INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus Mathématique

Lê Văn Thành

On weak laws of large numbers for maximal partial sums of pairwise independent random variables

Volume 361 (2023), p. 577-585

Published online: 2 March 2023

https://doi.org/10.5802/crmath.387

CENTRE MERSENNE

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 577-585
https://doi.org/10.5802/crmath.387

Probability theory / *Probabilités*

On weak laws of large numbers for maximal partial sums of pairwise independent random variables

Lê Văn Thành [a]

[a] Department of Mathematics, Vinh University, 182 Le Duan, Vinh, Nghe An, Vietnam
E-mail: levt@vinhuni.edu.vn

Abstract. This paper develops Rio's method [11] to prove the weak law of large numbers for maximal partial sums of pairwise independent random variables. The method allows us to avoid using the Kolmogorov maximal inequality. As an application, a weak law of large numbers for maximal partial sums of pairwise independent random variables under a uniform integrability condition is also established. The sharpness of the result is illustrated by an example.

2020 Mathematics Subject Classification. 60F05.

Manuscript received 12 February 2022, accepted 30 May 2022.

1. Introduction and results

A real-valued function $L(\cdot)$ is said to be *slowly varying* (at infinity) if it is a positive and measurable function on $[A,\infty)$ for some $A \ge 0$, and for each $\lambda > 0$,

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1.$$

In [5], de Bruijn proved that if $L(\cdot)$ is a slowly varying function, then there exists a slowly varying function $\widetilde{L}(\cdot)$, unique up to asymptotic equivalence, satisfying

$$\lim_{x\to\infty} L(x)\widetilde{L}(xL(x)) = 1 \quad \text{and} \quad \lim_{x\to\infty} \widetilde{L}(x)L\left(x\widetilde{L}(x)\right) = 1.$$

The function $\widetilde{L}(\cdot)$ is called the de Bruijn conjugate of $L(\cdot)$ ([2, p. 29]). Bojanić and Seneta [3] showed that for most of "nice" slowly varying functions, we can choose $\widetilde{L}(x) = 1/L(x)$. Especially, if $L(x) = \log^\gamma(x)$ or $L(x) = \log^\gamma(\log(x))$ for some $\gamma \in \mathbb{R}$, then $\widetilde{L}(x) = 1/L(x)$. Here and thereafter, for a real number x, $\log(x)$ denotes the natural logarithm (base e) of $\max\{x, \mathrm{e}\}$.

Let $L(\cdot)$ be a slowly varying function and let $r > 0$. By using a suitable asymptotic equivalence version (see Lemma 2.2 and Lemma 2.3 (i) in Anh et al. [1]), we can assume that $L(\cdot)$ is positive and differentiable on $[a,\infty)$, and $x^r L(x)$ is strictly increasing on $[a,\infty)$ for some large a. Next, let $L_1(\cdot)$ be a slowly varying function with $L_1(0)=0$ with a linear growth to $L(a)$ over $[0,a)$, and $L_1(x)\equiv L(x)$ on $[a,\infty)$. Then

(i) $L_1(x)$ is continuous on $[0,\infty)$ and differentiable on $[a,\infty)$, and

(ii) $x^r L_1(x)$ is strictly increasing on $[0,\infty)$.

In this paper, we will assume, without loss of generality, that properties (i) and (ii) are fulfilled for all slowly varying functions.

The starting point of the current research is the following weak law of large numbers (WLLN) which was proved by Gut [8]. Hereafter, $\mathbb{1}(A)$ denotes the indicator function of a set A.

Theorem 1 (Gut [8]). *Let $0<p\le 1$ and let $\{X,X_n,n\ge 1\}$ be a sequence of independent identically distributed (i.i.d.) random variables. Let $L(\cdot)$ be a slowly varying function and let $b_x=x^{1/p}L(x)$, $x\ge 0$. Then*

$$\frac{\sum_{i=1}^n X_i - n\mathbb{E}(X\mathbb{1}(|X|\le b_n))}{b_n}\xrightarrow{\mathbb{P}}0 \quad \textit{as } n\to\infty \tag{1}$$

if and only if

$$\lim_{n\to\infty} n\mathbb{P}(|X|>b_n)=0. \tag{2}$$

The above WLLN has been extended in several directions, see [12, 13] for WLLNs with random indices for arrays of independent random variables taking values in Banach spaces, and see [4, 6, 9, 10] and the references therein for WLLNs for dependent random variables and dependent random vectors. Boukhari [4, Theorem 1.2] showed that for $0<p<1$, condition (2) implies

