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Blow-up of nonradial solutions to the hyperbolic-elliptic chemotaxis system with logistic source

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Abstract. This paper is concerned with the blow-up of solutions to the following hyperbolic-elliptic chemotaxis system:

$$\begin{cases} u_t = -\nabla\cdot(\chi u\nabla v) + g(u), & x\in\Omega,\ t>0,\\ 0 = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases}$$

under homogeneous Neumann boundary conditions in a bounded domain $\Omega\subset\mathbb{R}^n, n\geq 1$, with smooth boundary and the function g is assumed to generalize the logistic source:

$$g(s)\leq as - bs^\gamma \ \text{ for } s>0$$

with $1<\gamma\leq 2$. For $b<\chi$ and some suitable conditions on parameters of problem, we prove that the solutions of this problem blow up in finite time. This result extend the obtained results for this problem.

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1. Introduction

The mathematical model which describes the movement of cells towards the gradient of substance produced by the cells themselves is in the following form:

$$\begin{cases} u_t = \nabla\cdot\big(\varphi(u)\nabla u - \psi(u)\nabla v\big) + g(u), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega,\ t>0, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$, $n\geq 1$, is a bounded domain with smooth boundary and $\tau\in\{0,1\}$. Also, the function g satisfies:

$$g(0)\geq 0,\quad g(s)\leq as - bs^\gamma \quad \text{for} \quad s>0 \tag{1}$$

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with constants $a \geq 0, b > 0$ and $\gamma > 1$. Here, $u = u(x,t)$ is the cell density and $v = v(x,t)$ denotes the concentration of the chemical substance. While the functions φ and ψ are the diffusivity and chemotactic sensitivity, respectively and g denotes the growth of cells [4,6].

This problem has been studied by many authors. In the absence of logistic source, i.e. $g \equiv 0$, if $\varphi \equiv 1, \psi(s) = \chi s$ with $\chi > 0$ and $\tau = 1$, it is known that: the blow-up can not occur in the one dimensional case [10]. In the two dimensional case, under the condition $\|u_0\|_{L^1(\Omega)} < \frac{4\pi}{\chi}$, all solutions are global and bounded [9]. The same result is true for radial solutions provided that $\|u_0\|_{L^1(\Omega)} < \frac{8\pi}{\chi}$ [9]. While for $\|u_0\|_{L^1(\Omega)} > \frac{8\pi}{\chi}$, there exist radial solutions that blow up in finite time [3]. In the higher dimensional case, solutions are global and bounded when the initial data are small enough [14], whereas, radial solutions become unbounded in finite time under some suitable conditions on initial data [17]. Also, in the absence of logistic source, when the second equation is replaced with $0 = \Delta v - M + u$, where M denotes the mean value of initial data u_0 and $\varphi(s) \geq c_1 s^{-p}$ and $\psi(s) \leq c_2 s^q$ for all $s > 0$ with $c_1, c_2 > 0, p \geq 0$ and $q \in \mathbb{R}$, then all solutions are global and bounded provided that $p + q < \frac{2}{n}$. While, for $p + q > \frac{2}{n}$, there exist radial solutions which become unbounded in finite time [19].

