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The Frobenius number associated with the number of representations for sequences of repunits

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Abstract. The generalized Frobenius number is the largest integer represented in at most p ways by a linear combination of nonnegative integers of given positive integers $a_1, a_2, \dots, a_k$. When $p = 0$, it reduces to the classical Frobenius number. In this paper, we give the generalized Frobenius number when $a_j = (b^{n+j-1} - 1)/(b-1)$ $(b \geq 2)$ as a generalization of the result of $p = 0$ in [16].

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1. Introduction

Let $a_1, a_2, \dots, a_k$ be positive integers such that their greatest common divisor is one. Denote the *Frobenius number* by $g(a_1, a_2, \dots, a_k)$, which is the largest integer not representable as a nonnegative integer combination of $a_1, a_2, \dots, a_k$.

On the other hand, to find the number of representations $d(n; a_1, a_2, \dots, a_k)$ to $a_1 x_1 + a_2 x_2 + \dots + a_k x_k = n$ for a given positive integer n is also one of the most important and interesting topics. This number is equal to the coefficient of x^n in $1/(1-x^{a_1})(1-x^{a_2})\cdots(1-x^{a_k})$. Sylvester [19] and Cayley [8] showed that $d(n; a_1, a_2, \dots, a_k)$ can be expressed as the sum of a polynomial in n of degree $k-1$ and a periodic function of period $a_1 a_2 \cdots a_k$. In [3], the explicit formula for the polynomial part is derived by using Bernoulli numbers. For two variables, a formula for $d(n; a_1, a_2)$ is obtained in [20]. For three variables in the pairwise coprime case $d(n; a_1, a_2, a_3)$, in [9], the periodic function part is expressed in terms of trigonometric functions. However, the calculation becomes very complicated for larger a_1, a_2, a_3. In [6], three variables case can be easily worked with in his formula using floor functions.

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In this paper, we are interested in one of the general types of Frobenius numbers, which focuses on the number of solutions. For a given nonnegative integer p, consider the largest integer $g_p(a_1,a_2,\dots,a_k)$ such that the number of expressions that can be represented by $a_1,a_2,\dots,a_k$ is at most p ways. That is, all integers larger than $g_p(a_1,a_2,\dots,a_k)$ have at least the number of representations of $p+1$ or more ways. When $p=0$, $g(a_1,a_2,\dots,a_k)=g_0(a_1,a_2,\dots,a_k)$ is the original Frobenius number. One can consider a slightly modified number, that is, the largest integer $g'_p(a_1,a_2,\dots,a_k)$ such that the number of expressions is exactly p ([4]). However, for some cases any positive integer does not have exactly p representations, in particular, when p becomes larger. In addition, $p_1<p_2$ does not necessarily imply $g'_{p_1}(a_1,a_2,\dots,a_k)<g'_{p_2}(a_1,a_2,\dots,a_k)$. Therefore, it is better to treat with $g_p(a_1,a_2,\dots,a_k)$. Of course, after knowing $g_p(a_1,a_2,\dots,a_k)$ and $g_{p-1}(a_1,a_2,\dots,a_k)$, we can also get $g'_p(a_1,a_2,\dots,a_k)$.

In the literature on the Frobenius problem, *Sylvester number* $n(a_1,a_2,\dots,a_k)$, which is the total number of nonrepresentable integers, also plays an important role as that of $g(a_1,a_2,\dots,a_k)$. This number is called *genus* of the set of representable integers. Similarly to the generalized Frobenius number, we can consider the generalized Sylvester number $n_p(a_1,a_2,\dots,a_k)$ as the cardinality of the set of integers which can be represented by $a_1,a_2,\dots,a_k$ at most p ways. When $p=0$, $n(a_1,a_2,\dots,a_k)=n_0(a_1,a_2,\dots,a_k)$ is the original Sylvester number. Recently, in [10] generalized Frobenius numbers (called p-Frobenius numbers) for sequences of triangular numbers are obtained.

In this paper, we mainly focus on repunit numbers. Generalized Frobenius numbers and Sylvester numbers for different numbers are treated in different papers by the author. In [16], explicit formulas of Frobenius and Sylvester numbers for repunits are found when $p=0$. More general numbers satisfying a linear recurrence relation are treated in [2]. Approaches from numerical semigroup are used in both papers.

2. Preliminaries

For two variables, it is very easy to see that

$$g_p(a_1,a_2)=(p+1)a_1a_2-a_1-a_2\,, \tag{1}$$

$$n_p(a_1,a_2)=\frac{1}{2}\left((2p+1)a_1a_2-a_1-a_2+1\right) \tag{2}$$

(see [4]). However, for three variables, no formula is known, not even for special triples. For $p=0$, some explicit forms have been discovered in some particular cases, including arithmetic, geometric-like, Fibonacci, Mersenne, and triangular (see [14, 15, 17] and references therein) and so on. However, for $p\ge 1$, no explicit form has been found even in these particular cases. In due course, by using our constructed framework, we can also find explicit forms of the total number of nonnegative integers that can only be expressed in at most p ways.

For a positive integer p and a set of positive integers $A=\{a_1,a_2,\dots,a_k\}$ with $\gcd(A)=1$, denote by $R_p(A)$ the set of all nonnegative integers whose representations in terms of $a_2,\dots,a_k$ with nonnegative integral coefficients have at least p ways. We introduce the p-Apéry set (see [1]) below in order to obtain the formulas for $g_p(A)$ and $n_p(A)$. Without loss of generality, we assume that $a_1=\min(A)$.

Definition 1. *For a set of positive integers* $A=\{a_1,a_2,\dots,a_k\}$ *with* $\gcd(A)=1$ *and* $a_1=\min(A)$ *we denote by*

$$\mathrm{Ap}_p(A)=\mathrm{Ap}_p(a_1,a_2,\dots,a_k)=\left\{m_0^{(p)},m_1^{(p)},\dots,m_{a_1-1}^{(p)}\right\},$$

the p-Apéry set of A, where $m_i^{(p)}$ is the least positive integer of $R_{p+1}(A)$ satisfying $m_i^{(p)}\equiv i \pmod{a_1}$ $(0\le i\le a_1-1)$. *Note that $m_0^{(0)}$ is defined to be* 0.

It follows that for each p,

$$\mathrm{Ap}_p(A) \equiv \{0, 1, \dots, a_1 - 1\} \pmod{a_1}.$$

It is hard to find any explicit form of $g_p(a_1, a_2, \dots, a_k)$ when $k \ge 3$. Nevertheless, the following convenient formulas are known (see [4, Lemma 2]). Though finding $m_j^{(p)}$ is enough hard in general, we can obtain it for some special sequences $(a_1, a_2, \dots, a_k)$. For $n_p(a_1, a_2, \dots, a_k)$, see [4, Lemma 3].

Lemma 2. *Let k and p be integers with $k \ge 2$ and $p \ge 0$. Assume that* $\gcd(a_1, a_2, \dots, a_k) = 1$. *We have*

$$g_p(a_1, a_2, \dots, a_k) = \max_{0 \le j \le a_1 - 1} m_j^{(p)} - a_1, \tag{3}$$

$$n_p(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1 - 1} m_j^{(p)} - \frac{a_1 - 1}{2}. \tag{4}$$

Remark 3. When $p = 0$, (3) is the formula by Brauer and Shockley [7]:

$$g(a_1, a_2, \dots, a_k) = \left(\max_{1 \le j \le a_1 - 1} m_j \right) - a_1, \tag{5}$$

where $m_j = m_j^{(0)}$ $(1 \le j \le a_1 - 1)$ with $m_0 = 0$. When $p = 0$, (4) is the formula by Selmer [18]:

$$n(a_1, a_2, \dots, a_k) = \frac{1}{a_1} \sum_{j=0}^{a_1 - 1} m_j - \frac{a_1 - 1}{2}. \tag{6}$$

More generalized formulas, including power sum and weighted sum, are given in [11].

