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Monotonicity and complete monotonicity of some functions involving the modified Bessel functions of the second kind

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Abstract. In this paper, we introduce some monotonicity rules for the ratio of integrals. Furthermore, we demonstrate that the function $-T_{\nu,\alpha,\beta}(s)$ is completely monotonic in s and absolutely monotonic in ν if and only if $\beta \geq 1$, where $T_{\nu,\alpha,\beta}(s) = K_\nu^2(s) - \beta K_{\nu-\alpha}(s)K_{\nu+\alpha}(s)$ defined on $s > 0$ and $K_\nu(s)$ is the modified Bessel function of the second kind of order ν. Finally, we determine the necessary and sufficient conditions for the functions $s \mapsto T_{\mu,\alpha,1}(s)/T_{\nu,\alpha,1}(s)$, $s \mapsto (T_{\mu,\alpha,1}(s) + T_{\nu,\alpha,1}(s))/(2T_{(\mu+\nu)/2,\alpha,1}(s))$, and $s \mapsto \frac{\mathrm{d}^{n_1}}{\mathrm{d}\nu^{n_1}} T_{\nu,\alpha,1}(s)/\frac{\mathrm{d}^{n_2}}{\mathrm{d}\nu^{n_2}} T_{\nu,\alpha,1}(s)$ to be monotonic in $s \in (0,\infty)$ by employing the monotonicity rules.

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1. Introduction

The modified Bessel function of the first kind $I_\nu(s)$ [25] was defined by the solution to the following second-order ordinary differential equation

$$s^2 X''(s) + sX'(s) - \left(s^2 + \nu^2\right) X(s) = 0,$$

which can be represented as a series [25]

$$I_\nu(s) = \sum_{k=0}^{\infty} \frac{1}{k!\Gamma(\nu+k+1)} \left(\frac{s}{2}\right)^{2k+\nu}, \quad s \in \mathbb{R}, \nu \in \mathbb{R}\backslash\{-1,-2,\cdots\}.$$

The modified Bessel function of the second kind $K_\nu(s)$ [25] was defined by

$$K_\nu(s) = \begin{cases} \pi \dfrac{I_{-\nu}(s) - I_\nu(s)}{2\sin(\pi\nu)}, & \text{if } \nu \notin \mathbb{Z}, \\ \lim\limits_{n\to\nu} \pi \dfrac{I_{-n}(s) - I_n(s)}{2\sin(\pi n)}, & \text{if } \nu \in \mathbb{Z}, \end{cases}$$

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which can be represented as an integral [25]

$$K_\nu(s)=\int_0^\infty e^{-s\cosh t}\cosh(\nu t)\mathrm{d}t,\quad s>0,\nu\in\mathbb{R}. \tag{1}$$

Moreover, the product of any two modified Bessel functions of the second kind [25] had the following integral representation

$$K_\mu(s)K_\nu(s)=2\int_0^\infty K_{\mu\pm\nu}\big(2s\cosh(t)\big)\cosh\big((\mu\mp\nu)\,t\big)\mathrm{d}t,\quad s>0,\mu,\nu\in\mathbb{R}. \tag{2}$$

Grosswald [16] proved that

$$\mathscr{L}\left(\frac{2}{\pi^2 s}\frac{1}{J_\nu^2\left(\sqrt{s}\right)+Y_\nu^2\left(\sqrt{s}\right)}\right)=\mathscr{L}^{-1}\left(\frac{K_{\nu-1}\left(\sqrt{s}\right)}{\sqrt{s}K_\nu\left(\sqrt{s}\right)}\right),\quad \nu\ge 0, \tag{3}$$

where $\mathscr{L}$ and $\mathscr{L}^{-1}$ are the Laplace transform and its inverse respectively, $J_\nu(s)$ and $Y_\nu(s)$ are the Bessel functions of the first and second kind of order ν, respectively. Ismail [17] further expressed the formula (3) as

$$\frac{K_{\nu-1}(s)}{K_\nu(s)}=\frac{4}{\pi^2}\int_0^\infty\frac{s\mathrm{d}t}{t\left(s^2+t^2\right)\left(J_\nu^2(t)+Y_\nu^2(t)\right)},\quad \nu>0.$$

The modified Bessel functions have an essential role in mathematical analysis and have wide applications in statistics [14, 17, 20] and physics [22, 24]. For more information about the Bessel functions, refer to [8, 12, 15, 27, 28, 30, 31].

Recall that the Turán type inequality was presented by Paul Turán in [23], which states as follows

$$P_n^2(s)-P_{n-1}(s)P_{n+1}(s)>0,\quad |s|<1, n=1,2,\cdots,$$

where P_n is the classical Legendre polynomial. Many scholars have discovered that the Turán type inequality also holds for numerous classical polynomials and special functions, such as Krätzel functions [6], Struve functions [9], Tricomi confluent hypergeometric functions [5] and Bessel functions [3, 7, 8, 18, 24]. Among these, Ismail and Muldoon [18] and Van Haeringen [24] independently proved the following Turán type inequality holds for the modified Bessel function of the second kind

$$K_\nu^2(s)-K_{\nu-1}(s)K_{\nu+1}(s)\le 0,\quad s>0,\nu\in\mathbb{R}. \tag{4}$$

After that, Segura [21] and Baricz [4] presented some interesting results about the lower and upper bounds of $K_\nu^2(s)-K_{\nu-1}(s)K_{\nu+1}(s)$.

Inspired by these, we will generalize the left hand side of (4) as follows

$$T_{\nu,\alpha,\beta}(s):=K_\nu^2(s)-\beta K_{\nu-\alpha}(s)K_{\nu+\alpha}(s),\quad s>0,\nu,\alpha,\beta\in\mathbb{R}, \tag{5}$$

and provide some properties of $T_{\nu,\alpha,\beta}(s)$. Besides, to further investigate $T_{\nu,\alpha,1}(s)$, we will consider the following functions involving $T_{\nu,\alpha,1}(s)$ that are defined on $(0,\infty)$

$$L^*_{\mu,\nu,\alpha}(s):=\frac{T_{\mu,\alpha,1}(s)}{T_{\nu,\alpha,1}(s)}, \tag{6}$$

$$L^{**}_{\mu,\nu,\alpha}(s):=\frac{T_{\mu,\alpha,1}(s)+T_{\nu,\alpha,1}(s)}{2T_{(\mu+\nu)/2,\alpha,1}(s)}, \tag{7}$$

and

$$W_{\nu,n_1,n_2,\alpha}(s):=\frac{\dfrac{\partial^{n_1}T_{\nu,\alpha,1}(s)}{\partial\nu^{n_1}}}{\dfrac{\partial^{n_2}T_{\nu,\alpha,1}(s)}{\partial\nu^{n_2}}}. \tag{8}$$

This paper is organized into four sections. Section 1 provides a brief background and the research motivation of this paper. Section 2 presents several monotonicity rules for the ratio of integrals, extending existing research results. Section 3 introduces two multiple integral representations and asymptotic formulas of $T_{\nu,\alpha,\beta}(s)$. Section 4 explores the complete monotonicity and absolute monotonicity of function $T_{\nu,\alpha,\beta}(s)$ and establishes the necessary and sufficient conditions for functions $L^{*}_{\mu,\nu,\alpha}(s)$, $L^{**}_{\mu,\nu,\alpha}(s)$, and $W_{\nu,n_1,n_2,\alpha}(s)$ to be monotonic.

2. Monotonicity rules for ratio of integrals

Monotonicity rules play a fundamental role in mathematical analysis [2, 29]. In 1955, Biernacki and Krzyż [11] provided a crucial monotonicity rule for the ratio of two power series.

Monotocity rule A. *Suppose that two convergent real power series $P(s)=\sum_{k=0}^{\infty} p_k s^k$ and $Q(s)=\sum_{k=0}^{\infty} q_k s^k$ are defined on $(-d,d)(d>0)$. Then the ratio $P(s)/Q(s)$ is increasing (decreasing) on $(0,d)$ if $\{p_k/q_k\}_{k\geq 0}$ is non-constant, increasing (decreasing) and $q_k>0$ for all $k\geq 0$.*

In 1982, Cheeger, Gromov, and Taylor presented an important monotonicity rule for the ratio of two integrals [13].