In the presence of logistic source, i.e. $g \neq 0$, if $\varphi \equiv 1$ and $\psi(s) = \chi s$ with $\chi > 0$, then solutions are global and bounded with $\tau = 0$ and $b > \frac{n-2}{n}\chi$ [12]. The same result holds with $\tau > 0$ if $n = 2$ [11] or $n \geq 3$ and $b > b^*$, where b^* is sufficiently large [15]. If $\varphi(s) \geq c_1 s^p, \psi(s) = \chi s$ and $\tau = 0$, where $p \in \mathbb{R}$ and χ is some positive constant, then solutions are global and bonded with $\gamma = 2$ under the condition $b > \chi(1 - \frac{2}{n(1-p)_+})$, or, equivalently, $p > 1 - \frac{2\chi}{n(\chi - b)_+}$ [2]. For $\gamma \geq 2$ and $p \geq 0$, under the condition $b > b^*$ with $b^* = 0$ for $p \geq 2 - \frac{2}{n}$ and $b^* = \frac{(2-p)n-2}{(2-p)n}\chi$ for $p < 2 - \frac{2}{n}$, then solutions are global and bounded [13]. Also, the same result is true for $1 < \gamma < 2$ and $p > 2 - \frac{2}{n}$ [13]. Recently, it is proved that in the case of $\gamma > 2$, solutions are global and bounded without any restrictions on p and b [7]. Other results about this problem are: under the conditions $\varphi(s) \geq (s+1)^{-p}$ and $\psi(s) \leq s^q$ with $p, q \in \mathbb{R}$, all solutions are global and bounded provided that $p + q < \frac{2}{n}$ and $\gamma > 1$ or $p + q \geq \frac{2}{n}, b > \frac{(p+q)n-2}{(p+q)n}\chi$ and $\gamma \geq q + 1$ with $q \geq 1$ [20]. Also, if $\varphi(s) \geq c_1 s^p$ and $c_2 s^q \leq \psi(s) \leq c_3 s^q$ with $c_i > 0, i = 1,2,3$, and $\tau > 0$, it is proved that if $q < 1$, then solutions are global and bounded [1]. Recent results have been shown that the blow-up phenomenon can occur in the presence of logistic source. The known results are: when Ω is a ball in $\mathbb{R}^n, n \geq 5$, and $\varphi \equiv 1$ and $\psi(s) = \chi s$ with $\chi > 0$, if the second equation is replaced with $0 = \Delta v + u - \frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx$ and g satisfies (1) with $a = 0$ and other additional conditions, then radially symmetric solutions become unbounded in finite time provided that $1 < \gamma < \frac{3}{2} + \frac{1}{2n-2}$ [16]. Also, for $n \geq 5$, if $\varphi(s) \leq s^{-p}$ and $\psi(s) = s^q$ for all $s > 0$ with $p, q \in \mathbb{R}$ and g as before with $a \geq 0$, then radially symmetric solutions blow up in finite time if $\frac{2}{n} - 1 < p < 1$ and $q > 1$ with $2p + 3q < 4$, also, $1 < \gamma < \frac{(3-p)n-2}{2n-2}$ [20]. The following hyperbolic-elliptic chemotaxis system is studied by Winkler [18]:

$$\begin{cases} u_t = -\nabla \cdot \left(\chi u \nabla v\right) + g(u), & x \in \Omega,\ t > 0, \\ 0 = \Delta v - v + u, & x \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0,\ v(x,0) = v_0, & x \in \Omega, \end{cases} \tag{2}$$

where $\Omega \subseteq \mathbb{R}^n, n \geq 1$, is a bounded domain with smooth boundary and ν denotes the unit outward normal vector to $\partial\Omega$. Here, g is a smooth function which satisfies (1) with $\gamma > 1$ and $u_0(x)$ and $v_0(x)$ are the initial value functions. For the one-dimensional case with $\chi = 1$ and $\gamma = 2$, it is proved that the solution of this problem is global in time with $b \geq 1$, and for $b < 1$ and $p > \frac{1}{1-b}$, there exist solutions which become unbounded in finite time with $\|u_0\|_{L^p(\Omega)}$ sufficiently large. In the the higher dimensional case, when Ω is a ball, solutions are global in time with $b \geq 1$, whereas $b < 1$ and $p > \frac{1}{1-b}$, there exist radial symmetric solutions that blow up in finite time with $\|u_0\|_{L^p(\Omega)}$ sufficiently large [8]. The authors in [5] studied this problem in bounded domains as well as whole space. In the case of $1<\gamma\le 2$, they obtained the similar results with one-dimensional case and extend the results in [18] to the higher dimensional case. Moreover, in the case of $\gamma>2$, they proved that solutions are global.

