xi\mathscr{L}} \right\|_{p\to p} \leq C \exp\left\{ \Re\xi + \Phi_p(\xi) \right\}. \tag{13}$$

Proof. For $\Re\xi \leq 0$, the result is already known (see [16, Theorem 1 (iii)]). Now we prove the result for $\Re\xi > 0$. It was proved in [16, Corollary 4] that for all non-zero ξ,

$$\left\| h_{-\xi} \right\|_{L^{p,1}(\mathfrak{X})} \leq C \frac{\exp\left\{ \gamma(0)\Re\xi \right\}}{|\xi|} \left(\sum_{d=0}^{\infty} d q^{d\delta_p} \left| h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d) \right| \right), \tag{14}$$

where

$$h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d)$$

denotes the heat kernel associated to the heat operator on $\mathbb{Z}$ (see [16, page 748]), and $\|h_{-\xi}\|_{L^{p,1}(\mathfrak{X})}$ denote the Lorentz $L^{p,1}$-norm of $h_{-\xi}$ (for details about Lorentz norm, we refer to [8, Chapter 1, Section 1.4]). By using [16, formula (2) at page 748], we also have

$$h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d) = e^{\xi(1-\gamma(0))} I_{|d|}\left(-\xi\left(1-\gamma(0) \right) \right),$$

where $I_n(\xi)$ denotes the modified Bessel function of order n. Since $\Re\{-\xi(1-\gamma(0))\} < 0$, the arguments given in [16] cannot be applied here. However, by using [12, formula (8.01) at page 379], we have

$$h^{\mathbb{Z}}_{-\xi(1-\gamma(0))}(d) = e^{i\pi d} e^{2\xi(1-\gamma(0))} h^{\mathbb{Z}}_{\xi(1-\gamma(0))}(d) \quad \forall\, d \in \mathbb{N} \cup \{0\}.$$

Now, by plugging into (14) the pointwise estimate of $h^{\mathbb{Z}}_{\xi(1-\gamma(0))}(d)$ given in [16, Lemma 6], the result follows by just imitating the proofs of [16, Theorem 8 (i)] and [16, Theorem 1 (iii)].

Now we will prove the lower bound of $\| e^{\xi\mathscr{L}} \|_{p\to p}$. For $p > 2$,

$$\left\| e^{\xi\mathscr{L}} \right\|_{p\to p} \geq \frac{\| e^{\xi\mathscr{L}} \phi_z \|_{L^p(\mathfrak{X})}}{\| \phi_z \|_{L^p(\mathfrak{X})}} = \exp\left\{\Re\left(\xi\gamma(z)\right)\right\} \quad \text{for all } z \in S_p^{\circ}.$$

By taking the supremum over all $z \in S_p^{\circ}$, we have $\sup\{\exp\{\Re(\xi\gamma(z))\} : z \in S_p^{\circ}\} = \exp\{\Re\xi + \Phi_p(\xi)\}$. This gives the desired lower bound. □

Now we investigate the hypercyclicity of the semigroup $e^{(a\mathscr{L}+b)t}$ when $2 < p < \infty$.

Lemma 11. *Suppose that $T(t) = e^{t(a\mathscr{L}+b)}$ where $t \geq 0$, a is a non-zero complex number and b is real. Then for $2 < p < \infty$, $T(t)$ is not hypercyclic on $L^p(\mathfrak{X})$ whenever $b \leq -\Re a - \Phi_p(a)$ or $b \geq -\Re a + \Phi_p(a)$.*

Proof. Fix $p \in (2,\infty)$ and let a be a non-zero complex number. To prove that the semigroup $T(t)$ is not hypercyclic is equivalent to show that the set $\{T(t)h : t \geq 0\}$ is not dense in $L^p(\mathfrak{X})$ for any $h \in L^p(\mathfrak{X})$. If $b \leq -\Re a - \Phi_p(a)$ then it follows from Lemma 10 that for every $h \in L^p(\mathfrak{X})$,

$$\| T(t)h \|_{L^p(\mathfrak{X})} = e^{bt} \left\| e^{at\mathscr{L}} h \right\|_{L^p(\mathfrak{X})} \leq C \exp\left\{t\left(b + \Re a + \Phi_p(a)\right)\right\} \| h \|_{L^p(\mathfrak{X})} \leq C \| h \|_{L^p(\mathfrak{X})}.$$

This show that the set $\{T(t)h : t \geq 0\}$ is bounded and hence it cannot be dense.

