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Procesi's Conjecture on the Formanek-Weingarten Function is False

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Abstract. In this paper, we disprove a recent monotonicity conjecture of C. Procesi on the generating function for monotone walks on the symmetric group, an object which is equivalent to the Weingarten function of the unitary group.

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1. Introduction

Let Γ_d be the Cayley graph of the symmetric group $\mathrm{S}(d)$ as generated by the conjugacy class of transpositions. Thus Γ_d is a $\binom{d}{2}$-regular graded graph with levels $L_0, \ldots, L_{d-1}$, where L_k is the set of permutations which factor into a product of $d-k$ disjoint cycles. Let us mark each edge of Γ_d corresponding to the transposition $(i\ j)$ with $j \in \{2, \ldots, d\}$, the larger of the two symbols interchanged. This edge labeling was first considered by Stanley [6] and Biane [1] in connection with noncrossing partitions and parking functions.

A walk on Γ_d is said to be *monotone* if the labels of the edges it traverses form a weakly increasing sequence. The combinatorics of such walks has been intensively studied in recent years, beginning with the discovery [4] that these trajectories play the role of Feynman diagrams for integration with respect to Haar measure on unitary groups. This is part of a broader subject nowadays known as Weingarten calculus, see [2].

Although non-obvious, it is a fact that the number of monotone walks of given length r between two given permutations $\rho, \sigma \in \mathrm{S}(d)$ depends only on the cycle type $\alpha \vdash d$ of the permutation $\rho^{-1}\sigma$. It is therefore sufficient to consider the number $m^r(\alpha)$ of r-step monotone

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walks on Γ_d beginning at the identity permutation and ending at a fixed permutation of cycle type α. To each partition $\alpha \vdash d$ we associate the generating function

$$M_\alpha(x) = \sum_{r=0}^{\infty} m^r(\alpha) x^r \tag{1}$$

enumerating monotone walks on Γ_d of arbitrary length and type α. It is known [3] that

$$M_\alpha(x) = \sum_{\lambda \vdash d} \frac{\chi_\alpha^\lambda}{\prod_{\Box \in \lambda} h(\Box)\,(1 - c(\Box)\,x)}, \tag{2}$$

where χ_α^λ are the irreducible characters of the symmetric group $\mathrm{S}(d)$, with $h(\Box)$ and $c(\Box)$ being, respectively, the hook length and content of a given cell $\Box$ in the Young diagram of λ (see [7] for definitions). In particular, $M_\alpha(x)$ is a rational function of x which may be considered as a continuous function of x on the interval $(0, \frac{1}{d-1})$ whose outputs are positive rational numbers. Up to a simple rescaling, the values $M_\alpha(\frac{1}{N})$ coincide with the values of the Weingarten function of the unitary group $\mathrm{U}(N)$; see [3, 4].

In a recent paper [5], Procesi has pointed out that the function $M_\alpha(x)$ was also studied from the perspective of classical invariant theory by Formanek, and that in this context the values $M_\alpha(\frac{1}{d})$ have special significance. Procesi tabulated these numbers for all diagrams $\alpha \vdash d \leq 8$, and on the basis of these computations made the following conjecture.

Conjecture 1. *If $\alpha > \beta$ in lexicographic order, then $M_\alpha(\frac{1}{d}) > M_\beta(\frac{1}{d})$.*

In this brief note we give explicit numerical examples which show that Conjecture 1 is false.

2. Small x

We first clarify that Procesi's Conjecture 1 refers to lexicographic order on partitions viewed as nondecreasing sequences of positive integers, with 1 the first letter in the alphabet, 2 the second letter, and so on. For example, the partitions of six listed in lexicographic order are

111111
11112
1113
1122
114
123
15
222
24
33
6,

and Conjecture 1 says that the numbers $M_\alpha(\frac{1}{6})$ strictly decrease as α moves down this list, and this is so. However, the pattern fails for sufficiently large degree d.

The first sign that Conjecture 1 might be false in general is that it is incompatible with the known $x \to 0$ asymptotics of $M_\alpha(x)$. The minimal length of a walk on Γ_d from the identity to a permutation of type α is $d - \ell(\alpha)$, and thus by parity the number $m^r(\alpha)$ can only be positive when $r = d - \ell(\alpha) + 2k$ with k a nonnegative integer. We may therefore reparameterize the counts $m^r(\alpha)$ as $m_k(\alpha) := m^{d-\ell(\alpha)+2k}(\alpha)$ for $k \in \mathbb{N}_0$. The generating function $M_\alpha(x)$ then becomes

$$M_\alpha(x) = x^{d-\ell(\alpha)} \sum_{k=0}^{\infty} m_k(\alpha) x^{2k}. \tag{3}$$

It is then clear that

$$\lim_{x\to 0} \frac{M_\beta(x)}{M_\alpha(x)} = 0 \tag{4}$$

whenever $\ell(\alpha) > \ell(\beta)$, which is incompatible with lexicographic order.