3. Repunits

A repunit is a number consisting of copies of the single digit 1 ([5, Ch.11]). In general, for some integer $b \ge 2$, a repunit in base b is given by

$$\frac{b^n - 1}{b - 1} \quad (n \ge 1).$$

When $b = 2$, there are the Mersenne numbers ([17]), which have been studied extensively for hundreds of years.

For the set $A = \{a_1, a_2, \dots, a_k\}$, denote by $\langle A \rangle$ the set of all the representable elements in the linear combination of $a_1, a_2, \dots, a_k$ with nonnegative coefficients, that is,

$$\langle A \rangle = \left\{ a_1 x_1 + a_2 x_2 + \cdots + a_k x_k \middle| x_1, x_2, \dots, x_k \ge 0 \right\}.$$

In [16, Corollary 6], the minimal system of generators of the numerical semigroup $S(b, n)$ is given as the set

$$\left\{ \frac{b^n - 1}{b - 1}, \frac{b^{n+1} - 1}{b - 1}, \dots, \frac{b^{2n-1} - 1}{b - 1} \right\}.$$

Therefore, when considering only whether or not there is a representation (solution), it is the same for both finite and infinite elements. However, in our case, in order to consider the number of representations (solutions) concretely, for example, in the case of p or less, and $p + 1$ or more, even though they are overlapping elements, their presences definitely affect the count of numbers. In other words, the situation is different for an infinite number of elements and a finite number of elements. Therefore, from now on, in this paper, we will limit our consideration to finite elements

$$A_n := \{a_1, a_2, \dots, a_n\}. \tag{7}$$

by fixing

$$a_j = a_j(n) = \frac{b^{n+j-1}-1}{b-1} \quad (j = 1, 2, \dots, n). \tag{8}$$

In [16], explicit formulas for the case $p = 0$ are found.

Lemma 4. *We have*

$$g_0(A_n) = g_0\big(S(b,n)\big) = \frac{b^n(b^n-1)}{b-1} - 1,$$

$$n_0(A_n) = n_0\big(S(b,n)\big) = \frac{b^n}{2}\left(\frac{b^n-b}{b-1} + n - 1\right) \quad (n \ge 1).$$

When $n = 2$, by using (1), we can obtain the explicit formula:

$$g_p\big(A_2\big) = \frac{\big(b^2-1\big)\big(b^2 + p\big(b^2+b+1\big)\big)}{b-1} - 1.$$

In this paper, we shall prove the following explicit formula when $n \ge 3$ and $0 \le p \le b$.

Theorem 5. *Let $b \ge 2$. When $n \ge 3$, we have*

$$g_p\big(S(b,n)\big) = \frac{b^n\,(b^n-1) + p\big(b^{n+2}-1\big)}{b-1} - 1.$$

When $n = 3$ and $0 \le p \le b$, we also give the formula for generalized Sylvester number $n_p(A_3)$.

3.1. *Three variables*

In this section, we consider the case for three variables as $n = 3$ in (7) with (8). When $p = 0$, the Apéry set is given for any integer $n \ge 2$ ([16, Theorem 12]).

Lemma 6. *We have*

$$\mathrm{Ap}_0(a_1, \dots, a_n) = \left\{x_2 a_2 + \cdots + x_n a_n \,\middle|\, (x_2, \dots, x_n) \in R(b,n)\right\},$$

where $R(b,n)$ denotes the set of all $(n-1)$-tuple satisfying the following conditions:

(1) *for every $i = 2, \dots, n$, $x_i \in \{0, 1, \dots, b\}$;*
(2) *if $i = 3, \dots, n$ and $x_i = b$ then $x_2 = \cdots = x_{i-1} = 0$.*

Table 1. Complete residue system for repunit

$r_{0,0}$	$r_{1,0}$	$\cdots$	$r_{b,0}$
$r_{0,1}$	$r_{1,1}$	$\cdots$	$r_{b,1}$
$\vdots$			$\vdots$
$r_{0,b-1}$	$r_{1,b-1}$	$\cdots$	$r_{b,b-1}$
$r_{0,b}$			

In particular, for $n = 3$, the Apéry set for $p = 0$ is given as in Lemma 7. See Table 1. For convenience, put

$$r_{x_2,x_3} = x_2 a_2 + x_3 a_3, \tag{9}$$

where x_2 and x_3 are nonnegative integers.

When a_j ($j = 1, 2, 3$; $n = 3$) as triple in (8), the Apéry set for $p = 0$ is given as follows (see also [16, Corollary 14, Example 17]).

Lemma 7. *For any integer $n \ge 2$, we have*

$$\mathrm{Ap}_0(a_1, a_2, a_3) = \left\{r_{0,0}, \dots, r_{b,0}, r_{0,1}, \dots, r_{b,1}, \dots, r_{0,b-1}, \dots, r_{b,b-1}, r_{0,b}\right\}.$$

When $p \ge 1$, it is not easy to give the Apéry set accurately for any $n \ge 3$. However, if $n = 3$, we can determine the Apéry set. We focus on the numbers of representations, so we have to see how they are distributed.

Lemma 8. *The sequence of the number of the representations* $\{d(i; a_2, a_3)\}_{i\ge 0}$ *is given by*

$$\left\{\left\{\overbrace{\overbrace{\underbrace{m\dots m}_{k}\underbrace{m-1\dots m-1}_{a_2-k}}\dots\overbrace{\underbrace{m\dots m}_{k}\underbrace{m-1\dots m-1}_{a_2-k}}}^{b}\right\}_{k=1}^{a_2-1}\underbrace{m\dots m}_{(b+1)a_2}\right\}_{m=1}^{\infty}$$

$$=\overbrace{\overbrace{1\underbrace{0\dots0}_{a_2-1}}\dots\overbrace{1\underbrace{0\dots0}_{a_2-1}}}^{b}\overbrace{\overbrace{11\underbrace{0\dots0}_{a_2-2}}\dots\overbrace{11\underbrace{0\dots0}_{a_2-2}}}^{b}\cdots\overbrace{\overbrace{\underbrace{1\dots1}_{a_2-1}0}\dots\overbrace{\underbrace{1\dots1}_{a_2-1}0}}^{b}\underbrace{1\dots1}_{(b+1)a_2}$$

$$\overbrace{\overbrace{2\underbrace{1\dots1}_{a_2-1}}\dots\overbrace{2\underbrace{1\dots1}_{a_2-1}}}^{b}\overbrace{\overbrace{22\underbrace{1\dots1}_{a_2-2}}\dots\overbrace{22\underbrace{1\dots1}_{a_2-2}}}^{b}\cdots\overbrace{\overbrace{\underbrace{2\dots2}_{a_2-1}1}\dots\overbrace{\underbrace{2\dots2}_{a_2-1}1}}^{b}\underbrace{2\dots2}_{(b+1)a_2}$$

$$\overbrace{\overbrace{3\underbrace{2\dots2}_{a_2-1}}\dots\overbrace{3\underbrace{2\dots2}_{a_2-1}}}^{b}\cdots.$$

Proof. We have

$$\frac{1}{(1-x^{a_2})(1-x^{a_3})}=\sum_{i=1}^{\infty} d(i; a_2, a_3)x^i\,.$$

Since $\gcd(a_2, a_3) = 1$, the length of the period is $a_2a_3 = a_2(ba_2 + 1)$. □

From Lemma 8, it can be seen that the positions of 1 appear by ascending order $(x_2, x_3) = (0,0), \dots, (b-1,0), (b,0), (0,1), \dots, (2b-1,0), (b-1,1), (2b,0), (b,1), (0,2), \dots, (3b-1), (2b-1,1), (b-1,2), \dots, (b^2-1), \dots, (2b-1, b-2), (b-1, b-1), (b^2,0), \dots, (2b, b-2), (b, b-1), (0, b)$. However, since $r_{b+j,l} \equiv r_{j-1,l+1} \pmod{a_1}$ $(j \ge 1, l \ge 0)$, the least nonnegative value, whose number of representations equals 2, is selected as each element of the Apéry set when the remainders with respect to a_1 are equal. Hence, we have the Apéry set in Lemma 9.