Now we consider the case when $b \geq -\Re a + \Phi_p(a)$. We prove this assertion by contradiction. Suppose that there exists a non-zero h in $L^p(\mathfrak{X})$ such that $\{T(t)h : t \geq 0\}$ is dense in $L^p(\mathfrak{X})$. Note that for all $t \geq 0$, $e^{at\mathscr{L}} e^{-at\mathscr{L}} h = e^{-at\mathscr{L}} e^{at\mathscr{L}} h = h$. By using the norm estimate (13), we have

$$\| h \|_{L^p(\mathfrak{X})} = \left\| e^{-at\mathscr{L}} e^{at\mathscr{L}} h \right\|_{L^p(\mathfrak{X})} \leq C \exp\left\{t\left(-\Re a + \Phi_p(a)\right)\right\} \left\| e^{at\mathscr{L}} h \right\|_{L^p(\mathfrak{X})},$$

which further implies that

$$\| T(t)h \|_{L^p(\mathfrak{X})} = e^{bt} \left\| e^{at\mathscr{L}} h \right\|_{L^p(\mathfrak{X})} \geq C \exp\left\{t\left(b + \Re a - \Phi_p(a)\right)\right\} \| h \|_{L^p(\mathfrak{X})} \geq C \| h \|_{L^p(\mathfrak{X})}.$$

Thus we conclude that the function '0' does not belong to the closure of $\{T(t)h : t \geq 0\}$ in $L^p(\mathfrak{X})$ and we finally arrive at a contradiction. This shows that $\{T(t)h : t \geq 0\}$ cannot be dense in $L^p(\mathfrak{X})$ for any $h \in L^p(\mathfrak{X})$, and completes the proof. □

4.1. *Proof of Theorem C*

Proof. It follows from the definition that (1) $\Longrightarrow$ (2). Moreover, (2) $\Longrightarrow$ (3) is a consequence of the Lemma 11. So we only need to show that (3) $\Longrightarrow$ (1). To prove our assertion, in view of Theorem A, it is enough to show that $T(t)$ has a non-trivial periodic point in $L^p(\mathfrak{X})$, or equivalently, $1 \in P\sigma_p(T(t))$, for some $t > 0$. Assume $z = s + i\delta_p$ and define $h(s) = \Re(a\gamma(s+i\delta_p)+b)$, for all $s \in [-\tau/2, \tau/2]$. A straightforward computation yield that the maximum and the minimum values of h on the interval $[-\tau/2, \tau/2]$ are $\Re a + \Phi_p(a) + b$ and $\Re a - \Phi_p(a) + b$ respectively. By applying the Maximum Modulus principle on the τ-periodic function $e^{t(a\gamma(\cdot)+b)}$, we obtain

$$\max_{z\in S_p} \left| e^{t(a\gamma(z)+b)} \right| = \exp\left\{\left(\Re a + \Phi_p(a) + b\right) t\right\} \quad \text{and} \quad \min_{z\in S_p} \left| e^{t(a\gamma(z)+b)} \right| = \exp\left\{\left(\Re a - \Phi_p(a) + b\right) t\right\}.$$

If $-\Re a - \Phi_p(a) < b < -\Re a + \Phi_p(a)$, then for any fixed $t > 0$,

$$\max_{z\in S_p}\left|e^{t(a\gamma(z)+b)}\right| > 1 \quad \text{and} \quad \min_{z\in S_p}\left|e^{t(a\gamma(z)+b)}\right| < 1.$$

Consequently it can be proved that there exists $z_0 \in S_p^\circ$ such that $e^{t(a\gamma(z_0)+b)} = 1$. Since $P\sigma_p(T(t)) = \exp\{(a\gamma(S_p^\circ) + b)t\}$ (by Proposition 5 (ii) and Theorem 7 (b)), $1 \in P\sigma_p(T(t))$, which establishes our claim. □

4.2. *Some Consequences*

There are some well-known examples of semigroups which are generated by the affine functions. As a consequence of Theorem C we have the following interesting results about the chaotic dynamics of these semigroups.