Monotocity rule B. *If functions $f(t), g(t)$ are positive, integrable and the ratio $f(t)/g(t)$ is decreasing, then the function*

$$s \mapsto \frac{\int_0^s f(t)\mathrm{d}t}{\int_0^s g(t)\mathrm{d}t}$$

is decreasing on $(0,\infty)$.

In 2022, Qi [19, Lemma 9] gave an interesting monotonicity rule as follows.

Monotocity rule C. *Suppose that the functions $f(t)$, $g(t)>0$, and $w(s,t)>0$ are integrable in $t\in(a,b)$ and $s\in\mathbb{R}$. If the ratios*

$$\frac{f(t)}{g(t)} \quad \text{and} \quad \frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$$

are both increasing or both decreasing in $t\in(a,b)$, then the function

$$s \mapsto \frac{\int_a^b f(t)w(s,t)\mathrm{d}t}{\int_a^b g(t)w(s,t)\mathrm{d}t} \tag{9}$$

is increasing on $\mathbb{R}$; if one of the ratios $\frac{f(t)}{g(t)}$ and $\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$ is increasing and the other is decreasing in $t\in(a,b)$, then the function (9) is decreasing on $\mathbb{R}$.

In this paper, we denote that $\mathbf{I}=[\alpha_1,\beta_1]\times[\alpha_2,\beta_2]\times\cdots\times[\alpha_n,\beta_n]$ and $\mathbf{t}=(t_1,t_2,\cdots,t_n)$, where $\alpha_i,\beta_i,t_i\in\mathbb{R}(i=1,2,\cdots,n)$; $D_0=\{(s,t):\alpha(s)<t<\beta(s), s>s_0\}$, $\widetilde{D}_0=(s_0,\infty)\times\mathbb{E}_0=(s_0,\infty)\times[\inf_{s\in(s_0,\infty)}\alpha(s),\sup_{s\in(s_0,\infty)}\beta(s)]$, $D_1=\{(s,t):\alpha_1(s)<t<\beta_1(s), s>s_0\}$ and $\mathbb{E}_i=[\inf_{s\in(s_0,\infty)}\alpha_i(s),\sup_{s\in(s_0,\infty)}\beta_i(s)](i=1,2)$, where s_0 is a given constant and $\alpha(s)$, $\beta(s)$, $\alpha_i(s),\beta_i(s)(i=1,2)$ are differentiable functions with $\beta(s)\geq\alpha(s)$ and $\beta_i(s)\geq\alpha_i(s)(i=1,2)$.

Now we will provide some monotonicity rules for the ratio of multiple integrals.

Monotonicity rule 1. *Suppose that the functions $f(t_1)$, $g(t_1)>0$, $w_1(s,t_1)>0$ and $w_2(\mathbf{t})>0$ are integrable in $\mathbf{t}\in\mathbf{I}$ and $s\in(s_0,\infty)$. If the ratios*

$$\frac{f(t_1)}{g(t_1)} \quad \text{and} \quad \frac{\frac{\partial}{\partial s}w_1(s,t_1)}{w_1(s,t_1)}$$

are both increasing or both decreasing in $t_1\in(\alpha_1,\beta_1)$, then the function

$$s \mapsto \frac{\int_{\mathbf{I}} f(t_1)w_1(s,t_1)w_2(\mathbf{t})\mathrm{d}\mathbf{t}}{\int_{\mathbf{I}} g(t_1)w_1(s,t_1)w_2(\mathbf{t})\mathrm{d}\mathbf{t}} \tag{10}$$

is increasing on (s_0,∞); if one of the ratios $\frac{f(t_1)}{g(t_1)}$ and $\frac{\frac{\partial}{\partial s}w_1(s,t_1)}{w_1(s,t_1)}$ is increasing and the other is decreasing in $t_1\in(\alpha_1,\beta_1)$, then the function (10) is decreasing on (s_0,∞).

Proof. Taking derivative of the function (10), we obtain the conclusion immediately. □

The following monotonicity rule can be obtained by reusing Monotonicity rule C.

Monotonicity rule 2. *Suppose that the functions $f(t_n)$, $g(t_n) > 0$, $w_1(s,t_1) > 0$, $w_j(t_{j-1},t_j) > 0$ $(j = 2,3,\cdots,n)$ are integrable in $t_i \in [\alpha_i,\beta_i] (i = 1,2,\cdots,n)$ and $s \in (s_0,\infty)$. If the function $R_j(t_{j-1},t_j)$ is monotonic in t_j for all $j = 2,3,\cdots,n$, $R_0(t_n)$ is monotonic, and $R_1(s,t_1)$ is monotonic in t_1, then the function*

$$s \mapsto \frac{\int_{\mathbf{I}} f(t_n) w_1(s,t_1) \prod_{k=1}^{n-1} w_{k+1}(t_k,t_{k+1}) \, \mathrm{d}\mathbf{t}}{\int_{\mathbf{I}} g(t_n) w_1(s,t_1) \prod_{k=1}^{n-1} w_{k+1}(t_k,t_{k+1}) \, \mathrm{d}\mathbf{t}} \tag{11}$$

is monotonic on (s_0,∞), where

$$R_0(t_n) := \frac{f(t_n)}{g(t_n)}, \quad R_1(s,t_1) := \frac{\frac{\partial}{\partial s} w_1(s,t_1)}{w_1(s,t_1)}, \quad R_j(t_{j-1},t_j) := \frac{\frac{\partial}{\partial t_{j-1}} w_j\left(t_{j-1},t_j\right)}{w_j\left(t_{j-1},t_j\right)}, j = 2,3,\cdots,n.$$

Furthermore, let τ represent the number of decreasing functions in $R_0(t_n)$, $R_1(s,t_1)$, $R_2(t_1,t_2)$, $\cdots$, $R_n(t_{n-1},t_n)$. If τ is even, then the function (11) is increasing; if τ is odd, then the function (11) is decreasing.

Next, we will consider two monotonicity rules for the ratio of variable limit integrals.

Monotonicity rule 3. *Suppose that the continuous functions $f(t)$, $g(t)$ defined on $\mathbb{E}_0$ satisfy that $g(t) > 0$, continuous function $w(s,t)$ defined on $\widetilde{D}_0$ satisfies that it is positive on D_0, $\partial w(s,t)/\partial s$ is continuous on D_0.*

(1) *If $\frac{f(t)}{g(t)}$ is increasing, $\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$ is increasing (decreasing) in t for arbitrary given $s \in (s_0,\infty)$, $\alpha(s)$ and $\beta(s)$ are increasing (decreasing), then the function*

$$s \mapsto \frac{\int_{\alpha(s)}^{\beta(s)} f(t) w(s,t) \mathrm{d}t}{\int_{\alpha(s)}^{\beta(s)} g(t) w(s,t) \mathrm{d}t} \tag{12}$$

is increasing (decreasing) on (s_0,∞).

(2) *If $\frac{f(t)}{g(t)}$ is decreasing, $\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$ is increasing (decreasing) in t for arbitrary given $s \in (s_0,\infty)$, $\alpha(s)$ and $\beta(s)$ are increasing (decreasing), then the function (12) is decreasing (increasing) on (s_0,∞).*