One might nevertheless hope that when we compare the small x behavior of $M_\alpha(x)$ and $M_\beta(x)$ with α and β being partitions of the same length, we find compatibility with lexicographic order. This too is false, as can be seen from the fact [3] that

$$m_0(\alpha) = \prod_{i=1}^{\ell(\alpha)} \mathrm{Cat}_{\alpha_i - 1}, \tag{5}$$

where $\mathrm{Cat}_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number. Then for $\alpha, \beta \vdash d$ partitions of the same length ℓ, we have

$$\lim_{x\to 0} \frac{M_\beta(x)}{M_\alpha(x)} = \prod_{i=1}^{\ell} \frac{\mathrm{Cat}_{\beta_i - 1}}{\mathrm{Cat}_{\alpha_i - 1}}. \tag{6}$$

For small values of d, it does indeed appear to be the case that this product is smaller than 1 when $\alpha > \beta$, but this is a law of small numbers. Consider the case where

$$\alpha = \Bigg(1, \underbrace{3, \ldots, 3}_{n}\Bigg) \quad \text{and} \quad \beta = \Bigg(\underbrace{2, \ldots, 2}_{n}, n+1\Bigg) \tag{7}$$

Then α and β are partitions of the same degree $d = 3n+1$, they have the same length $\ell(\alpha) = \ell(\beta) = n+1$, and α precedes β in the lexicographic order. However, the ratio of the corresponding Catalan products tends to infinity as $n \to \infty$,

$$\frac{\mathrm{Cat}_n}{2^n} \sim \frac{1}{\sqrt{\pi} n^{3/2}} \cdot 2^n. \tag{8}$$

3. Counterexamples

To give a counterexample to Conjecture 1 itself, we return to the character formula (2), which in fact yields counterexamples if one goes a bit farther than the data tabulated in [5]. Let α^+ denote the successor of α in the lexicographic order. The first value of d for which Conjecture 1 fails is the famously unlucky number $d = 13$, for which there exists precisely one violating pair α, α^+. This pair is

$$M_{(1^6,7)}\left(\frac{1}{13}\right) = \frac{13^{13}}{(13!)^2} \frac{30132115571}{1149266300} < \frac{13^{13}}{(13!)^2} \frac{426729597219}{16089728200} = M_{(1^5,2^4)}\left(\frac{1}{13}\right)$$

We have tested Conjecture 1 for $d \le 20$ and it fails for all $13 \le d \le 20$. Moreover the size of the set

$$G_d := \left\{\alpha \vdash d \colon M_\alpha\left(\frac{1}{d}\right) < M_{\alpha^+}\left(\frac{1}{d}\right)\right\}$$

of consecutive failures at rank d increases with d. For instance

$$\begin{aligned}
G_{14} &= \{(1^7,7), (1^5,2,7), (1^5,9)\},\\
G_{15} &= \{(1^8,7), (1^6,2,7), (1^6,9), (1^4,11), (1^3,2,10), (1^3,3,9)\},\\
G_{16} &= \{(1^{11},5), (1^9,7), (1^7,2,7), (1^7,9), (1^6,10), (1^5,2^2,7), (1^5,11), (1^4,2,10),\\
&\quad (1^4,3,9), (1^3,13), (1,4,11)\}.
\end{aligned}$$

Even though Conjecture 1 seems to fail for all $d \geq 13$ the structure of the failure set G_d appears to be very interesting: it seems that when d is large, the points in G_d form many short lexicographic intervals and one large lexicographic interval. For instance $|G_{20}| = 45$, so the proportion of the length of a typical interval on which $M_\alpha(\frac{1}{|\alpha|})$ is monotone is equal to $\frac{1}{45}$. Nevertheless, for the interval $((1,2^2,4,11),(2,5,13)]$, whose cardinality is equal to 151, one has $((1,2^2,4,11),(2,5,13)] \cap G_{20} = \{(2,5,13)\}$. The number of partitions of size 20 is 627, therefore there exists an interval on which $M_\alpha(\frac{1}{|\alpha|})$ is monotone and which is more than ten times longer than its expected length. This suggests that a weaker version of Conjecture 1 might be true. Let $\mathscr{P}_d$ denote the set of partitions of size d.

Question 2. Is it true that there exists constant $C > 0$ such that for every positive integer d there exists partitions $\alpha^d > \beta^d \in \mathscr{P}_d$ such that for every lexicographic sequence $\alpha^d \geq \alpha > \beta \geq \beta^d$ we have

$$M_\alpha\left(\frac{1}{d}\right) > M_\beta\left(\frac{1}{d}\right) \quad \text{and} \quad \frac{\left|[\alpha_d, \beta_d]\right|}{|\mathscr{P}_d|} \geq C?$$

We do not know the answer to this question and we leave it wide open. It would also be very interesting to find an explicit description of the set G_d, which appears to consists of very specific partitions which might be classifiable. Even though Conjecture 1 turned out to be false we believe that the research initiated by Procesi [5] on the behaviour of the function $M_\alpha(\frac{1}{|\alpha|})$ merits further investigation. Indeed, Procesi's work has added a new and largely unexplored dimension to Weingarten calculus.

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