When $p = 1$, the Apéry set is given as in Lemma 9. See Table 2. The largest element of the set is shown in a round circle.

Table 2. Two complete residue systems

$r_{0,0}$	$r_{1,0}$	$\cdots$	$r_{b,0}$	$r_{b+1,0}$	$r_{b+2,0}$	$\cdots$	$r_{2b+1,0}$
$r_{0,1}$	$r_{1,1}$	$\cdots$	$r_{b,1}$	$r_{b+1,1}$	$r_{b+2,1}$	$\cdots$	$r_{2b+1,1}$
$r_{0,2}$	$r_{1,2}$	$\cdots$	$r_{b,2}$	$\vdots$	$\vdots$		$\vdots$
$\vdots$	$\vdots$		$\vdots$	$r_{b+1,b-2}$	$r_{b+2,b-2}$	$\cdots$	$r_{2b+1,b-2}$
$r_{0,b-1}$	$r_{1,b-1}$	$\cdots$	$r_{b,b-1}$	$r_{b+1,b-1}$			
$r_{0,b}$	$r_{1,b}$	$\cdots$	$r_{b,b}$				
$r_{0,b+1}$							

Notice that some identities of $a_j = a_j(n)$ in (8) hold for any $n \ge 1$ but some others hold only for $n = 3$. Therefore, the main results hold only for $n = 3$.

Lemma 9. *Only for* $n = 3$*, we have*

$$\mathrm{Ap}_1(a_1, a_2, a_3) = \left\{r_{b+1,0}, \dots, r_{2b+1,0}, \dots, r_{b+1,b-2}, \dots, r_{2b+1,b-2}, r_{b+1,b-1}, r_{1,b}, \dots, r_{b,b}, r_{0,b+1}\right\}.$$

By the setting in (8) and (9), it is easy to find the following identities.

Sublemma 10. *For $n=3$, we have*

$$(b+1)a_3=\left(b^3+1\right)a_1+ba_2\,, \tag{10}$$

$$a_2+ba_3=\left(b^3+b+1\right)a_1\,, \tag{11}$$

$$\left(b^3+b+1\right)a_1+r_{x_2,0}=r_{x_2+1,b}\,, \tag{12}$$

$$\left(b^3+1\right)a_1+r_{b,0}=r_{0,b+1}\,. \tag{13}$$

For $n\ge 1$, we have

$$ba_1+r_{x_2,x_3}=r_{x_2+b+1,x_3-1}\,, \tag{14}$$

$$ba_1+r_{0,b}=r_{b+1,b-1}\,. \tag{15}$$

Proof of Lemma 9. By (10) and (11), for $n=3$ there is a different representation for the same integer. By (14), for $n\ge 1$ we have

$$r_{x_2,x_3}\equiv r_{x_2+b+1,x_3-1}\pmod{a_1}\quad(0\le x_2\le b,\,1\le x_3\le b-1)\,.$$

And by (15), for $n\ge 3$ we have

$$r_{0,b}\equiv r_{b+1,b-1}\pmod{a_1}\,.$$

This explains the situation where the leftmost block, except for the first row, is shifted to the second block as it is to the upper right by adding ba_1, as the part that is congruent modulo a_1. Only the first row (shown with a shadow in Table 2) is ordered from the end of the first block as the remainder modulo a_1, and the last one is at the bottom left. Namely, for $x_2=0,1,\dots,b-1$, by (12) for $n=3$ we have

$$r_{x_2,0}\equiv r_{x_2+1,b}\pmod{a_1}\,.$$

And by (13) for $n=3$ we have

$$r_{b,0}\equiv r_{0,b+1}\pmod{a_1}\,.$$

Therefore, all the elements in $\mathrm{Ap}_1(a_1,a_2,a_3)$ have exactly two different expressions each, and any such an element minus a_1 does not have two different expressions. □

When $p=2$, the Apéry set is given as in Lemma 11. See Table 3.

Lemma 11. *For $n=3$ we have*

$$\mathrm{Ap}_2(a_1,a_2,a_3)=\{r_{2b+2,0},\dots,r_{3b+2,0},\dots,r_{2b+2,b-3},\dots,r_{3b+2,b-3},r_{2b+2,b-2},\dots$$
$$r_{b+2,b-1},\dots,r_{2b+1,b-1},r_{b+1,b},r_{1,b+1},\dots,r_{b,b+1},r_{0,b+2}\}\,.$$

In addition to the identities in Sublemma 10, it is convenient to see the following.

Sublemma 12. *For $n=3$, we have*

$$\left(b^3+2b+1\right)a_1+r_{x_2,0}=ba_1+r_{1+x_2,b}=r_{b+2+x_2,b-1}\,, \tag{16}$$

$$\left(b^3+b+1\right)a_1+r_{b,0}=ba_1+r_{0,b+1}=r_{b+1,b}\,, \tag{17}$$

$$\left(b^3+b+1\right)a_1+r_{x_2,1}=r_{x_2+1,b+1}\,, \tag{18}$$

$$\left(b^3+1\right)a_1+r_{b,1}=r_{0,b+2}\,. \tag{19}$$

For $n\ge 1$, we have

$$2ba_1+r_{x_2,x_3}=ba_1+r_{b+1+x_2,x_3-1}=r_{2b+2+x_2,x_3-2}\,, \tag{20}$$

$$ba_1+r_{0,b}=r_{b+1,b-1}\,. \tag{21}$$

$$\left(b^3+b+1\right)a_1+r_{x_2,1}=\left(b^3+1\right)a_1+r_{b+1+x_2,0}\,, \tag{22}$$

$$\left(b^3+1\right)a_1+r_{b,1}=\left(b^3-b+1\right)a_1+r_{2b+1,0}\,. \tag{23}$$

Proof of Lemma 11. By (10), (11) and (20), for $0 \le x_2 \le b$ and $2 \le x_3 \le b-1$ or $(x_2, x_3) = (0, b)$, for $n \ge 1$ we have

$$r_{x_2,x_3} \equiv r_{b+1+x_2,x_3-1} \equiv r_{2b+2+x_2,x_3-2} \pmod{a_1}.$$

By (16) for $0 \le x_2 \le b-1$, for $n = 3$ we have

$$r_{x_2,0} \equiv r_{1+x_2,b} \equiv r_{b+2+x_2,b-1} \pmod{a_1}.$$

And by (17), for $n = 3$ we have

$$r_{b,0} \equiv r_{0,b+1} \equiv r_{b+1,b} \pmod{a_1}.$$

By (22) and (18) for $0 \le x_2 \le b-1$, only for $n = 3$ we have

$$r_{x_2,1} \equiv r_{b+1+x_2,0} \equiv r_{x_2+1,b+1} \pmod{a_1}.$$

And by (23) and (19), only for $n = 3$ we have

$$r_{b,1} \equiv r_{2b+1,0} \equiv r_{0,b+2} \pmod{a_1}.$$

Therefore, all the elements in $\mathrm{Ap}_2(a_1, a_2, a_3)$ have exactly three different expressions each, and any such an element minus a_1 does not have three different expressions. □