Proof. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\alpha(s)}^{\beta(s)} f(t) w(s,t) \mathrm{d}t\right) = \int_{\alpha(s)}^{\beta(s)} f(t) \frac{\partial w(s,t)}{\partial s} \mathrm{d}t + \beta'(s) f\left(\beta(s)\right) w(s,\beta(s)\right) - \alpha'(s) f(\alpha(s)) w(s,\alpha(s)),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\alpha(s)}^{\beta(s)} g(t) w(s,t) \mathrm{d}t\right) = \int_{\alpha(s)}^{\beta(s)} g(t) \frac{\partial w(s,t)}{\partial s} \mathrm{d}t + \beta'(s) g\left(\beta(s)\right) w(s,\beta(s)) - \alpha'(s) g(\alpha(s)) w(s,\alpha(s)),$$

we have

$$
\begin{aligned}
&\left(\int_{\alpha(s)}^{\beta(s)} g(t)w(s,t)\mathrm{d}t\right)^2 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\int_{\alpha(s)}^{\beta(s)} f(t)w(s,t)\mathrm{d}t}{\int_{\alpha(s)}^{\beta(s)} g(t)w(s,t)\mathrm{d}t}\right)\\
&\quad=\int_{\alpha(s)}^{\beta(s)} f(t)\frac{\partial w(s,t)}{\partial s}\mathrm{d}t\int_{\alpha(s)}^{\beta(s)} g(t)w(s,t)\mathrm{d}t-\int_{\alpha(s)}^{\beta(s)} f(t)w(s,t)\mathrm{d}t\int_{\alpha(s)}^{\beta(s)} g(t)\frac{\partial w(s,t)}{\partial s}\mathrm{d}t\\
&\quad+\left(\beta'(s)f\left(\beta(s)\right)w\left(s,\beta(s)\right)-\alpha'(s)f\left(\alpha(s)\right)w\left(s,\alpha(s)\right)\right)\int_{\alpha(s)}^{\beta(s)} g(t)w(s,t)\mathrm{d}t\\
&\quad-\left(\beta'(s)g\left(\beta(s)\right)w\left(s,\beta(s)\right)-\alpha'(s)g\left(\alpha(s)\right)w\left(s,\alpha(s)\right)\right)\int_{\alpha(s)}^{\beta(s)} f(t)w(s,t)\mathrm{d}t\\
&\quad=\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(t)\frac{\partial w(s,t)}{\partial s}g(x)w(s,x)\mathrm{d}t\mathrm{d}x\\
&\quad-\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(t)w(s,t)g(x)\frac{\partial w(s,x)}{\partial s}\mathrm{d}t\mathrm{d}x\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \beta'(s)g(\beta(s))g(t)w(s,\beta(s))w(s,t)\left(\frac{f(\beta(s))}{g(\beta(s))}-\frac{f(t)}{g(t)}\right)\mathrm{d}t\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(\alpha(s))g(t)w(s,\alpha(s))w(s,t)\left(\frac{f(t)}{g(t)}-\frac{f(\alpha(s))}{g(\alpha(s))}\right)\mathrm{d}t\\
&\quad=\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(t)g(x)w(s,t)w(s,x)\left(\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}-\frac{\frac{\partial w(s,x)}{\partial s}}{w(s,x)}\right)\mathrm{d}t\mathrm{d}x\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \beta'(s)g(\beta(s))g(t)w(s,\beta(s))w(s,t)\left(\frac{f(\beta(s))}{g(\beta(s))}-\frac{f(t)}{g(t)}\right)\mathrm{d}t\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(\alpha(s))g(t)w(s,\alpha(s))w(s,t)\left(\frac{f(t)}{g(t)}-\frac{f(\alpha(s))}{g(\alpha(s))}\right)\mathrm{d}t\\
&\quad=\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(x)g(t)w(s,x)w(s,t)\left(\frac{\frac{\partial w(s,x)}{\partial s}}{w(s,x)}-\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}\right)\mathrm{d}t\mathrm{d}x\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \beta'(s)g(\beta(s))g(t)w(s,\beta(s))w(s,t)\left(\frac{f(\beta(s))}{g(\beta(s))}-\frac{f(t)}{g(t)}\right)\mathrm{d}t\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(\alpha(s))g(t)w(s,\alpha(s))w(s,t)\left(\frac{f(t)}{g(t)}-\frac{f(\alpha(s))}{g(\alpha(s))}\right)\mathrm{d}t\\
&\quad=\frac{1}{2}\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} g(t)g(x)w(s,t)w(s,x)\left(\frac{f(t)}{g(t)}-\frac{f(x)}{g(x)}\right)\left(\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}-\frac{\frac{\partial w(s,x)}{\partial s}}{w(s,x)}\right)\mathrm{d}t\mathrm{d}x\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \beta'(s)g(\beta(s))g(t)w(s,\beta(s))w(s,t)\left(\frac{f(\beta(s))}{g(\beta(s))}-\frac{f(t)}{g(t)}\right)\mathrm{d}t\\
&\quad+\int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(\alpha(s))g(t)w(s,\alpha(s))w(s,t)\left(\frac{f(t)}{g(t)}-\frac{f(\alpha(s))}{g(\alpha(s))}\right)\mathrm{d}t,
\end{aligned}
$$

which implies the desired Monotonicity rule 3. □

Remark 4. The conclusion of Monotonicity rule 3 can be described in Table 1.

Table 1. Monotonicity of function (12)

Cases	$\frac{f(t)}{g(t)}$	$\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$	$\alpha(s), \beta(s)$	Function (12)
1	↗	↗	↗	↗
	↗	↘	↘	↘
2	↘	↗	↗	↘
	↘	↘	↘	↗

Remark 5. If $\alpha(s)$ and $\beta(s)$ are constants, then Monotonicity rule 3 reduces to Monotonicity rule C.

Monotonicity rule 6. *Suppose that the continuous functions $f(s,t)$, $g(s,t)$, and $w(s,t)$ defined on $\widetilde{D}_0$ satisfy $g(s,t) > 0$, $w(s,t)$ is positive on D_0, $\partial f(s,t)/\partial s$, $\partial g(s,t)/\partial s$ and $\partial w(s,t)/\partial s$ are continuous on D_0. Denote that*

$$M_{f,g}(s,t,x) := \frac{\partial f(s,t)}{\partial s} g(s,x) + \frac{\partial f(s,x)}{\partial s} g(s,t) - f(s,x)\frac{\partial g(s,t)}{\partial s} - f(s,t)\frac{\partial g(s,x)}{\partial s}.$$

(1) *If $\frac{f(s,t)}{g(s,t)}$ is increasing in t for arbitrary given $s \in (s_0,\infty)$, $\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$ is increasing (decreasing) in t for arbitrary given $s \in (s_0,\infty)$, $\alpha(s)$ and $\beta(s)$ are increasing (decreasing), $M_{f,g}(s,t,x) \geq (\leq) 0$ for all $s \in (s_0,\infty)$ and $t, x \in \mathbb{E}_0$, then the function*

$$s \mapsto \frac{\int_{\alpha(s)}^{\beta(s)} f(s,t)w(s,t)\mathrm{d}t}{\int_{\alpha(s)}^{\beta(s)} g(s,t)w(s,t)\mathrm{d}t} \tag{13}$$

is increasing (decreasing) on (s_0,∞).

(2) *If $\frac{f(s,t)}{g(s,t)}$ is decreasing in t for arbitrary given $s \in (s_0,\infty)$, $\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$ is decreasing (increasing) in t for arbitrary given $s \in (s_0,\infty)$, $\alpha(s)$ and $\beta(s)$ are decreasing (increasing), $M_{f,g}(s,t,x) \geq (\leq) 0$ for all $s \in (s_0,\infty)$ and $t, x \in \mathbb{E}_0$, then the function* (13) *is increasing (decreasing) on (s_0,∞).*

Proof. Noting that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\alpha(s)}^{\beta(s)} f(s,t)w(s,t)\mathrm{d}t\right) = \int_{\alpha(s)}^{\beta(s)} \frac{\partial}{\partial s}\left(f(s,t)w(s,t)\right)\mathrm{d}t + \beta'(s) f\left(s,\beta(s)\right) w\left(s,\beta(s)\right) - \alpha'(s) f(s,\alpha(s)) w(s,\alpha(s)),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\int_{\alpha(s)}^{\beta(s)} g(s,t)w(s,t)\mathrm{d}t\right) = \int_{\alpha(s)}^{\beta(s)} \frac{\partial}{\partial s}\left(g(s,t)w(s,t)\right)\mathrm{d}t + \beta'(s) g\left(s,\beta(s)\right) w\left(s,\beta(s)\right) - \alpha'(s) g(s,\alpha(s)) w(s,\alpha(s)),$$

we have

$$\left(\int_{\alpha(s)}^{\beta(s)} g(s,t)w(s,t)\mathrm{d}t\right)^2 \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\int_{\alpha(s)}^{\beta(s)} f(s,t)w(s,t)\mathrm{d}t}{\int_{\alpha(s)}^{\beta(s)} g(s,t)w(s,t)\mathrm{d}t}\right)$$