$$\frac{\max_{1\le j\le n}\left|\sum_{i=1}^j X_i\right|}{b_n}\xrightarrow{\mathbb{P}}0 \quad \text{as } n\to\infty, \tag{3}$$

irrespective of the joint distribution of the X_n's. Boukhari [4] presented an example showing that his result does not hold when $p=1$. The proof of the sufficient part of Theorem 1 in [8] works well with pairwise independent random variables since we do not involve the maximal partial sums. Krulov [10] and Chandra [6] established WLLNs for maximal partial sums for the case where the summands are negatively associated and asymptotically almost negatively associated, respectively. The authors in [6, 10] considered general normalizing constants, which showed that the sufficient part of Theorem 1 also holds for $1\le p<2$. However, the method used in [6, 10] requires a Kolmogorov-type maximal inequality (see [6, Lemma 1.2]) which does not hold for pairwise independent random variables.

The aim of this paper is to establish WLLNs for maximal partial sums of pairwise independent random variables thereby extending the sufficient part of Theorem 1 for the case $p=1$ to WLLN for maximal partial sums from sequences of pairwise independent random variables. We use a technique developed by Rio [11] to avoid using the Kolmogorov maximal inequality. In addition, we also establish a WLLN for maximal partial sums of pairwise independent random variables under a uniform integrability condition, and present an example to show that this result does not hold in general if the uniform integrability assumption is weakened to the uniform boundedness of the moments.

Let Λ be a nonempty index set. A family of random variables $\{X_\lambda,\lambda\in\Lambda\}$ is said to be *stochastically dominated* by a random variable X if

$$\sup_{\lambda\in\Lambda}\mathbb{P}(|X_\lambda|>t)\le\mathbb{P}(|X|>t)\quad \text{for all } t\ge 0. \tag{4}$$

Some authors use an apparently weaker definition of $\{X_\lambda,\lambda\in\Lambda\}$ being stochastically dominated by a random variable Y, namely that

$$\sup_{\lambda\in\Lambda}\mathbb{P}(|X_\lambda|>t)\le C_1\mathbb{P}(C_2|Y|>t)\quad \text{for all } t\ge 0 \tag{5}$$

for some constants $C_1, C_2 \in (0,\infty)$. It is shown recently by Rosalsky and Thành [14] that (4) and (5) are indeed equivalent. We note that if (4) is satisfied, then for all $t > 0$ and $r > 0$

$$\sup_{\lambda\in\Lambda}\mathbb{E}(|X_\lambda|^r\mathbb{1}(|X_\lambda|\le t)) \le \mathbb{E}(|X|^r\mathbb{1}(|X|\le t)) + t^r\mathbb{P}\{|X|>t\},$$

and

$$\sup_{\lambda\in\Lambda}\mathbb{E}(|X_\lambda|^r\mathbb{1}(|X_\lambda|> t)) \le \mathbb{E}(|X|^r\mathbb{1}(|X|> t)).$$

The following theorem is the main result of this paper.

Theorem 2. *Let $1 \le p < 2$ and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables which is stochastically dominated by a random variable X. Let $L(\cdot)$ be a slowly varying function and let $b_n = n^{1/p}L(n)$, $n \ge 1$. If*

$$\lim_{n\to\infty} n\mathbb{P}(|X| > b_n) = 0, \tag{6}$$

then

$$\frac{\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_i - \mathbb{E}(X_i\mathbb{1}(|X_i|\le b_n)))\right|}{b_n} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty. \tag{7}$$

We postpone the proof of Theorem 2 to Section 2. From Theorem 3.2 of Boukhari [4], we have that if $\{X_n, n \ge 1\}$ is a sequence of pairwise independent random variables, and $\{b_n, n \ge 1\}$ is a sequence of positive constants, then

$$\frac{\max_{1\le i\le n}|X_i|}{b_n} \xrightarrow{\mathbb{P}} 0 \quad \text{if and only if} \quad \sum_{i=1}^{n}\mathbb{P}(|X_i| > b_n\varepsilon)\to 0 \ \text{ for all } \varepsilon > 0. \tag{8}$$

By using Theorem 2 and (8), we obtain the following corollary.