Table 3. Three complete residue systems

$r_{0,0}$	$r_{1,0}$	$\cdots$	$r_{b,0}$	$r_{b+1,0}$	$r_{b+2,0}$	$\cdots$	$r_{2b+1,0}$	$r_{2b+2,0}$	$r_{2b+3,0}$	$\cdots$	$r_{3b+2,0}$
$r_{0,1}$	$r_{1,1}$	$\cdots$	$r_{b,1}$	$r_{b+1,1}$	$r_{b+2,1}$	$\cdots$	$r_{2b+1,1}$	$\vdots$	$\vdots$		$\vdots$
$r_{0,2}$	$r_{1,2}$	$\cdots$	$r_{b,2}$	$\vdots$	$\vdots$		$\vdots$	$r_{2b+2,b-3}$	$r_{2b+3,b-3}$	$\cdots$	$r_{3b+2,b-3}$
$\vdots$	$\vdots$		$\vdots$	$r_{b+1,b-2}$	$r_{b+2,b-2}$	$\cdots$	$r_{2b+1,b-2}$	$r_{2b+2,b-2}$			
$r_{0,b-1}$	$r_{1,b-1}$	$\cdots$	$r_{b,b-1}$	$r_{b+1,b-1}$	$r_{b+2,b-1}$	$\cdots$	$r_{2b+1,b-1}$				
$\boxed{r_{0,b}}$	$r_{1,b}$	$\cdots$	$r_{b,b}$	$r_{b+1,b}$							
$\boxed{r_{0,b+1}}$	$r_{1,b+1}$	$\cdots$	$r_{b,b+1}$								
$\boxed{r_{0,b+2}}$											

1st block	2nd block	3rd block
$\underbrace{\qquad}_{b+1}$	$\underbrace{\qquad}_{b+1}$	$\underbrace{\qquad}_{b+1}$

In Table 3, the one consisting of $b+1$ columns is regarded as one block, and the whole is divided into three blocks.

In general, for any nonnegative integer p with $p \le b$, the corresponding complete residue system is given as follows. See Table 4.

Lemma 13. *For $p \le b$, we have*

$$\begin{aligned}\mathrm{Ap}_p(a_1, a_2, a_3) = \{&r_{pb+p,0}, \dots, r_{(p+1)b+p,0}, \dots, r_{pb+p,b-p-1}, \dots, r_{(p+1)b+p,b-p-1}, r_{pb+p,b-p},\\ &r_{(p-1)b+p,b-p+1}, \dots, r_{pb+p-1,b-p+1}, r_{(p-1)b+p-1,b-p+2},\\ &r_{(p-2)b+p-1,b-p+3}, \dots, r_{(p-1)b+p-2,b-p+3}, r_{(p-2)b+p-2,b-p+4},\\ &\dots,\\ &r_{b+2,b+p-3}, \dots, r_{2b+1,b+p-3}, r_{b+1,b+p-2}, r_{1,b+p-1}, \dots, r_{b,b+p-1}, r_{0,b+p}\}.\end{aligned}$$

Proof. From the first block, we get the elements $r_{1,b+p-1},\dots,r_{b,b+p-1}$ and $r_{0,b+p}$. From the second block, we get the elements $r_{b+2,b+p-3},\dots,r_{2b+1,b+p-3}$ and $r_{b+1,b+p-2}$. Then, from the $(p-1)^{\text{th}}$ block, we get the elements $r_{(p-2)b+p-1,b-p+3},\dots,r_{(p-1)b+p-2,b-p+3}$ and $r_{(p-2)b+p-2,b-p+4}$. From the p^{th} block, we get the elements $r_{(p-1)b+p,b-p+1},\dots,r_{pb+p-1,b-p+1}$ and $r_{(p-1)b+p-1,b-p+2}$. Finally, from the $(p+1)^{\text{th}}$ block, we get the elements $r_{pb+p,0},\dots,r_{(p+1)b+p,0},\dots,r_{pb+p,b-p-1},\dots,r_{(p+1)b+p,b-p-1}$ and $r_{pb+p,b-p}$.

More precisely, each corresponding element modulo a_1 in $\mathrm{Ap}_p(a_1,a_2,a_3)$, $\mathrm{Ap}_{p-1}(a_1,a_2,a_3)$, $\dots$, $\mathrm{Ap}_0(a_1,a_2,a_3)$ are given as follows.

$$
\begin{aligned}
r_{pb+p+x_2,x_3}&\equiv r_{(p-1)b+(p-1)+x_2,x_3+1}\equiv r_{(p-2)b+(p-2)+x_2,x_3+2}\\
&\equiv\cdots\equiv r_{x_2,x_3+p}\quad\bigl(0\le x_2\le b,\ 0\le x_3\le b-p-1\quad\text{or}\quad (x_2,x_3)=(0,b-p)\bigr)\,,\\
r_{(p-1)b+p+x_2,b-p+1}&\equiv r_{(p-2)b+p-1+x_2,b-p+2}\equiv r_{(p-3)b+p-2+x_2,b-p+3}\\
&\equiv\cdots\equiv r_{1+x_2,b}\equiv r_{x_2,0}\quad(0\le x_2\le b-1)\,,\\
r_{(p-1)b+p-1,b-p+2}&\equiv r_{(p-2)b+p-2,b-p+3}\equiv\cdots\equiv r_{0,b+1}\equiv r_{b,0}\,,\\
r_{(p-2)b+p+x_2,b-p+3}&\equiv r_{(p-3)b+p-2+x_2,b-p+4}\equiv\cdots\equiv r_{1+x_2,b+1}\\
&\equiv r_{b+1+x_2,0}\equiv r_{x_2,1}\quad(0\le x_2\le b-1)\,,\\
r_{(p-2)b+p-2,b-p+4}&\equiv r_{(p-3)b+p-3,b-p+5}\equiv\cdots\equiv r_{0,b+2}\equiv r_{2b+1,0}\equiv r_{b,1}\,,\\
&\cdots\cdots\\
r_{b+2+x_2,b+p-3}&\equiv r_{1+x_2,b+p-2}\equiv r_{(p-2)b+p-2+x_2,0}\equiv r_{(p-3)b+p-3+x_2,1}\\
&\equiv\cdots\equiv r_{x_2,p-2}\,,\\
r_{b+1,b+p-2}&\equiv r_{0,b+p-1}\equiv r_{(p-1)b+p-2,0}\equiv r_{(p-2)b+p-3,1}\\
&\equiv\cdots\equiv r_{b,p-2}\,,\\
r_{1+x_2,b+p-1}&\equiv r_{(p-1)b+p-1+x_2,0}\equiv r_{(p-2)b+p-2+x_2,1}\\
&\equiv\cdots\equiv r_{x_2,p-1}\,,\\
r_{0,b+p}&\equiv r_{pb+p-1,0}\equiv r_{(p-1)b+p-2,1}\equiv\cdots\equiv r_{b,p-1}\pmod{a_1}\,.
\end{aligned}
$$

Note that in each congruence sequence, the elements are strictly decreasing.