In the present paper, we will study problem (2) under the condition (1) with $1<\gamma\le 2$. For $b<\chi$, we will prove that the solutions of this problem blow up in finite time under some suitable conditions. Our main result is stated in the following theorem:

Theorem 1. *Assume that the function g satisfies*

$$g(s)\le as-bs^{\gamma}\quad for\quad s>0$$

with $1<\gamma\le 2$. *Define:*

$$\begin{aligned}C_{k,\lambda,\mu,\varepsilon}=\lambda\Bigg(\chi(k-1)-bk-\mu\left(\frac{k}{\varepsilon(k+1)}\right)^{k}\Bigg)\\ -(1-\theta)\left(\frac{2C_{GN}^{2}}{k+1}\left(\frac{2\varepsilon^{k}}{\mu}\right)^{k}\left(\frac{k+1}{k}\right)^{k(k-1)}\big(\chi(k-1)\big)^{k+1}\right)^{\frac{1}{1-\theta}}\left(\frac{(k+1)\theta}{2k\mu}\right)^{\frac{\theta}{(1-\theta)}}\\ -\frac{2C_{GN}^{2}}{k+1}\left(\frac{2\varepsilon^{k}}{\mu}\right)^{k}\left(\frac{k+1}{k}\right)^{k(k-1)}\big(\chi(k-1)\big)^{k+1},\end{aligned}$$

where $k>\frac{\chi}{\chi-b}$, $0<\mu<((\chi-b)k-\chi)(\frac{k+1}{k})^{k}$, $\lambda\in[0,1)$ *and* $\frac{k}{k+1}(\frac{\mu}{(\chi-b)k-\chi})^{\frac{1}{k}}<\varepsilon\le 1,\theta=\frac{nk}{2+nk}$ *and* C_{GN} *is the constant in the Gagliardo–Nirenberg inequality. If there exist some constants* k,μ,λ *and* ε *such that the following conditions hold:*

$$\text{(i) } 1<\gamma<2:\begin{cases} b<\chi & \textit{and}\quad b\le a, & \textit{if}\quad C_{k,\lambda,\mu,\varepsilon}\ge 0,\\ b<\min\{a,\chi\} & \textit{and}\quad M\le\frac{k(a-b)|\Omega|^{1-k}}{-C_{k,\lambda,\mu,\varepsilon}}, & \textit{if}\quad C_{k,\lambda,\mu,\varepsilon}<0,\end{cases}\tag{3}$$

$$\text{(ii) } \gamma=2:\begin{cases} b<\chi, & & \textit{if}\quad C_{k,\lambda,\mu,\varepsilon}\ge 0,\\ b<\chi & \textit{and}\quad M\le\frac{ka|\Omega|^{1-k}}{-C_{k,\lambda,\mu,\varepsilon}}, & \textit{if}\quad C_{k,\lambda,\mu,\varepsilon}<0\end{cases}\tag{4}$$

with $M=(\min\{1,\frac{b}{a}\})^{-\frac{1}{\gamma-1}}\max\{\|u_0\|_{L^1(\Omega)},|\Omega|\}$. *Then the solution of problem* (2) *blows up in finite time.*

In the next section, we will prove the Theorem (1).

2. Blow up in finite time

Here, we state the well-posedness and solvability result.

Lemma 2. *Let* $\Omega\subseteq\mathbb{R}^n, n\ge 1$, *be a bounded convex domain with smooth boundary, also g satisfies* (1). *Moreover, we assume that* $0\le u_0\in W^{2,q}(\Omega)\cap L^1(\Omega)$ *for all* $1<q<\infty$. *Then for any* $1<p<\infty$, *there exists a maximal time* $T_{\max}\in(0,\infty]$ *such that unique non-negative solutions* $u\in L^p([0,t),W^{1,p}(\Omega))\cap L^\infty([0,t)\times\Omega)$ *and* $v\in L^p([0,t),W^{2,p}(\Omega))\cap L^\infty([0,t)\times\Omega)$ *exist for any time* $t<T_{\max}$. *In addition, if* $T_{\max}<+\infty$, *then*

$$\lim_{t\to T_{\max}}\|u(.,t)\|_{L^\infty(\Omega)}=\infty.$$

For details of the proof, we refer the reader to [5].