$$
\begin{aligned}
&= \int_{\alpha(s)}^{\beta(s)} \frac{\partial}{\partial s}\Big(f(s,t)w(s,t)\Big)\mathrm{d}t \int_{\alpha(s)}^{\beta(s)} g(s,t)w(s,t)\mathrm{d}t \\
&- \int_{\alpha(s)}^{\beta(s)} f(s,t)w(s,t)\mathrm{d}t \int_{\alpha(s)}^{\beta(s)} \frac{\partial}{\partial s}\Big(g(s,t)w(s,t)\Big)\mathrm{d}t \\
&+ \Big(\beta'(s)f\left(s,\beta(s)\right)w\left(s,\beta(s)\right) - \alpha'(s)f(s,\alpha(s))w(s,\alpha(s))\Big)\int_{\alpha(s)}^{\beta(s)} g(s,t)w(s,t)\mathrm{d}t \\
&- \Big(\beta'(s)g\left(s,\beta(s)\right)w\left(s,\beta(s)\right) - \alpha'(s)g(s,\alpha(s))w(s,\alpha(s))\Big)\int_{\alpha(s)}^{\beta(s)} f(s,t)w(s,t)\mathrm{d}t \\
&= \int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} \left(\frac{\partial f(s,t)}{\partial s}w(s,t) + f(s,t)\frac{\partial w(s,t)}{\partial s}\right)g(s,x)w(s,x)\mathrm{d}t\mathrm{d}x \\
&- \int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(s,t)w(s,t)\left(\frac{\partial g(s,x)}{\partial s}w(s,x) + g(s,x)\frac{\partial w(s,x)}{\partial s}\right)\mathrm{d}t\mathrm{d}x \\
&+ \int_{\alpha(s)}^{\beta(s)} \beta'(s)g\left(s,\beta(s)\right)g(s,t)w\left(s,\beta(s)\right)w(s,t)\left(\frac{f\left(s,\beta(s)\right)}{g\left(s,\beta(s)\right)} - \frac{f(s,t)}{g(s,t)}\right)\mathrm{d}t \\
&+ \int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(s,\alpha(s))g(s,t)w(s,\alpha(s))w(s,t)\left(\frac{f(s,t)}{g(s,t)} - \frac{f(s,\alpha(s))}{g(s,\alpha(s))}\right)\mathrm{d}t \\
&= \int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} \left(\frac{\partial f(s,t)}{\partial s}g(s,x) - f(s,t)\frac{\partial g(s,x)}{\partial s}\right)w(s,t)w(s,x)\mathrm{d}t\mathrm{d}x \\
&+ \int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(s,t)g(s,x)w(s,t)w(s,x)\left(\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)} - \frac{\frac{\partial w(s,x)}{\partial s}}{w(s,x)}\right)\mathrm{d}t\mathrm{d}x \\
&+ \int_{\alpha(s)}^{\beta(s)} \beta'(s)g\left(s,\beta(s)\right)g(s,t)w\left(s,\beta(s)\right)w(s,t)\left(\frac{f\left(s,\beta(s)\right)}{g\left(s,\beta(s)\right)} - \frac{f(s,t)}{g(s,t)}\right)\mathrm{d}t \\
&+ \int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(s,\alpha(s))g(s,t)w(s,\alpha(s))w(s,t)\left(\frac{f(s,t)}{g(s,t)} - \frac{f(s,\alpha(s))}{g(s,\alpha(s))}\right)\mathrm{d}t \\
&= \int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} \left(\frac{\partial f(s,x)}{\partial s}g(s,t) - f(s,x)\frac{\partial g(s,t)}{\partial s}\right)w(s,x)w(s,t)\mathrm{d}t\mathrm{d}x \\
&+ \int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} f(s,x)g(s,t)w(s,x)w(s,t)\left(\frac{\frac{\partial w(s,x)}{\partial s}}{w(s,x)} - \frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}\right)\mathrm{d}t\mathrm{d}x \\
&+ \int_{\alpha(s)}^{\beta(s)} \beta'(s)g\left(s,\beta(s)\right)g(s,t)w\left(s,\beta(s)\right)w(s,t)\left(\frac{f\left(s,\beta(s)\right)}{g\left(s,\beta(s)\right)} - \frac{f(s,t)}{g(s,t)}\right)\mathrm{d}t \\
&+ \int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(s,\alpha(s))g(s,t)w(s,\alpha(s))w(s,t)\left(\frac{f(s,t)}{g(s,t)} - \frac{f(s,\alpha(s))}{g(s,\alpha(s))}\right)\mathrm{d}t \\
&= \frac{1}{2}\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} M_{f,g}(s,t,x)w(s,t)w(s,x)\mathrm{d}t\mathrm{d}x \\
&+ \frac{1}{2}\int_{\alpha(s)}^{\beta(s)}\int_{\alpha(s)}^{\beta(s)} g(s,t)g(s,x)w(s,t)w(s,x)\left(\frac{f(s,t)}{g(s,t)} - \frac{f(s,x)}{g(s,x)}\right)\left(\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)} - \frac{\frac{\partial w(s,x)}{\partial s}}{w(s,x)}\right)\mathrm{d}t\mathrm{d}x \\
&+ \int_{\alpha(s)}^{\beta(s)} \beta'(s)g\left(s,\beta(s)\right)g(s,t)w\left(s,\beta(s)\right)w(s,t)\left(\frac{f\left(s,\beta(s)\right)}{g\left(s,\beta(s)\right)} - \frac{f(s,t)}{g(s,t)}\right)\mathrm{d}t \\
&+ \int_{\alpha(s)}^{\beta(s)} \alpha'(s)g(s,\alpha(s))g(s,t)w(s,\alpha(s))w(s,t)\left(\frac{f(s,t)}{g(s,t)} - \frac{f(s,\alpha(s))}{g(s,\alpha(s))}\right)\mathrm{d}t,
\end{aligned}
$$

which implies the conclusion of Monotonicity rule 6. □

Remark 7. The conclusion of Monotonicity rule 6 can be described in Table 2.

Table 2. Monotonicity of function (13)

Cases	$\frac{f(s,t)}{g(s,t)}$	$\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$	$\alpha(s),\beta(s)$	$M_{f,g}(s,t,x)$	Function (13)
1	↗	↗	↗	≥ 0	↗
	↗	↘	↘	≤ 0	↘
2	↘	↘	↘	≥ 0	↗
	↘	↗	↗	≤ 0	↘

Remark 8. If $f(s,t)$ and $g(s,t)$ are univariate functions with respect to the same variable, then Monotonicity rule 6 reduces to Monotonicity rule 3.

Taking $f(s,t)=\int_{\alpha_2(s)}^{\beta_2(s)} l(y)h(t,y)\mathrm{d}y$ and $g(s,t)=\int_{\alpha_2(s)}^{\beta_2(s)} r(y)h(t,y)\mathrm{d}y$ in Monotonicity rule 6, we have the following corollary.

Corollary 9. *Suppose that the continuous functions $l(y)$, $r(y)$ defined on $\mathbb{E}_2$ satisfy $r(y)>0$, function $w(s,t)$ defined on $(s_0,\infty)\times\mathbb{E}_1$ is continuous and satisfies that it is positive on D_1, positive function $h(t,y)$ defined on $\mathbb{E}_1\times\mathbb{E}_2$ satisfies that $\partial h(t,y)/\partial t$ is continuous, $\partial w(s,t)/\partial s$ is continuous on D_1.*

(1) *If $\frac{l(y)}{r(y)}$ and $\frac{\frac{\partial}{\partial t}h(t,y)}{h(t,y)}$ are increasing in y, $\frac{\frac{\partial}{\partial s}w(s,t)}{w(s,t)}$ is increasing (decreasing) in t for arbitrary given $s\in(s_0,\infty)$, $\alpha_1(s)$, $\beta_1(s)$, $\alpha_2(s)$ and $\beta_2(s)$ are increasing (decreasing), then the function*

$$s\mapsto\frac{\int_{\alpha_1(s)}^{\beta_1(s)}\int_{\alpha_2(s)}^{\beta_2(s)} l(y)w(s,t)h(t,y)\mathrm{d}y\mathrm{d}t}{\int_{\alpha_1(s)}^{\beta_1(s)}\int_{\alpha_2(s)}^{\beta_2(s)} r(y)w(s,t)h(t,y)\mathrm{d}y\mathrm{d}t} \tag{14}$$

is increasing (decreasing) on (s_0,∞).