4.2.1. *The Heat Semigroup*

It was already mentioned in the introduction that the chaotic dynamics of the heat semigroup generated by shifts of the Laplace–Beltrami operator are extensively studied on symmetric spaces [9, 13] and harmonic NA-groups [15]. Our goal is to formulate these results for the heat semigroup on homogeneous trees by using Theorem C as a tool. A detailed study of the heat semigroup on $\mathfrak{X}$ was carried out in [3], where comparable upper and lower bounds for the L^p-L^r operator norms of $e^{-t\mathscr{L}}$ were obtained (see [3, Theorem 2.2]). The heat semigroup on $\mathfrak{X}$ generated by shifts of $\mathscr{L}$ is defined by the formula

$$T(t) = e^{-t(\mathscr{L}-b)} \quad \text{where} \quad t \geq 0,\ b \in \mathbb{R}.$$

Using Theorem B, it is clear that for any $b \in \mathbb{R}$, $T(t)$ is neither hypercyclic nor does it have any non-trivial periodic point on $L^p(\mathfrak{X})$ whenever $1 \leq p \leq 2$. Moreover, by putting $\Psi(z) = -z+b$ and $a = -1$, respectively, the following result is an immediate consequence of Theorem A and Theorem C.

Theorem 12. *Suppose that $T(t) = e^{-t(\mathscr{L}-b)}$ where $t \geq 0$. Then for $2 < p < \infty$, the following are equivalent.*

(1) *$T(t)$ is chaotic on $L^p(\mathfrak{X})$.*
(2) *$T(t)$ has a non-trivial periodic point.*
(3) *b satisfies the relation $\gamma(i\delta_p) < b < \gamma(\tau/2 + i\delta_p)$.*
(4) *$T(t)$ is hypercyclic.*

The geometrical interpretation of the above result can also be seen from Figure 2. The figure on the left represents the L^p-point spectrum of $\mathscr{L}$ and the figure on right represents the L^p-point spectrum of $\mathscr{L} - b$. It is easy to see that the L^p-point spectrum of $\mathscr{L} - b$ cuts the imaginary axis at infinitely many points if and only if $\gamma(i\delta_p) < b < \gamma(\tau/2 + i\delta_p)$.

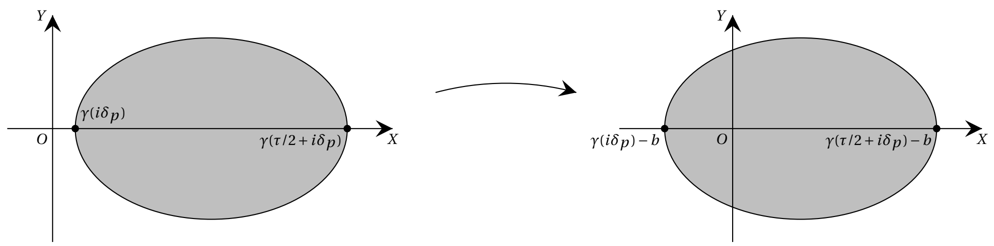

Figure 2.

4.2.2. *The Schrödinger Semigroup*

Now we consider the Schrödinger semigroup generated by the perturbation of $i\mathscr{L}$. Once again by using Theorem A and Theorem C with $\Psi(z) = iz + b$ and $a = i$, respectively, we have the following.

Theorem 13. *Suppose that* $T(t) = e^{t(i\mathscr{L}+b)}$ *where* $t \geq 0$. *Then for* $2 < p < \infty$, *the following are equivalent.*

(1) $T(t)$ *is chaotic on* $L^p(\mathfrak{X})$.
(2) $T(t)$ *has a non-trivial periodic point.*
(3) b *satisfies the relation* $\Im\gamma(\tau/4 + i\delta_p) < b < \Im\gamma(-\tau/4 + i\delta_p)$.
(4) $T(t)$ *is hypercyclic.*

In a similar way as above, Theorem 13 can also be described geometrically using the following figure.