Although the proof of the following lemma is given in [8, 18], but for completeness, we present it.

Lemma 3. *Assume that the function g satisfies* (1). *Then for all* $t \in (0, T_{\max})$, *there exists a constant* $M > 0$ *such that*

$$\|u(.,t)\|_{L^1(\Omega)} \leq M \tag{5}$$

with $M = (\min\{1, \frac{b}{a}\})^{-\frac{1}{\gamma-1}} \max\{\|u_0\|_{L^1(\Omega)}, |\Omega|\}$.

Proof. We integrate from the first equation in (2) and use (1) to get

$$\frac{d}{dt}\int_\Omega u\,dx = \int_\Omega g(u)\,dx \leq a\int_\Omega u\,dx - b\int_\Omega u^\gamma\,dx.$$

Making use of Hölder's inequality, we obtain

$$\int_\Omega u^\gamma\,dx \geq \left(\int_\Omega u\,dx\right)^\gamma |\Omega|^{1-\gamma}.$$

Hence, $y(t) = \int_\Omega u(x,t)\,dx$ satisfies

$$y'(t) \leq ay(t) - b|\Omega|^{1-\gamma} y^\gamma(t), \quad \text{for } t > 0.$$

Set $r(t) = (y(t))^{1-\gamma}$, thus we obtain

$$r'(t) + a(\gamma-1)r(t) \geq b(\gamma-1)|\Omega|^{1-\gamma},$$

which yields

$$r(t) \geq r(0)e^{-a(\gamma-1)t} + \frac{b}{a}|\Omega|^{1-\gamma}\left(1 - e^{-a(\gamma-1)t}\right).$$

Therefore,

$$y(t) \leq \left((y(0))^{1-\gamma}e^{-a(\gamma-1)t} + \frac{b}{a}|\Omega|^{1-\gamma}\left(1-e^{-a(\gamma-1)t}\right)\right)^{-\frac{1}{\gamma-1}}.$$

This inequality yields

$$y(t) \leq \frac{\max\left\{|\Omega|, y(0)\right\}}{\left(\min\left\{1, \frac{b}{a}\right\}\right)^{\frac{1}{\gamma-1}}}.$$

This inequality is desired result. □

We will use the Gagliardo–Nirenberg inequality in the proof of Theorem (1), for readers' convenience, we state this inequality in the following lemma [13].

Lemma 4. *Let* $\Omega \subseteq \mathbb{R}^n$, $n \geq 1$, *be a bounded domain with smooth boundary. For* $p(n-2) < 2n$, $r \in (0, p)$ *and* $\psi \in W^{1,2}(\Omega) \cap L^r(\Omega)$, *there exists a constant* $C_{GN} > 0$ *depending on n and* Ω *such that*

$$\|\psi\|_{L^p(\Omega)} \leq C_{GN}\left(\left\|\nabla\psi\right\|^\theta_{L^2(\Omega)}\left\|\psi\right\|^{(1-\theta)}_{L^r(\Omega)} + \left\|\psi\right\|_{L^r(\Omega)}\right)$$

with

$$\theta = \frac{\frac{n}{r} - \frac{n}{p}}{1 - \frac{n}{2} + \frac{n}{r}}.$$

We now prove our main result.

Proof of Theorem 1. We use of integrating by parts and the second equation in (2) to obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^k dx &= k\int_\Omega u^{k-1}u_t\,dx\\ &= k\int_\Omega u^{k-1}\left(-\nabla\cdot\left(\chi u\nabla v\right)+au-bu^\gamma\right)dx\\ &= \chi k(k-1)\int_\Omega u^{k-1}\nabla u\cdot\nabla v\,dx+ak\int_\Omega u^k\,dx-bk\int_\Omega u^{k+\gamma-1}\,dx\\ &= \chi(k-1)\int_\Omega \nabla u^k\cdot\nabla v\,dx+ak\int_\Omega u^k\,dx-bk\int_\Omega u^{k+\gamma-1}\,dx\\ &= -\chi(k-1)\int_\Omega u^k\Delta v\,dx+ak\int_\Omega u^k\,dx-bk\int_\Omega u^{k+\gamma-1}\,dx\\ &= -\chi(k-1)\int_\Omega u^k v\,dx+ak\int_\Omega u^k\,dx+\chi(k-1)\int_\Omega u^{k+1}\,dx-bk\int_\Omega u^{k+\gamma-1}\,dx.\end{aligned}\tag{6}$$