Corollary 3. *Let $\{X, X_n, n \ge 1\}$ be a sequence of pairwise i.i.d. random variables. Let p, $L(\cdot)$, b_n be as in Theorem 2. Then* (6) *and* (7) *are equivalent.*

Proof. If (6) holds, then (7) follows immediately from Theorem 2. Now, assume that (7) holds. By the symmetrization procedure, it suffices to check the case where the random variables X_n, $n \ge 1$ are symmetric. In this case, (7) becomes

$$\frac{\max_{1\le j\le n}\left|S_j\right|}{b_n} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty, \tag{9}$$

where $S_j = \sum_{i=1}^{j} X_i$, $j \ge 1$. Putting $S_0 = 0$, and applying (9) and inequality

$$\max_{1\le j\le n}|X_j| \le \max_{1\le j\le n}|S_j| + \max_{1\le j\le n}|S_{j-1}|,$$

we obtain

$$\frac{\max_{1\le j\le n}\left|X_j\right|}{b_n} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n\to\infty. \tag{10}$$

By combining (8) and (10), and using the identical distribution assumption, we obtain (6). □

Theorem 2 also enables us to establish a WLLN for maximal partial sums of pairwise independent random variables under a uniform integrability condition. After this paper was submitted, Thành [16, Corollary 4.10] established a similar WLLN for triangular arrays of random variables satisfying a Kolmogorov-type maximal inequality. Theorem 4 and Corollary 4.10 of Thành [16] do not imply each other.

Hereafter, we denote the de Bruijn conjugate of a slowly varying function $L(\cdot)$ by $\widetilde{L}(\cdot)$.

Theorem 4. *Let $1 \le p < 2$ and let $\{X_n, n \ge 1\}$ be a sequence of pairwise independent random variables. Let $L(\cdot)$ be a slowly varying function. If $\{|X_n|^p L(|X_n|^p), n \ge 1\}$ is uniformly integrable, then*

$$\frac{\max_{1\le j\le n}\left|\sum_{i=1}^{j}\left(X_i - \mathbb{E}\left(X_i \mathbb{1}(|X_i| \le b_n)\right)\right)\right|}{b_n} \xrightarrow{\mathbb{P}} 0 \quad \text{as } n \to \infty, \tag{11}$$

where $b_n = n^{1/p}\widetilde{L}^{1/p}(n)$, $n \ge 1$.

Proof. Let $f(x) = x^p L(x^p)$, $g(x) = x^{1/p}\widetilde{L}^{1/p}(x)$, $x \ge 0$. Recalling that we have assumed, without loss of generality, that f and g are strictly increasing on $[0,\infty)$. From Theorem 1.5.12 and Proposition 1.5.15 in Bingham et al. [2] (see also Lemma 2.1 in Anh et al. [1]),

$$\lim_{x\to\infty}\frac{f(g(x))}{x} = 1,$$

and therefore

$$f(g(n)) > n/2 \quad \text{for all large } n. \tag{12}$$

By the de La Vallée Poussin criterion for uniform integrability, there exists a nondecreasing function h defined on $[0,\infty)$ with $h(0) = 0$ such that

$$\lim_{x\to\infty}\frac{h(x)}{x} = \infty, \tag{13}$$

and

$$\sup_{i\ge1}\mathbb{E}(h(f(|X_i|))) = \sup_{i\ge1}\mathbb{E}(h(|X_i|^p L(|X_i|^p))) < \infty. \tag{14}$$

By using Theorem 2.5 (i) of Rosalsky and Thành [14], (14) implies that the sequence $\{X_n, n \ge 1\}$ is stochastically dominated by a nonnegative random variable X with distribution function

$$F(x) = 1 - \sup_{i\ge1}\mathbb{P}(|X_i| > x),\ x \in \mathbb{R}.$$

We thus have by (12), (13), (14) and the Markov inequality that

$$\begin{aligned}
\lim_{n\to\infty} n\mathbb{P}(X > b_n) &= \lim_{n\to\infty} n\sup_{i\ge1}\mathbb{P}(|X_i| > g(n))\\
&= \lim_{n\to\infty} n\sup_{i\ge1}\mathbb{P}(f(|X_i|) > f(g(n)))\\
&\le \lim_{n\to\infty} n\sup_{i\ge1}\mathbb{P}(f(|X_i|) \ge n/2)\\
&\le \lim_{n\to\infty} n\sup_{i\ge1}\mathbb{P}(h(f(|X_i|)) \ge h(n/2))\\
&\le \lim_{n\to\infty} n\sup_{i\ge1}\frac{\mathbb{E}(h(f(|X_i|)))}{h(n/2))}\\
&= 2\sup_{i\ge1}\mathbb{E}(h(f(|X_i|)))\lim_{n\to\infty}\frac{n/2}{h(n/2)} = 0.
\end{aligned}$$