(2) *If $\frac{l(y)}{r(y)}$ and $-\frac{\frac{\partial}{\partial t}h(t,y)}{h(t,y)}$ is increasing in y, $\frac{\frac{\partial}{\partial s}w(s,t)}{w(s,t)}$ is decreasing (increasing) in t for arbitrary given $s\in(s_0,\infty)$, $-\alpha_1(s)$, $-\beta_1(s)$, $\alpha_2(s)$ and $\beta_2(s)$ are increasing (decreasing), then the function* (14) *is increasing (decreasing) on (s_0,∞).*

(3) *If $-\frac{l(y)}{r(y)}$ and $\frac{\frac{\partial}{\partial t}h(t,y)}{h(t,y)}$ is increasing in y, $\frac{\frac{\partial}{\partial s}w(s,t)}{w(s,t)}$ is decreasing (increasing) in t for arbitrary given $s\in(s_0,\infty)$, $\alpha_1(s)$, $\beta_1(s)$, $\alpha_2(s)$ and $\beta_2(s)$ are decreasing (increasing), then the function* (14) *is increasing (decreasing) on (s_0,∞).*

(4) *If $\frac{l(y)}{r(y)}$ and $\frac{\frac{\partial}{\partial t}h(t,y)}{h(t,y)}$ are decreasing in y, $\frac{\frac{\partial}{\partial s}w(s,t)}{w(s,t)}$ is increasing (decreasing) in t for arbitrary given $s\in(s_0,\infty)$, $\alpha_1(s)$, $\beta_1(s)$, $-\alpha_2(s)$ and $-\beta_2(s)$ are increasing (decreasing), then the function* (14) *is increasing (decreasing) on (s_0,∞).*

Remark 10. The conclusion of Corollary 9 can be described in Table 3.

Table 3. Monotonicity of function (14)

Cases	$\frac{l(y)}{r(y)}$	$\frac{\frac{\partial w(s,t)}{\partial s}}{w(s,t)}$	$\frac{\frac{\partial h(t,y)}{\partial s}}{h(t,y)}$	$\alpha_1(s),\beta_1(s)$	$\alpha_2(s),\beta_2(s)$	Function (14)
1	↗	↗	↗	↗	↗	↗
	↗	↘	↗	↘	↘	↘
2	↗	↘	↘	↘	↗	↗
	↗	↗	↘	↗	↘	↘
3	↘	↘	↗	↘	↘	↗
	↘	↗	↗	↗	↗	↘
4	↘	↗	↘	↗	↘	↗
	↘	↘	↘	↘	↗	↘

3. Integral representations and asymptotic formulas of $T_{\nu,\alpha,\beta}(s)$

In this section, we investigate the integral representations of $T_{\nu,\alpha,\beta}(s)$, as well as its asymptotic formulas.

Proposition 11. *Let $s>0$, $\nu,\alpha,\beta\in\mathbb{R}$. Then*

$$T_{\nu,\alpha,\beta}(s)=2\int_0^\infty\int_0^\infty e^{-2s\cosh(t)\cosh(y)}\cosh(2\nu y)\big(1-\beta\cosh(2\alpha t)\big)\mathrm{d}y\mathrm{d}t \tag{15}$$

$$=2\int_{2s}^\infty\int_0^\infty e^{-t\cosh(y)}\cosh(2\nu y)\frac{1-\beta\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t. \tag{16}$$

Proof. Combining formulas (1) and (2), we have

$$\begin{aligned}K_\mu(s)K_\nu(s)&=2\int_0^\infty K_{\mu+\nu}\big(2s\cosh(t)\big)\cosh\big((\mu-\nu)t\big)\mathrm{d}t\\&=2\int_0^\infty\int_0^\infty e^{-2s\cosh(t)\cosh(y)}\cosh\big((\mu+\nu)y\big)\cosh\big((\mu-\nu)t\big)\mathrm{d}y\mathrm{d}t,\end{aligned}$$

and

$$\begin{aligned}T_{\nu,\alpha,\beta}(s)&=K_\nu^2(s)-\beta K_{\nu-\alpha}(s)K_{\nu+\alpha}(s)\\&=2\int_0^\infty K_{2\nu}\big(2s\cosh(t)\big)\big(1-\beta\cosh(2\alpha t)\big)\mathrm{d}t\\&=2\int_0^\infty\int_0^\infty e^{-2s\cosh(t)\cosh(y)}\cosh(2\nu y)\big(1-\beta\cosh(2\alpha t)\big)\mathrm{d}y\mathrm{d}t.\end{aligned}$$

Taking $t=2s\cosh(x)$ in

$$K_\mu(s)K_\nu(s)=2\int_0^\infty K_{\mu+\nu}\big(2s\cosh(x)\big)\cosh\big((\mu-\nu)x\big)\mathrm{d}x,$$

we have

$$\begin{aligned}&K_\mu(s)K_\nu(s)\\&=2\int_{2s}^\infty K_{\mu+\nu}(t)\cosh\left((\mu-\nu)\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)\frac{1}{\sqrt{t^2-4s^2}}\mathrm{d}t\\&=2\int_{2s}^\infty\int_0^\infty e^{-t\cosh(y)}\cosh\big((\mu+\nu)y\big)\frac{\cosh\left((\mu-\nu)\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t,\end{aligned}$$

and

$$T_{\nu,\alpha,\beta}(s)=2\int_{2s}^\infty\int_0^\infty e^{-t\cosh(y)}\cosh(2\nu y)\frac{1-\beta\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t.$$

□

Now, we will give the n^{th} derivative of $T_{\nu,\alpha,\beta}(s)$ with respect to s and ν by using formulas (15) and (16), respectively.

Proposition 12. *Let $s>0$, $n\in\mathbb{N}$ and $\nu,\alpha,\beta\in\mathbb{R}$. Then*

$$\frac{\mathrm{d}^nT_{\nu,\alpha,\beta}(s)}{\mathrm{d}s^n}=(-1)^{n-1}2^{n+1}s^n\int_0^\infty\int_0^\infty\begin{pmatrix}\big(\cosh(t)\cosh(y)\big)^ne^{-2s\cosh(t)\cosh(y)}\\\times\cosh(2\nu y)\big(\beta\cosh(2\alpha t)-1\big)\end{pmatrix}\mathrm{d}y\mathrm{d}t,$$

and

$$
\begin{aligned}
&\frac{\partial^n T_{\nu,\alpha,\beta}(s)}{\partial \nu^n}\\
&\qquad = \begin{cases} 2^{n+1}\int_0^\infty\int_0^\infty y^n e^{-2s\cosh(t)\cosh(y)}\sinh(2\nu y)\big(1-\beta\cosh(2\alpha t)\big)\mathrm{d}y\mathrm{d}t & \text{if } n \text{ is odd}\\ 2^{n+1}\int_0^\infty\int_0^\infty y^n e^{-2s\cosh(t)\cosh(y)}\cosh(2\nu y)\big(1-\beta\cosh(2\alpha t)\big)\mathrm{d}y\mathrm{d}t & \text{if } n \text{ is even}\end{cases}\\
&\qquad = \begin{cases} 2^{n+1}\int_{2s}^\infty\int_0^\infty y^n e^{-t\cosh(y)}\sinh(2\nu y)\frac{1-\beta\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t & \text{if } n \text{ is odd,}\\ 2^{n+1}\int_{2s}^\infty\int_0^\infty y^n e^{-t\cosh(y)}\cosh(2\nu y)\frac{1-\beta\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t & \text{if } n \text{ is even.}\end{cases}
\end{aligned}
$$

Next, we will consider the asymptotic formulas of function $T_{\nu,\alpha,\beta}(s)$ at the points $s=0$ and $s=\infty$.