Making use of Young 's inequality to the first term on the right hand side of (6), we obtain

$$\chi(k-1)\int_\Omega u^k v\,dx \le \frac{\mu}{2}\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx+c_1\int_\Omega v^{k+1}\,dx,\tag{7}$$

where $\frac{k}{k+1}(\frac{\mu}{(\chi-b)k-\chi})^{\frac{1}{k}}<\varepsilon\le 1$ and μ is some positive constant, also:

$$c_1=\frac{1}{k+1}\left(\frac{2\varepsilon^k}{\mu}\right)^k\left(\frac{k+1}{k}\right)^{k(k-1)}\left(\chi(k-1)\right)^{k+1}.$$

We need to obtain an upper bound to the second term on the right hand side of (7). In order to do this, we apply the Gagliardo–Nirenberg inequality to get

$$c_1\int_\Omega v^{k+1}\,dx=c_1\left\|v^{\frac{k+1}{2}}\right\|^2_{L^2(\Omega)}\le 2c_1C^2_{GN}\left(\left\|\nabla v^{\frac{k+1}{2}}\right\|^{2\theta}_{L^2(\Omega)}\left\|v^{\frac{k+1}{2}}\right\|^{2(1-\theta)}_{L^{\frac{2}{k+1}}(\Omega)}+\left\|v^{\frac{k+1}{2}}\right\|^2_{L^{\frac{2}{k+1}}(\Omega)}\right)$$

with

$$\theta=\frac{\frac{n(k+1)}{2}-\frac{n}{2}}{1-\frac{n}{2}+\frac{n(k+1)}{2}}=\frac{nk}{2+nk}.$$

Because of $\theta=\frac{nk}{2+nk}<1$, then we can apply the Young inequality to obtain

$$\begin{aligned}c_1\int_\Omega v^{k+1}\,dx &=\le \frac{2k\mu}{k+1}\left\|\nabla v^{\frac{k+1}{2}}\right\|^2_{L^2(\Omega)}+c_2\left\|v^{\frac{k+1}{2}}\right\|^2_{L^{\frac{2}{k+1}}(\Omega)}+c_3\left\|v^{\frac{k+1}{2}}\right\|^2_{L^{\frac{2}{k+1}}(\Omega)}\\ &= \frac{2k\mu}{k+1}\left\|\nabla v^{\frac{k+1}{2}}\right\|^2_{L^2(\Omega)}+c_4\left\|v^{\frac{k+1}{2}}\right\|^2_{L^{\frac{2}{k+1}}(\Omega)}\end{aligned}\tag{8}$$

with $c_2=(1-\theta)(2c_1C^2_{GN})^{\frac{1}{1-\theta}}(\frac{(k+1)\theta}{2k\mu})^{\frac{\theta}{(1-\theta)}}$, $c_3=2c_1C^2_{GN}$ and $c_4=c_2+c_3$.

In the following, we obtain an upper bound to the first term on the right hand side of (8). In order to do this, as in [16, Lemma 2.2], we multiply the second equation of (2) by v^k to get

$$\begin{aligned}0&=\int_\Omega v^k\Delta v\,dx-\int_\Omega v^{k+1}\,dx+\int_\Omega v^k u\,dx\\ &=-k\int_\Omega v^{k-1}\left|\nabla v\right|^2dx-\int_\Omega v^{k+1}\,dx+\int_\Omega v^k u\,dx\\ &=-\frac{4k}{(k+1)^2}\int_\Omega\left|\nabla v^{\frac{k+1}{2}}\right|^2dx-\int_\Omega v^{k+1}\,dx+\int_\Omega v^k u\,dx.\end{aligned}$$