Applying Theorem 2, we obtain (11). □

The following example shows that in Theorem 4, the assumption that $\{|X_n|^p L(|X_n|^p), n \ge 1\}$ is uniformly integrable cannot be weakened to

$$\sup_{n\ge1}\mathbb{E}(|X_n|^p L(|X_n|^p)) < \infty. \tag{15}$$

Example 5. Let $1 \le p < 2$, and let $\{X_n, n \ge 1\}$ be a sequence of independent symmetric random variables with

$$\mathbb{P}(X_n = 0) = 1 - \frac{1}{n}, \quad \mathbb{P}(X_n = n^{1/p}) = \mathbb{P}(X_n = -n^{1/p}) = \frac{1}{2n}, \quad n \ge 1.$$

Consider the case where the slowly varying function $L(x) \equiv 1$. Then it is clear that

$$\sup_{n\ge1}\mathbb{E}(|X_n|^p L(|X_n|^p)) = \sup_{n\ge1}\mathbb{E}(|X_n|^p) = 1 < \infty$$

and

$$\sup_{n\geq 1}\mathbb{E}(|X_n|^pL(|X_n|^p)\mathbb{1}(|X_n|^pL(|X_n|^p)>a))=\sup_{n\geq 1}\mathbb{E}(|X_n|^p\mathbb{1}(|X_n|>a^{1/p}))=1$$

for all $a>0$. Therefore (15) is satisfied but $\{|X_n|^pL(|X_n|^p), n\geq 1\}$ is not uniformly integrable. For a real number x, let $\lfloor x\rfloor$ denote the greatest integer that is smaller than or equal to x. Then for $0<\varepsilon<1/4$ and for $n\geq 2$, we have

$$\sum_{i=1}^{n}\mathbb{P}(|X_i|>\varepsilon n^{1/p})\geq\sum_{i=\lfloor n/2\rfloor}^{n}\mathbb{P}(|X_i|>\varepsilon n^{1/p})\geq\sum_{i=\lfloor n/2\rfloor}^{n}\frac{1}{n}\geq\frac{1}{2}. \tag{16}$$

Combining (8) and (16) yields

$$\frac{\max_{1\leq i\leq n}|X_i|}{n^{1/p}}\overset{\mathbb{P}}{\nrightarrow}0.$$

This implies that (11) (with $b_n\equiv n^{1/p}$) fails.

2. Proof of Theorem 2

The following lemma plays an important role in the proof of Theorem 2. It gives a general approach to the WLLN. In this lemma, we do not require any dependence structure. Throughout this section, we use the symbol C to denote a universal positive constant which is not necessarily the same in each appearance.

Lemma 6. *Let $\{b_n, n\geq 1\}$ be a nondecreasing sequence of positive numbers satisfying*

$$\sup_{m\geq 1}\frac{b_{2^m}}{b_{2^{m-1}}}<\infty. \tag{17}$$

Let $\{X_n, n\geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X and let $X_{i,n}=X_i\mathbb{1}(|X_i|\leq b_n)$, $n\geq 1, i\geq 1$. Assume that

$$\lim_{n\to\infty}n\mathbb{P}(|X|>b_n)=0. \tag{18}$$

Then

$$\frac{\max_{1\leq j\leq n}\left|\sum_{i=1}^{j}\left(X_i-\mathbb{E}\left(X_{i,n}\right)\right)\right|}{b_n}\overset{\mathbb{P}}{\to}0\quad \text{as } n\to\infty \tag{19}$$

if and only if

$$\frac{\max_{1\leq j<2^n}\left|\sum_{i=1}^{j}\left(X_{i,2^n}-\mathbb{E}\left(X_{i,2^n}\right)\right)\right|}{b_{2^n}}\overset{\mathbb{P}}{\to}0\quad \text{as } n\to\infty. \tag{20}$$

Proof. We firstly prove under (18) that

$$\frac{\max_{1\leq j<2^n}\left|\sum_{i=1}^{j}\left(X_i-X_{i,2^n}\right)\right|}{b_{2^n}}\overset{\mathbb{P}}{\to}0\quad \text{as } n\to\infty. \tag{21}$$

To see this, let $\varepsilon>0$ be arbitrary. Then

$$\begin{aligned}\mathbb{P}\left(\frac{\max_{1\leq j<2^n}\left|\sum_{i=1}^{j}\left(X_i-X_{i,2^n}\right)\right|}{b_{2^n}}>\varepsilon\right)&\leq\mathbb{P}\left(\bigcup_{i=1}^{2^n-1}\left(X_i\neq X_{i,2^n}\right)\right)\\&\leq\sum_{i=1}^{2^n-1}\mathbb{P}\left(X_i\neq X_{i,2^n}\right)\\&\leq 2^n\mathbb{P}\left(|X|>b_{2^n}\right)\to 0\quad\text{as } n\to\infty\quad\text{(by (18))}\end{aligned}$$

thereby proving (21) since $\varepsilon>0$ is arbitrary.