Each element in $\mathrm{Ap}_p(a_1,a_2,a_3)$ has $p+1$ different expressions for (x_1,x_2,x_3) as follows. Since for $n\ge 1$

$$
ba_1+a_3=(b+1)a_2\,,\tag{24}
$$

we get

$$
\begin{aligned}
r_{pb+p+x_2,x_3}&=ba_1+r_{(p-1)b+p-1+x_2,1+x_3}=2ba_1+r_{(p-2)b+p-2+x_2,2+x_3}\\
&=\cdots=pba_1+r_{x_2,p+x_3}\bigl(0\le x_2\le b,\ 0\le x_3\le b-p-1\quad\text{or}\quad (x_2,x_3)=(0,b-p)\bigr)\,.
\end{aligned}
$$

By (10) and (11), we get

$$
\begin{aligned}
r_{(p-1)b+p+x_2,b-p+1}&=ba_1+r_{(p-2)b+p-1+x_2,b-p+2}=2ba_1+r_{(p-3)b+p-2+x_2,b-p+3}\\
&=\cdots=(p-1)ba_1+r_{1+x_2,b}=\bigl(b^3+pb+1\bigr)a_1+r_{x_2,0}\,(0\le x_2\le b-1)\,,\\
r_{(p-1)b+p-1,b-p+2}&=ba_1+r_{(p-2)b+p-2,b-p+3}=2ba_1+r_{(p-3)b+p-3,b-p+4}\\
&=\cdots=(p-1)ba_1+r_{0,b+1}=\bigl(b^3+(p-2)b+1\bigr)a_1+r_{b,0}\,,\\
r_{(p-2)b+p-1+x_2,b-p+3}&=ba_1+r_{(p-3)b+p-2+x_2,b-p+4}=2ba_1+r_{(p-4)b+p-3+x_2,b-p+5}\\
&=\cdots=(p-2)ba_1+r_{1+x_2,b+1}\\
&=\bigl(b^3+(p-2)b+1\bigr)a_1+r_{b+1+x_2,0}\\
&=\bigl(b^3+(p-1)b+1\bigr)a_1+r_{x_2,1}\quad(0\le x_2\le b-1)\,,\\
r_{(p-2)b+p-2,b-p+4}&=ba_1+r_{(p-3)b+p-3,b-p+5}=2ba_1+r_{(p-4)b+p-4,b-p+6}
\end{aligned}
$$

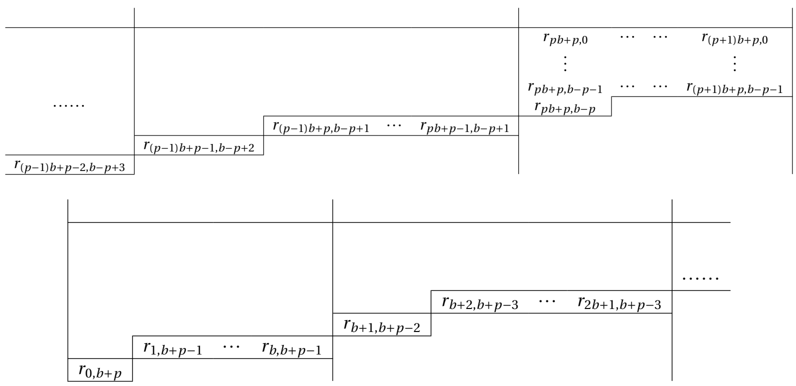
$$
\begin{aligned}
&= \cdots = (p-2)ba_1 + r_{0,b+2} = \left(b^3 + (p-3)b + 1\right) a_1 + r_{2b+1,0}\\
&= \left(b^3 + (p-2)b + 1\right) a_1 + r_{b,1}\,.
\end{aligned}
$$

By (24) again, we get

$$
\begin{aligned}
r_{b+2+x_2,b+p-3} &= ba_1 + r_{1+x_2,b+p-2} = \left(b^3 - (p-4)b + 1\right) a_1 + r_{(p-2)b+p-2+x_2,0}\\
&= \left(b^3 - (p-5)b + 1\right) a_1 + r_{(p-3)b+p-3+x_2,1} = \cdots\\
&= \left(b^3 + 2b + 1\right) a_1 + r_{x_2,p-2}\ (0 \le x_2 \le b-1)\\
r_{b+1,b+p-2} &= ba_1 + r_{0,b+p-1} = \left(b^3 - (p-3)b + 1\right) a_1 + r_{(p-1)b+p-2,0}\\
&= \left(b^3 - (p-4)b + 1\right) a_1 + r_{(p-2)b+p-3,1}\\
&= \cdots = \left(b^3 + b + 1\right) a_1 + r_{b,p-2}\,,\\
r_{1+x_2,b+p-1} &= \left(b^3 - (p-2)b + 1\right) a_1 + r_{(p-1)b+p-1+x_2,0}\\
&= \left(b^3 - (p-3)b + 1\right) a_1 + r_{(p-2)b+p-2+x_2,1}\\
&= \cdots = \left(b^3 + b + 1\right) a_1 + r_{x_2,p-1}\ (0 \le x_2 \le b-1)\\
r_{0,b+p} &= \left(b^3 - (p-1)b + 1\right) a_1 + r_{pb+p-1,0}\\
&= \left(b^3 - (p-2)b + 1\right) a_1 + r_{(p-1)b+p-2,1}\\
&= \cdots = \left(b^3 + 1\right) a_1 + r_{b,p-1}\,.
\end{aligned}
$$

□

Table 4. $\mathrm{Ap}_p(a_1,a_2,a_3)$



Theorem 14. *Let $b \ge 2$. Then for $0 \le p \le b$, we have*

$$
\begin{aligned}
g_p(a_1,a_2,a_3) &= \frac{b^3\left(b^3-1\right) + p\left(b^5-1\right)}{b-1} - 1\,,\\
n_p(a_1,a_2,a_3) &= \frac{b^3}{2}\left(\frac{b^3-b}{b-1} + 2\right) + \frac{p(b+1)}{2}\left(2b^3 + 2b^2 - (p-1)b + 2\right).
\end{aligned}
$$

Proof. The maximum element in $\mathrm{Ap}_p(a_1,a_2,a_3)$ is $r_{0,b+p}$. Thus, by Lemma 2 (3), we have

$$
\begin{aligned}
g_p(a_1,a_2,a_3) &= (b+p)a_3 - a_1\\
&= b^3 a_1 + pa_3 - 1\\
&= \frac{b^3\left(b^3-1\right) + p\left(b^5-1\right)}{b-1} - 1\,.
\end{aligned}
$$

We consider the sum of the elements in $\mathrm{Ap}_p(a_1,a_2,a_3)$. For the coefficients of a_2, we have

$$
\begin{aligned}
&(b-p)\left(\frac{\big((p+1)b+p\big)\big((p+1)b+p+1\big)}{2}-\frac{(pb+p-1)(pb+p)}{2}\right)\\
&\quad+(pb+p)+\left(\frac{(pb+p-1)(pb+p)}{2}-\frac{\big((p-1)b+p-1\big)\big((p-1)b+p\big)}{2}\right)\\
&\quad+\big((p-1)b+p-1\big)+\left(\frac{\big((p-1)b+p-1\big)\big((p-1)b+p\big)}{2}-\frac{\big((p-2)b+p-2\big)\big((p-2)b+p-1\big)}{2}\right)\\
&\quad+\big((p-2)b+p-2\big)+\cdots+\left(\frac{(2b+1)(2b+2)}{2}-\frac{b(b+1)}{2}\right)+(b+1)+\frac{b(b+1)}{2}\\
&\quad=\frac{1}{2}(b+1)\big((2p+1)b^2-p(p-1)b-p(p-1)\big)\,.
\end{aligned}
$$