Making use of Young 's inequality, we obtain

$$\frac{4k}{(k+1)^2}\int_\Omega\left|\nabla v^{\frac{k+1}{2}}\right|^2dx+\int_\Omega v^{k+1}\,dx=\int_\Omega v^k u\,dx\le\frac{1}{k+1}\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx+\varepsilon\int_\Omega v^{k+1}\,dx.$$

This inequality implies that

$$\frac{4k}{(k+1)^2}\int_\Omega \left|\nabla v^{\frac{k+1}{2}}\right|^2 dx+(1-\varepsilon)\int_\Omega v^{k+1}\,dx \le \frac{1}{k+1}\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx.$$

Since $\varepsilon \le 1$, then we have

$$\frac{4k}{k+1}\int_\Omega \left|\nabla v^{\frac{k+1}{2}}\right|^2 dx \le \left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx.$$

Combining this inequality with (8) gives

$$c_1\int_\Omega v^{k+1}\,dx \le \frac{\mu}{2}\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx + c_4\left(\int_\Omega v\,dx\right)^{k+1}. \tag{9}$$

Integrating of the second equation in (2) and using (1), we get

$$\int_\Omega v\,dx=\int_\Omega u\,dx.$$

We now insert this equality in (9) to obtain

$$c_1\int_\Omega v^{k+1}\,dx \le \frac{\mu}{2}\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx + c_4\left(\int_\Omega u\,dx\right)^{k+1}.$$

This inequality along with (7) yields

$$\chi(k-1)\int_\Omega u^k v\,dx \le \mu\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx + c_4\left(\int_\Omega u\,dx\right)^{k+1}.$$

Thus,

$$-\chi(k-1)\int_\Omega u^k v\,dx \ge -\mu\left(\frac{k}{\varepsilon(k+1)}\right)^k\int_\Omega u^{k+1}\,dx - c_4\left(\int_\Omega u\,dx\right)^{k+1}.$$

Combining this inequality with (6), we can write

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^k dx\\ &\quad\ge \left[\chi(k-1)-\mu\left(\frac{k}{\varepsilon(k+1)}\right)^k\right]\int_\Omega u^{k+1}\,dx - c_4\left(\int_\Omega u\,dx\right)^{k+1}+ak\int_\Omega u^k\,dx-bk\int_\Omega u^{k+\gamma-1}\,dx.\end{aligned}$$

We set $\sigma=\chi(k-1)-\mu(\frac{k}{\varepsilon(k+1)})^k$. Thus, we have

$$\frac{d}{dt}\int_\Omega u^k dx \ge \sigma\int_\Omega u^{k+1}\,dx - c_4\left(\int_\Omega u\,dx\right)^{k+1}+ak\int_\Omega u^k\,dx-bk\int_\Omega u^{k+\gamma-1}\,dx. \tag{10}$$

In the rest of the proof, we consider the following two cases:

Case 1. $\gamma=2$**.** The inequality (10) for $\gamma=2$ gives

$$\frac{d}{dt}\int_\Omega u^k dx \ge (\sigma-bk)\int_\Omega u^{k+1}\,dx + ak\int_\Omega u^k\,dx - c_4\left(\int_\Omega u\,dx\right)^{k+1}. \tag{11}$$

Because of $\chi>b$ and $k>\frac{\chi}{\chi-b}$, then we can take

$$0<\mu<\big((\chi-b)k-\chi\big)\left(\frac{k+1}{k}\right)^k.$$

Now

$$\frac{k}{k+1}\left(\frac{\mu}{\big(\chi-b\big)k-\chi}\right)^{\frac{1}{k}}<\varepsilon\le 1$$

implies that

$$\sigma-bk=\chi(k-1)-bk-\mu\left(\frac{k}{\varepsilon(k+1)}\right)^k>0.$$

We apply the Hölder inequality to the first and second terms on the right hand side of (11) as follows:

$$\int_\Omega u\,dx \le \left(\int_\Omega u^k\,dx\right)^{\frac{1}{k}} |\Omega|^{\frac{k-1}{k}}, \qquad \int_\Omega u\,dx \le \left(\int_\Omega u^{k+1}\,dx\right)^{\frac{1}{k+1}} |\Omega|^{\frac{k}{k+1}},$$
$$\int_\Omega u^k\,dx \le |\Omega|^{\frac{1}{k+1}} \left(\int_\Omega u^{k+1}\,dx\right)^{\frac{k}{k+1}}.$$