Next, assume that (19) holds. Then

$$\frac{\max_{1\leq j<2^n}\left|\sum_{i=1}^{j}\left(X_i-\mathbb{E}\left(X_{i,2^n}\right)\right)\right|}{b_{2^n}}\overset{\mathbb{P}}{\to}0\quad \text{as } n\to\infty. \tag{22}$$

Combining (21) and (22), we obtain (20).

Finally, assume that (20) holds. It follows from (20) and (21) that

$$\frac{\max_{1\le j<2^n}\left|\sum_{i=1}^{j}\left(X_i-\mathbb{E}\left(X_{i,2^n}\right)\right)\right|}{b_{2^n}}\xrightarrow{\mathbb{P}}0\quad\text{as } n\to\infty. \tag{23}$$

Now, for $m\ge 1$, set

$$K_m=\max_{2^{m-1}\le n<2^m}\frac{\max_{1\le j<2^m}\left|\sum_{i=1}^{j}\mathbb{E}(X_{i,2^m}-X_{i,n})\right|}{b_{2^{m-1}}}.$$

Then by using (17), (18) and the stochastic domination assumption, we have

$$\begin{aligned}K_m&\le\frac{\sum_{i=1}^{2^m}\mathbb{E}\left(|X_i|\mathbb{1}(b_{2^{m-1}}<|X_i|\le b_{2^m})\right)}{b_{2^{m-1}}}\\&\le\frac{\sum_{i=1}^{2^m}b_{2^m}\mathbb{P}(|X_i|>b_{2^{m-1}})}{b_{2^{m-1}}}\\&\le C2^m\mathbb{P}(|X|>b_{2^{m-1}})\to 0\quad\text{as } m\to\infty.\end{aligned} \tag{24}$$

For $n\ge 1$, let $m\ge 1$ be such that $2^{m-1}\le n<2^m$. Then by (17), (23) and (24)), we have

$$\begin{aligned}\frac{\max_{1\le j\le n}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}(X_{i,n}))\right|}{b_n}&\le\frac{\max_{1\le j<2^m}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}(X_{i,2^m}))\right|}{b_{2^{m-1}}}\\&\quad+\frac{\max_{1\le j<2^m}\left|\sum_{i=1}^{j}\mathbb{E}(X_{i,2^m}-X_{i,n})\right|}{b_{2^{m-1}}}\\&\le\frac{C\max_{1\le j<2^m}\left|\sum_{i=1}^{j}(X_i-\mathbb{E}(X_{i,2^m}))\right|}{b_{2^m}}+K_m\\&\xrightarrow{\mathbb{P}}0\quad\text{as } m\to\infty\end{aligned}$$

thereby establishing (19). □

Proof of Theorem 2. Let

$$X_{i,n}=X_i\mathbb{1}(|X_i|\le b_n),\ n\ge 1, i\ge 1.$$

It is clear that the sequence $\{b_n, n\ge 1\}$ satisfies (17). By Lemma 6, it suffices to show that

$$\frac{\max_{1\le j<2^n}\left|\sum_{i=1}^{j}\left(X_{i,2^n}-\mathbb{E}\left(X_{i,2^n}\right)\right)\right|}{b_{2^n}}\xrightarrow{\mathbb{P}}0\quad\text{as } n\to\infty. \tag{25}$$

For $m\ge 0$, set $S_{0,m}=0$ and

$$S_{j,m}=\sum_{i=1}^{j}(X_{i,2^m}-\mathbb{E}X_{i,2^m}),\ j\ge 1.$$

We will use techniques developed by Rio [11] (see also [15] for the case of regular varying normalizing sequences) as follows. For $n\ge 1$, $1\le j<2^n$ and for $0\le m\le n$, let $k=\lfloor j/2^m\rfloor$ be the greatest integer which is less than or equal to $j/2^m$. Then $0\le k<2^{n-m}$ and $k2^m\le j<(k+1)2^m$. Let $j_m=k2^m$, and

$$Y_{i,m}=\left|X_{i,2^m}-X_{i,2^{m-1}}\right|-\mathbb{E}\left(\left|X_{i,2^m}-X_{i,2^{m-1}}\right|\right).$$