For the coefficients of a_3, we have

$$
\begin{aligned}
(b+1)\frac{(b-p-1)(b-p)}{2}&+(b-p)+b(b-p+1)+(b-p+2)+b(b-p+3)\\
&+\cdots+b(b+p-3)+(b+p-2)+b(b+p-1)+(b+p)\\
&=(b+1)\frac{(b-p-1)(b-p)}{2}+pb^2+(p+1)b\\
&=\frac{1}{2}\left(b^3+\left(p^2+p+1\right)b+\left(p^2+p\right)\right).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sum_{j=1}^{a_1}m_j^{(p)}&=\frac{1}{2}(b+1)\big((2p+1)b^2-p(p-1)b-p(p-1)\big)a_2+\frac{1}{2}\left(b^3+\left(p^2+p+1\right)b+\left(p^2+p\right)\right)a_3\\
&=\frac{1}{2}\frac{b^3-1}{b-1}\left(b^5+(2p+1)b^4+2(2p+1)b^3-\left(p^2-3p-1\right)b^2-\left(p^2-3p-1\right)b+2p\right).
\end{aligned}
$$

Therefore, by Lemma 2 (4), we have

$$
\begin{aligned}
n_p(a_1,a_2,a_3)&=\frac{1}{a_1}\sum_{j=1}^{a_1}m_j^{(p)}-\frac{a_1-1}{2}\\
&=\frac{1}{2}\left(b^5+(2p+1)b^4+2(2p+1)b^3-p(p-3)b^2-p(p-3)b+2p\right)\\
&=\frac{b^3}{2}\left(\frac{b^3-b}{b-1}+2\right)+\frac{p(b+1)}{2}\left(2b^3+2b^2-(p-1)b+2\right).
\end{aligned}
$$

□

3.2. *The case $p>b$*

When $p>b$, it is not easy to give a general formula. As the value of p increases, it becomes more and more complicated to accurately determine the Apéry set. Nevertheless, as long as the value of p is small, it is possible to determine $\mathrm{Ap}_p(A)$, $\mathrm{Ap}_{p+1}(A)$ and so on from $\mathrm{Ap}_{p-1}(A)$ ($p-1=b$). Here, for simplicity, put $A=A_3=\{a_1,a_2,a_3\}$.

All $\mathrm{Ap}_j(A)$ ($j\ge 0$) elements except the element in the top row shift up one row to the block on the right and move as is. These are the elements of $\mathrm{Ap}_{j+1}(A)$ that are equal modulo a_1 respectively. Only the element of $\mathrm{Ap}_j(A)$ in the top row corresponds to the leftmost block in order from the continuation. This procedure is repeated as $\mathrm{Ap}_{j+2}(A)$, $\mathrm{Ap}_{j+3}(A)$ and so on.

For example, let $b=2$. The position of $\mathrm{Ap}_j(A)$ is denoted by ⓙ as shown in Table 6.

We have

$$
\mathrm{Ap}_0(A)=\left\{r_{0,0},r_{1,0},r_{2,0},r_{0,1},r_{1,1},r_{2,1},r_{0,2}\right\},
$$

Table 5. $\mathrm{Ap}_j(A)$ for $p > b$

$r_{0,0}$	$r_{1,0}$	$r_{2,0}$	$r_{3,0}$	$r_{4,0}$	$r_{5,0}$	$r_{6,0}$	$r_{7,0}$	$r_{8,0}$	$r_{9,0}$	$r_{10,0}$	$r_{11,0}$
$r_{0,1}$	$r_{1,1}$	$r_{2,1}$	$r_{3,1}$	$r_{4,1}$	$r_{5,1}$	$r_{6,1}$	$r_{7,1}$	$r_{8,1}$	$r_{9,1}$		
$r_{0,2}$	$r_{1,2}$	$r_{2,2}$	$r_{3,2}$	$r_{4,2}$	$r_{5,2}$	$r_{6,2}$	$r_{7,2}$				
$r_{0,3}$	$r_{1,3}$	$r_{2,3}$	$r_{3,3}$	$r_{4,3}$	$r_{5,3}$						
$r_{0,4}$	$r_{1,4}$	$r_{2,4}$	$r_{3,4}$								
$r_{0,5}$	$r_{1,5}$										

Table 6. The position of $\mathrm{Ap}_j(A)$

			①			②	③	④	⑤
⓪				②		③	④	⑤	
	①		②	③		④	⑤		
①	②		③	④	⑤				
②	③	④	⑤						
④	⑤								

$$
\begin{aligned}
\mathrm{Ap}_1(A) &= \{r_{1,2}, r_{2,2}, r_{0,3}, r_{3,0}, r_{4,0}, r_{5,0}, r_{3,1}\},\\
\mathrm{Ap}_2(A) &= \{r_{4,1}, r_{5,1}, r_{3,2}, r_{1,3}, r_{2,3}, r_{0,4}, r_{6,0}\},\\
\mathrm{Ap}_3(A) &= \{r_{7,0}, r_{8,0}, r_{6,1}, r_{4,2}, r_{5,2}, r_{3,3}, r_{1,4}\},\\
\mathrm{Ap}_4(A) &= \{r_{2,4}, r_{0,5}, r_{9,0}, r_{7,1}, r_{8,1}, r_{6,2}, r_{4,3}\},\\
\mathrm{Ap}_5(A) &= \{r_{5,3}, r_{3,4}, r_{1,5}, r_{10,0}, r_{11,0}, r_{9,1}, r_{7,2}\},\\
\mathrm{Ap}_6(A) &= \{r_{8,2}, r_{6,3}, r_{4,4}, r_{2,5}, r_{0,6}, r_{12,0}, r_{10,1}\},\\
\mathrm{Ap}_7(A) &= \{r_{11,1}, r_{9,2}, r_{7,3}, r_{5,4}, r_{3,5}, r_{1,6}, r_{13,0}\},\\
\mathrm{Ap}_8(A) &= \{r_{14,0}, r_{12,1}, r_{10,2}, r_{8,3}, r_{6,4}, r_{4,5}, r_{2,6}\},\\
\mathrm{Ap}_9(A) &= \{r_{0,7}, r_{15,0}, r_{13,1}, r_{11,2}, r_{9,3}, r_{7,4}, r_{5,5}\},\\
\mathrm{Ap}_{10}(A) &= \{r_{3,6}, r_{1,7}, r_{16,0}, r_{14,1}, r_{12,2}, r_{10,3}, r_{8,4}\}.
\end{aligned}
$$

All the above sets are congruent to $\{0,1,2,3,4,5,6\} \pmod{a_1}$ ($a_1 = 2^3 - 1 = 7$), with the elements in order. Finding the maximum element, we obtain that

$$
\begin{aligned}
g_0(A) &= r_{0,2} - a_1 = 62 - 7 = 55,\\
g_1(A) &= r_{0,3} - a_1 = 93 - 7 = 86,\\
g_2(A) &= r_{0,4} - a_1 = 124 - 7 = 117,\\
g_3(A) &= r_{1,4} - a_1 = 139 - 7 = 132,\\
g_4(A) &= r_{0,5} - a_1 = 155 - 7 = 148,\\
g_5(A) &= r_{1,5} - a_1 = 170 - 7 = 163,\\
g_6(A) &= r_{0,6} - a_1 = 186 - 7 = 179,\\
g_7(A) &= r_{1,6} - a_1 = 201 - 7 = 194,\\
g_8(A) &= r_{2,6} - a_1 = 209 - 7 = 202,
\end{aligned}
$$

$$g_9(A) = r_{5,5} - a_1 = 230 - 7 = 223\,,$$
$$g_{10}(A) = r_{8,4} - a_1 = 244 - 7 = 237\,.$$

Note that g_0, g_1, g_2 can be yielded from Theorem 14.