These inequalities yield

$$\begin{aligned}&\int_\Omega u^k\,dx \ge |\Omega|^{1-k}\left(\int_\Omega u\,dx\right)^k, \qquad \int_\Omega u^{k+1}\,dx \ge |\Omega|^{-k}\left(\int_\Omega u\,dx\right)^{k+1},\\ &\int_\Omega u^{k+1}\,dx \ge |\Omega|^{-\frac{1}{k}}\left(\int_\Omega u^k\,dx\right)^{\frac{k+1}{k}}.\end{aligned} \tag{12}$$

By inserting these inequalities in (11), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^k dx &\ge (1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\left(\int_\Omega u^k\,dx\right)^{\frac{k+1}{k}}\\ &\quad + \left(\lambda(\sigma-bk)|\Omega|^{-k} - c_4\right)\left(\int_\Omega u\,dx\right)^{k+1} + ak|\Omega|^{1-k}\left(\int_\Omega u\,dx\right)^k\\ &= (1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\left(\int_\Omega u^k\,dx\right)^{\frac{k+1}{k}} + C_{k,\lambda,\mu,\varepsilon}\left(\int_\Omega u\,dx\right)^{k+1} + ak|\Omega|^{1-k}\left(\int_\Omega u\,dx\right)^k\end{aligned}$$

with $\lambda \in [0,1)$. If there exist some constants $k, \lambda, \mu, \varepsilon$ such that $C_{k,\lambda,\mu,\varepsilon} \ge 0$, then we can write:

$$\frac{d}{dt}\int_\Omega u^k dx \ge (1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\left(\int_\Omega u^k\,dx\right)^{\frac{k+1}{k}}. \tag{13}$$

We now set $z(t) = \int_\Omega u^k\,dx$, then we can write

$$z'(t) \ge (1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}(z(t))^{\frac{k+1}{k}}.$$

Then,

$$z^{-1-\frac{1}{k}}\,dz \ge (1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\,dt. \tag{14}$$

Integrating (14) from 0 to t, we obtain

$$z^{-\frac{1}{k}}(t) \le z^{-\frac{1}{k}}(0) - \frac{1}{k}(1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\,t.$$

Since $\sigma - bk > 0$ and $\lambda < 1$, thus this inequality can not hold for all $t > 0$. Also, this inequality gives an upper bound for the blow-up time, i.e.

$$T_{\max} \le T = \frac{(z(0))^{-\frac{1}{k}}}{\frac{1}{k}(1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}}.$$

Also, if for some constants $k, \lambda, \mu, \varepsilon$; $C_{k,\lambda,\mu,\varepsilon} < 0$, then from Lemma (3) and condition (4), for $y(t) = \int_\Omega u\,dx$, we obtain

$$\begin{aligned}C_{k,\lambda,\mu,\varepsilon}\left(y(t)\right)^{k+1} + ak|\Omega|^{1-k}\left(y(t)\right)^k &= \left(y(t)\right)^k\left(C_{k,\lambda,\mu,\varepsilon}\,y(t) + ak|\Omega|^{1-k}\right)\\ &\ge \left(y(t)\right)^k\left(MC_{k,\lambda,\mu,\varepsilon} + ak|\Omega|^{1-k}\right) \ge 0.\end{aligned}$$

Thus, we get again the inequality (13). Hence blow-up occurs in finite time.