Then we can show that (see [15, p. 1236])

$$\begin{aligned}\max_{1\le j<2^n}\left|S_{j,n}\right|\le\sum_{m=1}^{n}\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}}\left(X_{i,2^{m-1}}-\mathbb{E}(X_{i,2^{m-1}})\right)\right|+\sum_{m=1}^{n}\max_{0\le k<2^{n-m}}\left|\sum_{i=k2^m+1}^{(k+1)2^m}Y_{i,m}\right|\\+\sum_{m=1}^{n}2^{m+1}b_{2^m}\mathbb{P}\left(|X|>b_{2^{m-1}}\right).\end{aligned} \tag{26}$$

By (6), we have

$$2^{m+1}\mathbb{P}\left(|X| > b_{2^{m-1}}\right) \to 0 \quad \text{as } m \to \infty. \tag{27}$$

It follows from Karamata's theorem (see, e.g., [2, p. 30]) that

$$\sum_{k=1}^{n} \alpha^k L\left(\beta^k\right) \le C\alpha^n L(\beta^n) \quad \text{for all } \alpha > 1, \beta \ge 1. \tag{28}$$

By using (27), (28) and Toeplitz's lemma, we have

$$\lim_{n\to\infty} \frac{\sum_{m=1}^{n} b_{2^m} 2^{m+1}\mathbb{P}\left(|X| > b_{2^{m-1}}\right)}{b_{2^n}} = \lim_{n\to\infty} \frac{\sum_{m=1}^{n} 2^{m/p} L(2^m) 2^{m+1}\mathbb{P}\left(|X| > b_{2^{m-1}}\right)}{2^{n/p} L(2^n)} = 0. \tag{29}$$

Let $\varepsilon_1 > 0$ be arbitrary, and let a and b be positive constants satisfying

$$1/2 < a < 1/p, \; a + b = 1/p.$$

For $n \ge 1$, $0 \le m \le n$, let

$$\lambda_{m,n} = \varepsilon_1 2^{bm} 2^{an} L(2^n). \tag{30}$$

An elementary calculation shows (see [15, p. 1236])

$$\sum_{m=1}^{n} \lambda_{m,n} \le \frac{2^b \varepsilon_1 b_{2^n}}{2^b - 1} := C_1(b)\varepsilon_1 b_{2^n}. \tag{31}$$

By (26), (29), the proof of (25) is completed if we show that

$$I_n := \mathbb{P}\left(\sum_{m=1}^{n} \max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right| \ge C_1(b)\varepsilon_1 b_{2^n}\right) \to 0 \quad \text{as } n \to \infty, \tag{32}$$

and

$$J_n := \mathbb{P}\left(\sum_{m=1}^{n} \max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}} \left(X_{i,2^{m-1}} - \mathbb{E}(X_{i,2^{m-1}})\right)\right| \ge C_1(b)\varepsilon_1 b_{2^n}\right) \to 0 \quad \text{as } n \to \infty. \tag{33}$$

We note that for each $m \ge 1$, $Y_{i,m}, i \ge 1$ are mean 0 and pairwise independent random variables. Therefore

$$\begin{aligned}
I_n &\le \sum_{m=1}^{n} \mathbb{P}\left(\max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right| \ge \lambda_{m,n}\right) \quad \text{(by (31))}\\
&\le \sum_{m=1}^{n} \lambda_{m,n}^{-2} \mathbb{E}\left(\max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right|\right)^2 \quad \text{(by Markov's inequality)}\\
&\le \sum_{m=1}^{n} \lambda_{m,n}^{-2} \sum_{k=0}^{2^{n-m}-1} \mathbb{E}\left(\sum_{i=k2^m+1}^{(k+1)2^m} Y_{i,m}\right)^2\\
&= \sum_{m=1}^{n} \lambda_{m,n}^{-2} \sum_{k=0}^{2^{n-m}-1} \sum_{i=k2^m+1}^{(k+1)2^m} \mathbb{E}\left(Y_{i,m}\right)^2\\
&\le \sum_{m=1}^{n} \lambda_{m,n}^{-2} \sum_{k=0}^{2^{n-m}-1} \sum_{i=k2^m+1}^{(k+1)2^m} \mathbb{E}\left(X_{i,2^m} - X_{i,2^{m-1}}\right)^2\\
&\le \sum_{m=1}^{n} 2^n \lambda_{m,n}^{-2} b_{2^m}^2 \mathbb{P}(|X| > b_{2^{m-1}}) \quad \text{(by the stochastic domination assumption)}\\
&= \varepsilon_1^{-2} \frac{1}{2^{n(2a-1)} L^2(2^n)} \left(\sum_{m=1}^{n} 2^{m(2a-1)} L^2(2^m) 2^m \mathbb{P}(|X| > b_{2^{m-1}})\right) \quad \text{(by (30))}\\
&\to 0 \quad \text{as } n \to \infty \quad \text{(by noting } 2a-1>0 \text{ and using (27), (28), and Toeplitz's lemma).}
\end{aligned} \tag{34}$$