However, from the value when $p = 11$, the correspondence rules mentioned above begin to collapse. Namely, according to the rule, it is expected that

$$\mathrm{Ap}_{11}(A) = \left\{r_{6,5}, r_{4,6}, r_{2,7}, \mathbf{r}_{17,0}, r_{15,1}, r_{13,2}, r_{11,3}\right\}\,,$$
$$\mathrm{Ap}_{12}(A) = \left\{r_{9,4}, r_{7,5}, r_{5,6}, \mathbf{r}_{0,8}, \mathbf{r}_{18,0}, r_{16,1}, r_{14,2}\right\}\,,$$

and so on. However, in fact,

$$\mathrm{Ap}_{11}(A) = \left\{r_{6,5}, r_{4,6}, r_{2,7}, \mathbf{r}_{0,8}, r_{15,1}, r_{13,2}, r_{11,3}\right\}\,,$$
$$\mathrm{Ap}_{12}(A) = \left\{r_{9,4}, r_{7,5}, r_{5,6}, \mathbf{r}_{17,0}, \mathbf{r}_{1,8}, r_{16,1}, r_{14,2}\right\}\,,$$

and so on. That is, the elements in bold letters actually appear interchangeably.

4. Four variables

The situation becomes more complicated for four variables, because there are many overlapped elements in $\mathrm{Ap}_j(A_4)$ and in $\mathrm{Ap}_{j+1}(A_4)$ and even more. That is, there exist i such that $m_j^{(i)} = m_{j+1}^{(i)}$.

In this section, we consider the case for four variables as $n = 4$ in (7) with (8).

Put $r_{x_2,x_3,x_4} = x_2 a_2 + x_3 a_3 + x_4 a_4$, where x_2, x_3 and x_4 are nonnegative integers. We illustrate the case where $b = 3$. In Table 7, denote r_{x_2,x_3,x_4} simply by (x_2, x_3, x_4), and the integers are modulo $a_1 = 40$.

As in Table 7, the elements in $\mathrm{Ap}_0(A_4)$ are divided into $b+1$ parts (blocks). The first b parts (blocks) have the $(b+1)b+1$ elements and the last part (block) has only one element, so that $\#\mathrm{Ap}_0(A_4) = b((b+1)b+1)+1 = (b^4-1)/(b-1)$.

As seen in Table 7, the values that take $x_4 = 0$ start from the first line, and continues to $x_3 = 0,1,2,3,4,\dots$ and the following lines. The values that take $x_4 = 1$ start from the $(b+1)^{\mathrm{th}}$ line, and continues to $x_3 = 0,1,2,3,4,\dots$ and the following lines. The valued that take $x_4 = 2$ start from the $(2b+2)^{\mathrm{th}}$ line, and continues to $x_3 = 0,1,2,3,4,\dots$ and the following lines. Finally, there is the value that take $x_4 = b$. Therefore, there are overlapped elements (shown as shaded cells) as $r_{0,b+1,0} = r_{b,0,1}, \dots, r_{0,2b,0} = r_{b,b-1,1}$, and $r_{0,b+1,1} = r_{b,0,2}, \dots, r_{0,2b,1} = r_{b,b-1,2}$ and so on.

Then as seen in Table 8, from $\mathrm{Ap}_0(A_4)$ to $\mathrm{Ap}_1(A_4)$, the corresponding element with the same modulo moves to the next right block to fill the gap. Only the elements in the first line go down to the last lower left block. However, unlike the case of $n = 3$, the elements in the shaded cell do not move. As mentioned above, they have the same value, but they have different expressions in terms of $a_2, \dots, a_k$.

The situation is similar for the case from $\mathrm{Ap}_1(A_4)$ to $\mathrm{Ap}_2(A_4)$. As in Table 9, the corresponding element with the same modulo moves to the next right block to fill the gap. Only the elements in the first line go down to the last lower left block. However, the elements in the shaded cell do not move.

Such a situation continues as long as $p \le b$. When $p = b+1$ and p becomes even larger, the element of $\mathrm{Ap}_0(A_4)$ that keeps moving to the upper right disappears, so the regularity is gradually lost and the situation becomes more difficult and complicated.

Therefore, the maximum element of $\mathrm{Ap}_p(A_4)$ is given by

$$r_{0,p,b} = \frac{p\left(b^6-1\right)}{b-1} + \frac{b\left(b^7-1\right)}{b-1} \equiv p(b+1) - 1\left(\left(b^4-1\right)/(b-1)\right)\,.$$

Hence, by Lemma 2 (3), for $b \ge 2$ and $0 \le p \le b$, we have

$$g_p(A_4) = r_{0,p,b} - \left(b^4-1\right)/(b-1)$$

Table 7. Complete residue system $\text{Ap}_0(A_4)$ for $n = 4$ and $b = 3$

									(0,0,0)	(1,0,0)	(2,0,0)	(3,0,0)
									(0,1,0)	(1,1,0)	(2,1,0)	(3,1,0)
									(0,2,0)	(1,2,0)	(2,2,0)	(3,2,0)
									(0,3,0)			
						(0,0,1)	(1,0,1)	(2,0,1)	(3,0,1)			
						(0,1,1)	(1,1,1)	(2,1,1)	(3,1,1)			
						(0,2,1)	(1,2,1)	(2,2,1)	(3,2,1)			
						(0,3,1)						
			(0,0,2)	(1,0,2)	(2,0,2)	(3,0,2)						
			(0,1,2)	(1,1,2)	(2,1,2)	(3,1,2)						
			(0,2,2)	(1,2,2)	(2,2,2)	(3,2,2)						
			(0,3,2)									
(0,0,3)												

									0	1	2	3
									4	5	6	7
									8	9	10	11
									12			
						13	14	15	16			
						17	18	19	20			
						21	22	23	24			
						25						
			26	27	28	29						
			30	31	32	33						
			34	35	36	37						
			38									
39												

Table 8. Complete residue system $\text{Ap}_1(A_4)$ for $n = 4$ and $b = 3$

													4	5	6	7
													8	9	10	11
													12			
										13	14	15				
									16	17	18	19				
									20	21	22	23				
									24	25						
							26	27	28							
						29	30	31	32							
						33	34	35	36							
						37	38									
				39												
	0	1	2													
3																

Table 9. Complete residue system $\text{Ap}_2(A_4)$ for $n = 4$ and $b = 3$

																	8	9	10	11
																	12			
														13	14	15				
													16							
										17	18	19	20							
										21	22	23	24							
										25	26	27								
									28	29										
							30	31	32	33										
							34	35	36	37										
							38	39												
					0	1														
			2	3																
	4	5	6																	
7																				

$$= \frac{b^4\left(b^4-1\right)+p\left(b^6-1\right)}{b-1}-1\,.$$

Table 10. Complete residue system $\mathrm{Ap}_3(A_4)$ for $b=3$

																					12
																		13	14	15	
																	16				
														17	18	19					
													20	21	22	23					
													24	25							
											26	27	28								
										29	30	31									
									32	33	34	35									
									36	37	38										
								39	0												
						1	2														
				3	4	5															
			6	7																	
	8	9	10																		
11																					

5. The cases $n \geq 5$

When $n \geq 5$, the exact structure of $\mathrm{Ap}_p(A_n)$, where A_n with a_j $(j = 1, 2, \dots, n)$ is given in (7) with (8), becomes more complicated. Nevertheless, it is still possible to prove that for $0 \leq p \leq b$

$$g_p(A_n) = \frac{b^n(b^n - 1) + p\left(b^{n+2} - 1\right)}{b-1} - 1\,.$$

When $n = 5$, put $r_{x_2,x_3,x_4,x_5} = x_2a_2 + x_3a_3 + x_4a_4 + x_5a_5$, where x_2, x_3, x_4 and x_5 are nonnegative integers. Or, denote r_{x_2,x_3,x_4,x_5} simply by (x_2, x_3, x_4, x_5).