Case 2. $1<\gamma<2$**.** In this case, we need to estimate a lower bound for the forth term on the right hand side of (10). In order to obtain this, for each $t>0$, we divide Ω into two sets,

$$\Omega_{\{<1\}}=\{x\in\Omega:\ u(x,t)<1\},\quad \Omega_{\{\geq 1\}}=\{x\in\Omega:\ u(x,t)\geq 1\}.$$

Thus, we can write

$$\begin{aligned}-bk\int_\Omega u^{k+\gamma-1}\,dx&=-bk\int_{\Omega_{\{<1\}}}u^{k+\gamma-1}\,dx-bk\int_{\Omega_{\{\geq 1\}}}u^{k+\gamma-1}\,dx\\&\geq -bk\int_{\Omega_{\{<1\}}}u^{k}\,dx-bk\int_{\Omega_{\{\geq 1\}}}u^{k+1}\,dx.\end{aligned}$$

Now, the last inequality along with (10) gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^k dx+c_4\left(\int_\Omega u\,dx\right)^{k+1}\\&\quad\geq \sigma\int_\Omega u^{k+1}\,dx+ak\int_\Omega u^k\,dx-bk\int_{\Omega_{\{<1\}}}u^k\,dx-bk\int_{\Omega_{\{\geq 1\}}}u^{k+1}\,dx\\&\quad=\sigma\int_{\Omega_{\{<1\}}}u^{k+1}\,dx+ak\int_{\Omega_{\{<1\}}}u^k\,dx-bk\int_{\Omega_{\{<1\}}}u^k\,dx\\&\qquad+\sigma\int_{\Omega_{\{\geq 1\}}}u^{k+1}\,dx+ak\int_{\Omega_{\{\geq 1\}}}u^k\,dx-bk\int_{\Omega_{\{\geq 1\}}}u^{k+1}\,dx\\&\quad=(\sigma-bk)\int_{\Omega_{\{\geq 1\}}}u^{k+1}\,dx+k(a-b)\int_{\Omega_{\{<1\}}}u^k\,dx+\sigma\int_{\Omega_{\{<1\}}}u^{k+1}\,dx+ak\int_{\Omega_{\{\geq 1\}}}u^k\,dx\\&\quad\geq(\sigma-bk)\int_{\Omega_{\{\geq 1\}}}u^{k+1}\,dx+k(a-b)\int_{\Omega_{\{<1\}}}u^k\,dx\\&\qquad+(\sigma-bk)\int_{\Omega_{\{<1\}}}u^{k+1}\,dx+k(a-b)\int_{\Omega_{\{\geq 1\}}}u^k\,dx\\&\quad=(\sigma-bk)\int_\Omega u^{k+1}\,dx+k(a-b)\int_\Omega u^k\,dx.\end{aligned}$$

Thus,

$$\frac{d}{dt}\int_\Omega u^k dx\geq(\sigma-bk)\int_\Omega u^{k+1}\,dx-c_4\left(\int_\Omega u\,dx\right)^{k+1}+k(a-b)\int_\Omega u^k\,dx.$$

Combining this inequality with (12), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^k dx&\geq(1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\left(\int_\Omega u^k\,dx\right)^{\frac{k+1}{k}}+\lambda(\sigma-bk)|\Omega|^{-k}\left(\int_\Omega u\,dx\right)^{k+1}\\&\quad-c_4\left(\int_\Omega u\,dx\right)^{k+1}+k(a-b)|\Omega|^{1-k}\left(\int_\Omega u\,dx\right)^{k}.\end{aligned}$$

Similar to Case 1, if there exist some constants $k,\lambda,\mu,\varepsilon$ such that $C_{k,\lambda,\mu,\varepsilon}\geq 0$, or $C_{k,\lambda,\mu,\varepsilon}<0$, we can conclude that

$$\frac{d}{dt}\int_\Omega u^k dx\geq(1-\lambda)(\sigma-bk)|\Omega|^{-\frac{1}{k}}\left(\int_\Omega u^k\,dx\right)^{\frac{k+1}{k}}.$$

This inequality gives the desired result.

□

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