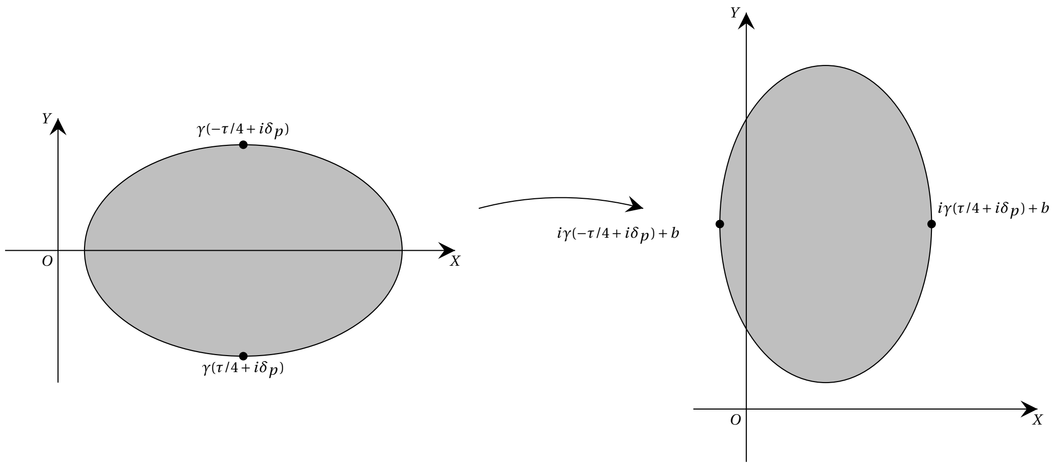

Figure 3.

Acknowledgements

The authors would like to thank an unknown referee for several important suggestions which improved an earlier draft.

References

[1] P. Boggarapu, S. Thangavelu, "On the chaotic behavior of the Dunkl heat semigroup on weighted L^p spaces", *Isr. J. Math.* **217** (2017), no. 1, p. 57-92.

[2] J. M. Cohen, M. Pagliacci, M. A. Picardello, "Universal properties of the isotropic Laplace operator on homogeneous trees", *Adv. Math.* **401** (2022), article no. 108311 (9 pages).

[3] M. Cowling, S. Meda, A. G. Setti, "Estimates for functions of the Laplace operator on homogeneous trees", *Trans. Am. Math. Soc.* **352** (2000), no. 9, p. 4271-4293.

[4] W. Desch, W. Schappacher, G. F. Webb, "Hypercyclic and chaotic semigroups of linear operators", *Ergodic Theory Dyn. Syst.* **17** (1997), no. 4, p. 793-819.

[5] R. L. Devaney, *An introduction to chaotic dynamical systems*, 2nd ed., Addition-Wesley Studies in Nonlinearity, Addison-Wesley Publishing Group, 1989.

[6] A. Figà-Talamanca, M. A. Picardello, "Spherical functions and harmonic analysis on free groups", *J. Funct. Anal.* **47** (1982), no. 3, p. 281-304.

[7] ———, *Harmonic analysis on free groups*, Lecture Notes in Pure and Applied Mathematics, vol. 87, Marcel Dekker, 1983.

[8] L. Grafakos, *Classical Fourier Analysis*, third ed., Graduate Texts in Mathematics, vol. 249, Springer, 2014.

[9] L. Ji, A. Weber, "Dynamics of the heat semigroup on symmetric spaces", *Ergodic Theory Dyn. Syst.* **30** (2010), no. 2, p. 457-468.

[10] P. Kumar, S. K. Rano, "A characterization of weak L^p-eigenfunctions of the Laplacian on Homogeneous trees", *Ann. Mat. Pura Appl.* **200** (2021), no. 2, p. 721-736.

[11] R. J. de Laubenfels, H. Emamirad, "Chaos for functions of discrete and continuous weighted shift operators", *Ergodic Theory Dyn. Syst.* **21** (2001), no. 5, p. 1411-1427.

[12] F. W. J. Olver, *Asymptotics and special functions*, Computer Science and Applied Mathematics, Academic Press Inc., 1974.

[13] M. Pramanik, R. P. Sarkar, "Chaotic dynamics of the heat semigroup on Riemannian symmetric spaces", *J. Funct. Anal.* **266** (2014), no. 5, p. 2867-2909.

[14] W. Rudin, *Functional analysis*, McGraw-Hill Series in Higher Mathematics, McGraw-Hill Book Co., 1973.

[15] R. P. Sarkar, "Chaotic dynamics of the heat semigroup on the Damek-Ricci spaces", *Isr. J. Math.* **198** (2013), no. 1, p. 487-508.

[16] A. G. Setti, "L^p and operator norm estimates for the complex time heat operator on homogeneous trees", *Trans. Am. Math. Soc.* **350** (1998), no. 2, p. 743-768.