Proposition 13. *Let $s>0$, and $\nu,\alpha,\beta\in\mathbb{R}$. Then*

$$T_{\nu,\alpha,\beta}(s)\sim\frac{2^{2\nu-2}}{s^{2\nu}}\Gamma^2(\nu)-\beta\frac{2^{|\nu-\alpha|-1}}{s^{|\nu-\alpha|}}\frac{2^{|\nu+\alpha|-1}}{s^{|\nu+\alpha|}}\Gamma(|\nu-\alpha|)\Gamma(|\nu+\alpha|),\qquad \text{as } s\to 0 \text{ if } \nu>0, \tag{17}$$

and

$$T_{\nu,\alpha,\beta}(s)\sim\frac{\pi}{2s}e^{-2s}\left(\big(1-\beta\big)+\frac{-4\alpha^2\beta-4\beta\nu^2+\beta+4\nu^2-1}{4s}\right),\qquad \text{as } s\to\infty. \tag{18}$$

Proof. The modified Bessel function of the second kind has the following two approximate formulas [1]

$$K_\nu(s)\sim\frac{2^{\nu-1}}{s^\nu}\Gamma(\nu),\qquad \text{as } s\to 0 \text{ if } \nu>0, \tag{19}$$

and

$$K_\nu(s)\sim\sqrt{\frac{\pi}{2s}}e^{-s}\left(1+\frac{4\nu^2-1}{8s}\right),\qquad \text{as } s\to\infty. \tag{20}$$

From asymptotic formulas (19) and (20), we obtain the asymptotic formulas (17) and (18). □

4. Monotonicity of $T_{\nu,\alpha,\beta}(s)$, $L^*_{\mu,\nu,\alpha}(s)$, $L^{**}_{\mu,\nu,\alpha}(s)$, and $W_{\nu,n_1,n_2,\alpha}(s)$

It is easy to check that $T_{-\nu,\alpha,\beta}(s)=T_{\nu,\alpha,\beta}(s)$ and $T_{\nu,-\alpha,\beta}(s)=T_{\nu,\alpha,\beta}(s)$, which indicates the function $T_{\nu,\alpha,\beta}(s)$ are even with respect to ν and α respectively.

4.1. *Complete monotonicity and absolute monotonicity of $T_{\nu,\alpha,\beta}(s)$*

Recall that completely monotonic functions and absolutely monotonic functions were given by the following definitions.

Definition 14 ([10, 26]). *If a function $\phi\colon(0,\infty)\to\mathbb{R}$ has derivatives of all orders and satisfies*

$$(-1)^n\frac{\mathrm{d}^n\phi(s)}{\mathrm{d}s^n}\ge 0$$

for all $s>0$ and $n\in\mathbb{N}$, then ϕ is said to be a completely monotonic function.

Definition 15 ([26]). *If a function $\phi\colon(0,\infty)\to\mathbb{R}$ has derivatives of all orders and satisfies*

$$\frac{\mathrm{d}^n\phi(s)}{\mathrm{d}s^n}\ge 0$$

for all $s>0$ and $n\in\mathbb{N}$, then ϕ is said to be an absolutely monotonic function.

In this subsection, we will provide the complete monotonicity and absolute monotonicity of $T_{\nu,\alpha,\beta}(s)$ and give some corresponding corollaries.

Theorem 16. *Let the function $T_{\nu,\alpha,\beta}(s)$ be defined by* (5). *Then $-T_{\nu,\alpha,\beta}(s)$ is completely monotonic in s and is absolutely monotonic in ν if and only if $\beta \geq 1$.*

Proof. Sufficiency can be deduced directly from Proposition 12, and Necessity can be deduced directly from the asymptotic formula (18). □

Taking $\alpha = 1$ and $\frac{1}{2}$ in Theorem 16 respectively, we obtain Corollary 17.

Corollary 17. *The functions*

$$s \mapsto K_\nu^2(s) - \beta K_{\nu-1}(s)K_{\nu+1}(s), \qquad s \mapsto K_\nu^2(s) - \beta K_{\nu-\frac{1}{2}}(s)K_{\nu+\frac{1}{2}}(s)$$

are completely monotonic on $(0,\infty)$ *for all $\nu \in \mathbb{R}$ if and only if $\beta \geq 1$.*

The following corollary can be reduced from Theorem 16.

Corollary 18. *The function*

$$s \mapsto K_{\nu-2}(s)K_{\nu+1}(s) + K_{\nu-1}(s)K_{\nu+2}(s) - K_\nu(s)K_{\nu+1}(s) - K_{\nu-1}(s)K_\nu(s)$$

is completely monotonic on $(0,\infty)$ *for all $\nu \in \mathbb{R}$.*

Proof. Using the recursive formulas [25] for the modified Bessel function of the second kind

$$K_\nu'(s) = -K_{\nu-1}(s) - \frac{\nu}{s}K_\nu(s),$$

and

$$s\big(K_{\nu+1}(s) - K_{\nu-1}(s)\big) = 2\nu K_\nu(s),$$

we obtain

$$\begin{aligned}
\frac{\mathrm{d}T_{\nu,1,1}(s)}{\mathrm{d}s} &= 2K_\nu(s)K_\nu'(s) - K_{\nu-1}(s)K_{\nu+1}'(s) - K_{\nu+1}(s)K_{\nu-1}'(s) \\
&= K_\nu(s)\Big(-K_{\nu-1}(s) - K_{\nu+1}(s)\Big) - \frac{1}{2}\Big(-K_{\nu-2}(s) - K_\nu(s)\Big)K_{\nu+1}(s) \\
&\quad - \frac{1}{2}K_{\nu-1}(s)\Big(-K_\nu(s) - K_{\nu+2}(s)\Big) \\
&= \frac{1}{2}\Big(K_{\nu-2}(s)K_{\nu+1}(s) + K_{\nu-1}(s)K_{\nu+2}(s) - K_\nu(s)K_{\nu+1}(s) - K_{\nu-1}(s)K_\nu(s)\Big).
\end{aligned}$$

In addition, we know that $\frac{dT_{\nu,1,1}(s)}{ds}$ is completely monotonic in s by using Theorem 16. The proof is completed. □

By employing the monotonicity of $T_{\nu,1,1}(s)$ in ν, an upper bound for $K_\nu^2(s) - K_{\nu-1}(s)K_{\nu+1}(s)$ is obtained.

Corollary 19. *Let $s > 0$ and $\nu \geq 0$. Then*

$$K_\nu^2(s) - K_{\nu-1}(s)K_{\nu+1}(s) \leq K_0^2(s) - K_1^2(s). \tag{21}$$

Remark 20. Obviously, inequality (21) is sharper than inequality (4).

4.2. *Monotonicity of functions $L^*_{\mu,\nu,\alpha}(s)$, $L^{**}_{\mu,\nu,\alpha}(s)$, and $W_{\nu,n_1,n_2,\alpha}(s)$*

In order to prove Theorems 24, 25, and 26, we first present three lemmas.

By taking derivations, the following lemma can be directly proved.

Lemma 21. *If $a_1, a_2, \cdots, a_n \in \mathbb{R}$, $b_1 \geq b_2 \geq \cdots \geq b_n$, and $m_{i,a,b} \geq 0$ for all $i = 0, 1, 2, \cdots, n$, then the function*

$$s \mapsto a_1 s^{b_1} + a_2 s^{b_2} + \cdots + a_n s^{b_n}$$

is non-negative for all $s \geq 1$, where

$$m_{0,a,b} := \sum_{j=1}^{n} a_j, \quad m_{1,a,b} := \sum_{j=1}^{n} a_j b_j,$$

and

$$m_{i,a,b} := \sum_{j=1}^{n-i+1} \left(a_j b_j \prod_{k=1}^{i-1} \left(b_j - b_{n+1-k} \right) \right), \qquad i = 2, 3, \cdots, n.$$

Lemma 22. *The function*

$$Y(z,\alpha) := \alpha^2 \left(z^2-1\right)^2 \left(z^2+1\right) z^{2\alpha-2} - \left(z^2+1\right)\left(z^{2\alpha}-1\right)^2 + \alpha\left(z^2-1\right)\left(z^{4\alpha}-1\right) \geq 0 \tag{22}$$

for all $z \geq 1$ and $\alpha \geq 0$.

Proof. Clearly, we have

$$\begin{aligned} &\alpha^2 \left(z^2-1\right)^2 \left(z^2+1\right) z^{2\alpha-2} - \left(z^2+1\right)\left(z^{2\alpha}-1\right)^2 + \alpha\left(z^2-1\right)\left(z^{4\alpha}-1\right) \\ &= (\alpha-1) z^{4\alpha+2} - (\alpha+1) z^{4\alpha} + \alpha^2 z^{2\alpha+4} + \left(2-\alpha^2\right) z^{2\alpha+2} + \left(2-\alpha^2\right) z^{2\alpha} + \alpha^2 z^{2\alpha-2} - (\alpha+1) z^2 + (\alpha-1). \end{aligned}$$

This proof is divided into the following five cases.