Similarly,

$$\begin{aligned}
J_n &\le \sum_{m=1}^{n} \mathbb{P}\left(\max_{0\le k<2^{n-m}} \left|\sum_{i=k2^m+1}^{k2^m+2^{m-1}} \left(X_{i,2^{m-1}} - \mathbb{E}(X_{i,2^{m-1}})\right)\right| \ge \lambda_{m,n}\right)\\
&\le \sum_{m=1}^{n} 2^n \lambda_{m,n}^{-2}\left(\mathbb{E}X^2 \mathbb{1}(|X| \le b_{2^{m-1}}) + b_{2^{m-1}}^2 \mathbb{P}(|X| > b_{2^{m-1}})\right)\\
&= \sum_{m=1}^{n} 2^n \lambda_{m,n}^{-2} \mathbb{E}X^2 \mathbb{1}(|X| \le b_{2^{m-1}}) + o(1), \qquad (35)
\end{aligned}$$

where we have applied (34) in the final step. By using integration by parts, and proceeding in a similar manner as the last two lines of (34), we have

$$\begin{aligned}
\sum_{m=1}^{n} 2^n \lambda_{m,n}^{-2} \mathbb{E}X^2 \mathbb{1}(|X| \le b_{2^{m-1}}) &\le \sum_{m=1}^{n} 2^n \lambda_{m,n}^{-2} \int_0^{b_{2^{m-1}}} 2x\mathbb{P}(|X| > x)\mathrm{d}x\\
&\le \varepsilon_1^{-2} 2^{n(1-2a)} L^{-2}(2^n) \sum_{m=1}^{n}\left(2^{-2mb}\sum_{k=1}^{m}\int_{b_{2^{k-1}}}^{b_{2^k}} 2x\mathbb{P}(|X| > x)\mathrm{d}x + 2^{-2mb}\int_0^{b_1} 2x\mathrm{d}x\right)\\
&\le \varepsilon_1^{-2} 2^{n(1-2a)} L^{-2}(2^n)\left(\sum_{k=1}^{n}\left(\sum_{m=k}^{n} 2^{-2bm}\right) b_{2^k}^2 \mathbb{P}\left(|X| > b_{2^{k-1}}\right) + \sum_{m=1}^{n} 2^{-2mb} b_1^2\right)\\
&\le \frac{C}{2^{n(2a-1)} L^2(2^n)}\left(\sum_{k=1}^{n} 2^{k(2a-1)} L^2(2^k) 2^k \mathbb{P}\left(|X| > b_{2^{k-1}}\right) + 1\right) \to 0 \quad \text{as } n \to \infty. \qquad (36)
\end{aligned}$$

Combining (34)–(36), we obtain (32) and (33) thereby completing the proof of (25). □

3. Concluding remarks

This paper establishes WLLNs for maximal partial sums of pairwise independent random variables without using the Kolmogorov maximal inequality. The method can be easily adapted to dependent random variables. We have the following result:

Theorem 7. *Let $\{X_n, n \ge 1\}$ be a sequence of random variables and let p, $L(\cdot)$ and b_n be as in Theorem 2. Assume that there exists a constant C such that for all nondecreasing functions f_i, $i \ge 1$ we have*

$$\operatorname{Var}\left(\sum_{i=k+1}^{k+\ell} f_i(X_i)\right) \le C \sum_{i=k+1}^{k+\ell} \operatorname{Var}(f_i(X_i)), \quad k \ge 0,\ \ell \ge 1, \qquad (37)$$

provided the variances exist. If $\{X_n, n \ge 1\}$ is stochastically dominated by a random variable X such that (6) *is satisfied, then we obtain WLLN* (7).