As in Table 7, the whole array is regarded as one group, the one with 0 added at the end, $(x_2, x_3, x_4, 0)$, is the first block, and the one with 1 added at the end, $(x_2, x_3, x_4, 1)$, is the next block, and the first column of block 0 and the last column of block 1 are overlapped. In addition, place block 1 just below block 0. Similarly, block 2, …, block $(b-1)$ are placed one on top of the other, assuming that the elements are $(x_2, x_3, x_4, 2), \dots, (x_2, x_3, x_4, b-1)$, respectively, and the last element $(0, 0, 0, b)$ is placed at the bottom left of the end. So, $\#\mathrm{Ap}_0(A_5) = bb((b+1)b+1)+1)+1 = (b^5-1)/(b-1)$.

In Table 11, we illustrate the case $n = 5$ and $b = 2$ by using the residues modulo $a_1 = (2^5 - 1)/(2-1) = 31$. The inside of the frame is when $p = 0$, and the outside of the frame is when $p = 1$. Shadowed cells represent duplicates, that is $m_j^{(0)} = m_j^{(1)}$. The first group corresponding the case $n = 4$ and $b = 2$ is

$$\big\{\{0,1,2,3,4,5,6\},\{7,8,9,10,11,12,13\},\{14\}\big\},$$

where each value is expressed of the form $(x_2, x_3, x_4, 0)$. The second group corresponding the case $n = 4$ and $b = 2$ is

$$\big\{\{15,16,17,18,19,20,21\},\{22,23,24,25,26,27,28\},\{29\}\big\},$$

where each value is expressed of the form $(x_2, x_3, x_4, 1)$. The last one is $\{30\}$, which value is of the form $(0, 0, 0, 2)$. When $p = 1$, the corresponding residue moves to fill the gap in the block on the right, but only the overlapping part does not move. Also, only the elements in the top row are moved to fill the gap in the bottom left block. The corresponding residue moves according to the same principle as $p = 2$ and $p = 3$, and the regularity continues until $p = b$. When $p = b+1$, all the elements of the upper right block move completely, and the regularity begins to be broken.

In Table 12, only the last and lower left block is shown for general b when $n = 5$. Here, obviously the bottom left element takes the maximum value. The numbers in the second diagram are the corresponding residues modulo $a_1 := (b^5 - 1)/(b-1)$.

Similarly, when $n = 6$, for all the quadruple (x_2, x_3, x_4, x_5) when $n = 5$, the first group is in the form of $(x_2, x_3, x_4, x_5, 0)$, the second group is of $(x_2, x_3, x_4, x_5, 1)$, …, the b^{th} group is of $(x_2, x_3, x_4, x_5, b-1)$ and the last one is of $(0, 0, 0, 0, b)$. In Table 13, only the last and lower left

Table 11. Complete residue systems for $n=5$ and $b=2$

												0	1	2	3	4	5
												3	4	5	6		
												6	7	8			
										7	8	9	10	11			
										10	11	12	13				
										13	14						
								14	15	16							
						15	16	17	18	19							
						18	19	20	21								
						21	22	23									
				22	23	24	25	26									
				25	26	27	28										
				28	29												
		29	30														
30	0	1															
2																	

Table 12. Complete residue systems for $\mathrm{Ap}_p(A_5)$

				$(0,0,b,b-1)$	$p=0$
$p=0$	$(0,0,0,b)$	$(1,0,0,b)$	$\dots$	$(b,0,0,b)$	$p=1$
$p=1$	$(0,1,0,b)$	$(1,1,0,b)$	$\dots$	$(b,1,0,b)$	$p=2$
$p=2$	$(0,2,0,b)$		$\dots$		
	$\dots$		$\dots$		
$p=b-1$	$(0,b-1,0,b)$	$(1,b-1,0,b)$	$\dots$	$(b,b-1,0,b)$	$p=b$
$p=b$	$(0,b,0,b)$				

				-2	$p=0$
$p=0$	-1	0	$\dots$	$b-1$	$p=1$
$p=1$	b	$b+1$	$\dots$	$2b$	$p=2$
$p=2$	$2b+1$		$\dots$		
	$\dots$		$\dots$		
$p=b-1$	$(b-1)(b+1)-1$	$(b-1)(b+1)$	$\dots$	$b(b+1)-2$	$p=b$
$p=b$	$b(b+1)-1$				

block is shown for general b and general n. Here, obviously the bottom left element takes the maximum value. The numbers in the second diagram are the corresponding residues modulo $a_1 := (b^n-1)/(b-1)$.

Table 13. Complete residue systems $\mathrm{Ap}_p(A_n)$ for general n

				$(\underbrace{0,\dots,0}_{n-3},b,b-1)$	$p=0$
$p=0$	$(\underbrace{0,\dots,0}_{n-2},b)$	$(1,\underbrace{0,\dots,0}_{n-3},b)$	$\dots$	$(b,\underbrace{0,\dots,0}_{n-3},b)$	$p=1$
$p=1$	$(0,1,\underbrace{0,\dots,0}_{n-4},b)$	$(1,1,\underbrace{0,\dots,0}_{n-4},b)$	$\dots$	$(b,1,\underbrace{0,\dots,0}_{n-4},b)$	$p=2$
$p=2$	$(0,2,\underbrace{0,\dots,0}_{n-4},b)$		$\dots$		
	$\dots$		$\dots$		
$p=b-1$	$(0,b-1,\underbrace{0,\dots,0}_{n-4},b)$	$(1,b-1,\underbrace{0,\dots,0}_{n-4},b)$	$\dots$	$(b,b-1,\underbrace{0,\dots,0}_{n-4},b)$	$p=b$
$p=b$	$(0,b,\underbrace{0,\dots,0}_{n-4},b)$				

				-2	$p=0$
$p=0$	-1	0	$\dots$	$b-1$	$p=1$
$p=1$	b	$b+1$	$\dots$	$2b$	$p=2$
$p=2$	$2b+1$		$\dots$		
	$\dots$		$\dots$		
$p=b-1$	$(b-1)(b+1)-1$	$(b-1)(b+1)$	$\dots$	$b(b+1)-2$	$p=b$
$p=b$	$b(b+1)-1$				

Concerning the element $(j, b-1, \underbrace{0, \dots, 0}_{n-4}, b)$ $(j = 0, 1, \dots, b)$, we see that

$$\frac{j\left(b^{n+1}-1\right)}{b-1}+\frac{(p-1)\left(b^{n+2}-1\right)}{b-1}+\frac{b\left(b^{2n-1}-1\right)}{b-1}$$
$$=\frac{b^n-1}{b-1}\left(b^n+1+(p-1)b^2+jb\right)+\left((p-1)(b+1)+j-1\right)\equiv(p-1)(b+1)+j-1\left(\frac{b^n-1}{b-1}\right).$$

Concerning the last element, we see that

$$\frac{p\left(b^{n+2}-1\right)}{b-1}+\frac{b\left(b^{2n-1}-1\right)}{b-1}=\frac{b^n-1}{b-1}\left(b^n+1+pb^2\right)+\left(p(b+1)-1\right)\equiv p(b+1)-1\left(\frac{b^n-1}{b-1}\right).$$

By Lemma 2 (3), we obtain the result in Theorem 5.

6. Further problems

When $p > b$, in particular, when p is comparatively bigger than b, the situation seems to be very complicated. Is it possible to give any simplified formula for $g_p(A_n)$?

From another point of view, what general formula holds for $n_p(A_n)$? Furthermore, there are more generalized concepts. The generalized Sylvester sum $s_p(a_1, a_2, \dots, a_k)$ is the total sum of all the elements which can be represented by $a_1, a_2, \dots, a_k$ at most p ways. When $p = 0$, $s(a_1, a_2, \dots, a_k) = s_0(a_1, a_2, \dots, a_k)$ is the original Sylvester sum. Recently, the Sylvester weighted sum [12, 13] is also introduced and studied.

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