(i) Assume that $\alpha \geq 2$. In this case, we have $b_1 = 4\alpha+2 \geq b_2 = 4\alpha \geq b_3 = 2\alpha+4 \geq b_4 = 2\alpha+2 \geq b_5 = 2\alpha \geq b_6 = 2\alpha-2 \geq b_7 = 2 \geq b_8 = 0$ and $a_1 = \alpha-1$, $a_2 = -1-\alpha$, $a_3 = \alpha^2$, $a_4 = 2-\alpha^2$, $a_5 = 2-\alpha^2$, $a_6 = \alpha^2$, $a_7 = -1-\alpha$, $a_8 = -1+\alpha$. Noting that

$$m_{0,a,b} = 0, \qquad m_{1,a,b} = 0, \qquad m_{2,a,b} = 16\alpha^2 \geq 0, \qquad m_{3,a,b} = 16\alpha^2(6\alpha+1) \geq 0,$$

$$m_{4,a,b} = 32\alpha^2\left(8\alpha^2+11\alpha+4\right) \geq 0, \qquad m_{5,a,b} = 64\alpha^2\left(8\alpha^3+18\alpha^2+17\alpha+7\right) \geq 0,$$
$$m_{6,a,b} = 128\alpha^2\left(10\alpha^4+14\alpha^3+7\alpha^2+13\alpha+10\right) \geq 0,$$
$$m_{7,a,b} = 256\alpha^2\left(14\alpha^5+3\alpha^4-25\alpha^3-5\alpha^2+11\alpha+2\right) \geq 0,$$

and

$$m_{8,a,b} = 512(\alpha-1)^2\alpha^2(\alpha+1)(\alpha+2)(2\alpha+1)^2 \geq 0,$$

we have $Y(z,\alpha) \geq 0$ by employing Lemma 21.

(ii) Assume that $1 < \alpha \leq 2$. In this case, we have $b_1 = 4\alpha+2 \geq b_2 = 2\alpha+4 \geq b_3 = 4\alpha \geq b_4 = 2\alpha+2 \geq b_5 = 2\alpha \geq b_6 = 2 \geq b_7 = 2\alpha-2 \geq b_8 = 0$ and $a_1 = \alpha-1$, $a_2 = \alpha^2$, $a_3 = -1-\alpha$, $a_4 = 2-\alpha^2$, $a_5 = 2-\alpha^2$, $a_6 = -1-\alpha$, $a_7 = \alpha^2$, $a_8 = -1+\alpha$. It is easy to check that

$$m_{0,a,b} = 0, \qquad m_{1,a,b} = 0, \qquad m_{2,a,b} = 16\alpha^2 \geq 0, \qquad m_{3,a,b} = 16\alpha^2(4\alpha+5) \geq 0,$$
$$m_{4,a,b} = 32\alpha^2\left(8\alpha^2+11\alpha+4\right) \geq 0, \qquad m_{5,a,b} = 64\alpha^2\left(8\alpha^3+18\alpha^2+17\alpha+7\right) \geq 0,$$
$$m_{6,a,b} = 128\alpha^2\left(10\alpha^4+14\alpha^3+7\alpha^2+13\alpha+10\right) \geq 0,$$
$$m_{7,a,b} = 256\alpha^2\left(4\alpha^5+9\alpha^4-4\alpha^3-4\alpha^2+27\alpha+22\right) \geq 0,$$

and

$$m_{8,a,b} = 512(\alpha-1)^2\alpha^2(\alpha+1)(\alpha+2)(2\alpha+1)^2 \geq 0,$$

then $Y(z,\alpha) \geq 0$ by employing Lemma 21.

(iii) Assume that $1/2 < \alpha \le 1$. In this case, $b_1 = 2\alpha+4 \ge b_2 = 4\alpha+2 \ge b_3 = 2\alpha+2 \ge b_4 = 4\alpha \ge b_5 = 2 \ge b_6 = 2\alpha \ge b_7 = 0 \ge b_8 = 2\alpha-2$ and $a_1 = \alpha^2$, $a_2 = \alpha-1$, $a_3 = 2-\alpha^2$, $a_4 = -1-\alpha$, $a_5 = -1-\alpha$, $a_6 = 2-\alpha^2$, $a_7 = -1+\alpha$, $a_8 = \alpha^2$. We verify that

$$m_{0,a,b} = 0, \qquad m_{1,a,b} = 0, \qquad m_{2,a,b} = 16\alpha^2 \ge 0, \qquad m_{3,a,b} = 16\alpha^2(4\alpha+5) \ge 0,$$
$$m_{4,a,b} = 32\alpha^2\left(4\alpha^2+10\alpha+9\right) \ge 0, \qquad m_{5,a,b} = 64\alpha^2\left(8\alpha^3+18\alpha^2+17\alpha+7\right) \ge 0,$$
$$m_{6,a,b} = 128\alpha^2\left(2\alpha^4+4\alpha^3+8\alpha^2+23\alpha+17\right) \ge 0,$$
$$m_{7,a,b} = 256\alpha^2\left(4\alpha^5+9\alpha^4-4\alpha^3-4\alpha^2+27\alpha+22\right) \ge 0,$$

and

$$m_{8,a,b} = 1536(\alpha-2)(\alpha-1)\alpha^2(\alpha+1)(\alpha+2)^2 \ge 0,$$

then $Y(z,\alpha) \ge 0$ by employing Lemma 21.

(iv) Assume that $0 < \alpha \le 1/2$. In this case, we have $b_1 = 2\alpha+4 \ge b_2 = 4\alpha+2 \ge b_3 = 2\alpha+2 \ge b_4 = 2 \ge b_5 = 4\alpha \ge b_6 = 2\alpha \ge b_7 = 0 \ge b_8 = 2\alpha-2$ and $a_1 = \alpha^2$, $a_2 = \alpha-1$, $a_3 = 2-\alpha^2$, $a_4 = -1-\alpha$, $a_5 = -1-\alpha$, $a_6 = 2-\alpha^2$, $a_7 = -1+\alpha$, $a_8 = \alpha^2$. We confirm that

$$m_{0,a,b} = 0, \qquad m_{1,a,b} = 0, \qquad m_{2,a,b} = 16\alpha^2 \ge 0, \qquad m_{3,a,b} = 16\alpha^2(4\alpha+5) \ge 0,$$
$$m_{4,a,b} = 32\alpha^2\left(4\alpha^2+10\alpha+9\right) \ge 0, \qquad m_{5,a,b} = 64\alpha^2\left(2\alpha^2+9\alpha+16\right) \ge 0,$$
$$m_{6,a,b} = 128\alpha^2\left(2\alpha^4+4\alpha^3+8\alpha^2+23\alpha+17\right) \ge 0,$$
$$m_{7,a,b} = 256\alpha^2\left(4\alpha^5+9\alpha^4-4\alpha^3-4\alpha^2+27\alpha+22\right) \ge 0,$$

and

$$m_{8,a,b} = 1536(\alpha-2)(\alpha-1)\alpha^2(\alpha+1)(\alpha+2)^2 \ge 0,$$

then $Y(z,\alpha) \ge 0$ by employing Lemma 21.

(v) Assume that $\alpha = 0$. In this case, the function (26) identically equal to 0.

□

Lemma 23. *Let*

$$\mathscr{M}_{\alpha,\beta}(s,t) := \frac{1-\beta\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}. \tag{23}$$

Then the inequality

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial s}\mathscr{M}_{\alpha,\beta}(s,t)}{\mathscr{M}_{\alpha,\beta}(s,t)}\right) \ge 0$$

holds for all $(s,t) \in D_2$ *and* $\alpha \in \mathbb{R}$ *if and only if* $\beta = 1$, *where* $D_2 = \{(s,t) : t > 2s, s > 0\}$.

Proof. Since $\mathscr{M}_{\alpha,\beta}(s,t)$ is even with respect to α, we focus on $\alpha \ge 0$.