Theorem 7 can be proved by assuming that $X_n \ge 0$, $n \ge 1$ since we can use identity $X_n = X_n^+ - X_n^-$ in the general case. We then use truncation $X_{i,n} = X_i \mathbb{1}(X_i \le b_n) + b_n \mathbb{1}(X_i > b_n)$, $n \ge 1, i \ge 1$ (to ensure that the truncated sequence $\{X_{n,i}, i \ge 1\}$ satisfies (37)), and modify the arguments given in the proofs of Lemma 6 and Theorem 2 accordingly. The details are straightforward and, hence, are omitted.

Many dependence structures satisfy (37), including m-pairwise negative dependence, extended negative dependence, φ-mixing, etc (see, e.g., [7, 11]).

It is obvious that (7) implies (1). For the i.i.d case and $1 \le p < 2$, Kruglov [10, Theorem 2] proved that (7) and (1) are equivalent. It is an open problem as to whether or not this also holds for the pairwise i.i.d. case.

Acknowledgements

The author would like to thank the Editor for offering useful comments and suggestions which enabled him to improve the paper. The author is also grateful to Professor Fakhreddine Boukhari for his interest in this work.

References

[1] V. T. N. Anh, N. T. T. Hien, L. V. Thành, V. T. H. Van, "The Marcinkiewicz–Zygmund-type strong law of large numbers with general normalizing sequences", *J. Theor. Probab.* **34** (2021), no. 1, p. 331-348.

[2] N. H. Bingham, C. M. Goldie, J. L. Teugels, *Regular variation*, Encyclopedia of Mathematics and Its Applications, vol. 27, Cambridge University Press, 1989.

[3] R. Bojanic, E. Seneta, "Slowly varying functions and asymptotic relations", *J. Math. Anal. Appl.* **34** (1971), no. 2, p. 302-315.

[4] F. Boukhari, "On a weak law of large numbers with regularly varying normalizing sequences", *J. Theor. Probab.* **35** (2022), no. 3, p. 2068-2079.

[5] N. G. de Bruijn, "Pairs of slowly oscillating functions occurring in asymptotic problems concerning the Laplace transform.", *Nieuw. Arch. Wisk.* **7** (1959), no. 3, p. 20-26.

[6] T. K. Chandra, "On an extension of the weak law of large numbers of Kolmogorov and Feller", *Stochastic Anal. Appl.* **32** (2014), no. 3, p. 421-426.

[7] N. C. Dzung, L. V. Thành, "On the complete convergence for sequences of dependent random variables via stochastic domination conditions and regularly varying functions theory", 2021, https://arxiv.org/abs/2107.12690.

[8] A. Gut, "An extension of the Kolmogorov–Feller weak law of large numbers with an application to the St. Petersburg game", *J. Theor. Probab.* **17** (2004), no. 3, p. 769-779.

[9] N. T. T. Hien, L. V. Thành, "On the weak laws of large numbers for sums of negatively associated random vectors in Hilbert spaces", *Stat. Probab. Lett.* **107** (2015), p. 236-245.

[10] V. M. Kruglov, "A generalization of weak law of large numbers", *Stochastic Anal. Appl.* **29** (2011), no. 4, p. 674-683.

[11] E. Rio, "Vitesses de convergence dans la loi forte pour des suites dépendantes (Rates of convergence in the strong law for dependent sequences)", *C. R. Math. Acad. Sci. Paris* **320** (1995), no. 4, p. 469-474.

[12] A. Rosalsky, L. V. Thành, "On the weak law of large numbers with random indices for randomly weighted row sums from arrays of random elements in Banach spaces", *J. Probab. Stat. Sci.* **4** (2006), no. 2, p. 123-135.

[13] ———, "Weak laws of large numbers for double sums of independent random elements in Rademacher type p and stable type p Banach spaces", *Nonlinear Anal., Theory Methods Appl.* **71** (2009), no. 12, p. e1065-e1074.

[14] ———, "A note on the stochastic domination condition and uniform integrability with applications to the strong law of large numbers", *Stat. Probab. Lett.* **178** (2021), p. 109181.

[15] L. V. Thành, "On the Baum–Katz theorem for sequences of pairwise independent random variables with regularly varying normalizing constants", *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 11–12, p. 1231-1238.

[16] ———, "A new concept of stochastic domination and the laws of large numbers", *TEST* (2022), https://doi.org/10.1007/s11749-022-00827-w.