Sufficiency. Since $\beta = 1$, the function $\mathscr{M}_{\alpha,\beta}(s,t)$ reduces to

$$\mathscr{M}_{\alpha,1}(s,t) = \frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}.$$

By a simple calculation, we can obtain

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial s}\mathcal{M}_{\alpha,1}(s,t)}{\mathcal{M}_{\alpha,1}(s,t)}\right) = \frac{2\left(\begin{array}{l} -2\left(2\alpha^2-1\right)s^2t+\alpha^2t^3-2s^2t\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right) \\ \quad +2\alpha s^2\sqrt{t^2-4s^2}\sinh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)\end{array}\right)\operatorname{csch}^2\left(\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{s\left(t^2-4s^2\right)^2}. \tag{24}$$

Taking $t = xs$, where $x \geq 2$, the right hand side of (24) is changed to

$$\frac{2\left(\begin{array}{l} x\sqrt{x^2-4}\left(\alpha^2\left(x^2-4\right)+2\right) \\ \quad -2x\sqrt{x^2-4}\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{x}{2}\right)\right) \\ \quad +2\alpha(x^2-4)\sinh\left(2\alpha\operatorname{arccosh}\left(\frac{x}{2}\right)\right)\end{array}\right)\operatorname{csch}^2\left(\alpha\operatorname{arccosh}\left(\frac{x}{2}\right)\right)}{s^2\left(x^2-4\right)^{5/2}}. \tag{25}$$

Let

$$\mathcal{N}(x,\alpha) := x\sqrt{x^2-4}\left(\alpha^2\left(x^2-4\right)+2\right)-2x\sqrt{x^2-4}\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{x}{2}\right)\right) + 2\alpha(x^2-4)\sinh\left(2\alpha\operatorname{arccosh}\left(\frac{x}{2}\right)\right).$$

We have

$$\mathcal{N}(x,\alpha) = 4^{-\alpha}\left(\sqrt{x^2-4}+x\right)^{-2\alpha}\times \left(\begin{array}{l} 4^{\alpha}\alpha^2x\left(x^2-4\right)^{3/2}\left(\sqrt{x^2-4}+x\right)^{2\alpha}-\alpha(x^2-4)\left(16^{\alpha}-\left(\sqrt{x^2-4}+x\right)^{4\alpha}\right) \\ \quad -x\sqrt{x^2-4}\left(4^{\alpha}-\left(\sqrt{x^2-4}+x\right)^{2\alpha}\right)^2 \end{array}\right). \tag{26}$$

Taking $x = (z^2+1)/z$ in (26), then $\mathcal{N}(x,\alpha)$ becomes

$$\mathcal{N}\left(\frac{z^2+1}{z},\alpha\right) = z^{-2-2\alpha}\left(z^2-1\right)\left(\alpha^2\left(z^2-1\right)^2\left(z^2+1\right)z^{2\alpha-2}-\left(z^2+1\right)\left(z^{2\alpha}-1\right)^2+\alpha\left(z^2-1\right)\left(z^{4\alpha}-1\right)\right),$$

where $z \geq 1$.

According to Lemma 22, we know that $\mathcal{N}((z^2+1)/z,\alpha) \geq 0$ for all $z \geq 1$ and $\alpha \geq 0$. Sufficiency is proven.

Necessity. Since

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial s}\mathcal{M}_{\alpha,\beta}(s,t)}{\mathcal{M}_{\alpha,\beta}(s,t)}\right) \geq 0$$

for all $(s,t)\in D_2$ and $\alpha\in\mathbb{R}$, we have

$$0\le \frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial s}\mathcal{M}_{\alpha,\beta}(s,t)}{\mathcal{M}_{\alpha,\beta}(s,t)}\right)$$
$$= \frac{4\left(\begin{array}{l} -t\left(s^2\left(\left(1-4\alpha^2\right)\beta^2+2\right)+\alpha^2\beta^2t^2\right)\sqrt{t^2-4s^2} \\ \quad +\beta t\left(\alpha^2t^2-4\left(\alpha^2-1\right)s^2\right)\sqrt{t^2-4s^2}\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right) \\ \quad -\beta^2s^2t\sqrt{t^2-4s^2}\cosh\left(4\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right) \\ \quad +\alpha\beta^2s^2\left(t^2-4s^2\right)\sinh\left(4\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right) \\ \quad +2\alpha\beta s^2\left(4s^2-t^2\right)\sinh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right) \end{array}\right)}{s(t-2s)^2(2s+t)^2\sqrt{t^2-4s^2}\left(\beta\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)-1\right)^2}. \tag{27}$$

Taking $t=xs$ in (27), we get

$$0\le \frac{-4\alpha^2\beta^2s^3x^3+4s^3x\left(\left(4\alpha^2-1\right)\beta^2-2\right)}{s\left(s^2x^2-4s^2\right)^2\left(\beta\cosh\left(2\alpha\operatorname{arcsech}\left(\frac{2}{x}\right)\right)-1\right)^2}$$
$$+4\beta\frac{\left(\begin{array}{l} sx\left(\alpha^2s^2x^2-4\left(\alpha^2-1\right)s^2\right)\cosh\left(2\alpha\operatorname{arcsech}(2/x)\right)-\beta s^3x\cosh\left(4\alpha\operatorname{arcsech}(2/x)\right) \\ \quad +\alpha s^3\sqrt{x^2-4}\left(\beta\sinh\left(4\alpha\operatorname{arcsech}(2/x)\right)-2\sinh\left(2\alpha\operatorname{arcsech}(2/x)\right)\right) \end{array}\right)}{s\left(s^2x^2-4s^2\right)^2\left(\beta\cosh\left(2\alpha\operatorname{arcsech}(2/x)\right)-1\right)^2}.$$

Thus, we obtain

$$4s^3x\left(\left(4\alpha^2-1\right)\beta^2-2\right)-4\alpha^2\beta^2s^3x^3$$
$$+4\beta\left(\begin{array}{l} sx\left(\alpha^2s^2x^2-4\left(\alpha^2-1\right)s^2\right)\cosh\left(2\alpha\operatorname{arcsech}(2/x)\right)-\beta s^3x\cosh\left(4\alpha\operatorname{arcsech}(2/x)\right) \\ \quad +\alpha s^3\sqrt{x^2-4}\left(\beta\sinh\left(4\alpha\operatorname{arcsech}(2/x)\right)-2\sinh\left(2\alpha\operatorname{arcsech}(2/x)\right)\right) \end{array}\right)\ge 0$$

for all $s,\alpha\ge 0$ and $x\ge 2$. Taking $\alpha=0$ in the above inequality gives to

$$-8(\beta-1)^2s^3x\ge 0,$$

which means $\beta=1$. □

Now we will investigate the necessary and sufficient conditions for the monotonicity of functions $L^*_{\mu,\nu,\alpha}(s)$, $L^{**}_{\mu,\nu,\alpha}(s)$, and $W_{\nu,n_1,n_2,\alpha}(s)$. Since $L^*_{\mu,\nu,\alpha}(s)$, $L^{**}_{\mu,\nu,\alpha}(s)$, and $W_{\nu,n_1,n_2,\alpha}(s)$ reduce to constants when $|\mu|=|\nu|$ and $n_1=n_2$, we only focus on $|\mu|\ne|\nu|$ and $n_1\ne n_2$ in this section.

Theorem 24. *The function $L^*_{\mu,\nu,\alpha}(s)$ decreases from $(0,\infty)$ to $(1,A_{\mu,\nu}(\alpha))$ if and only if $|\mu|>|\nu|$ and increases from $(0,\infty)$ to $(A^{-1}_{\mu,\nu}(\alpha),1)$ if and only if $|\mu|<|\nu|$, where*

$$A_{\mu,\nu}(\alpha):=\begin{cases}+\infty & \text{if } \alpha\le|\mu|,\\ \frac{\Gamma(\alpha-\mu)\Gamma(\alpha+\mu)}{\Gamma(\alpha-\nu)\Gamma(\alpha+\nu)} & \text{if } |\mu|<\alpha,\end{cases}\qquad A^{-1}_{\mu,\nu}(\alpha):=\begin{cases}0 & \text{if } \alpha\le|\nu|,\\ \frac{\Gamma(\alpha-\mu)\Gamma(\alpha+\mu)}{\Gamma(\alpha-\nu)\Gamma(\alpha+\nu)} & \text{if } |\nu|<\alpha.\end{cases}$$

Proof. We first prove that the function $L^*_{\mu,\nu,\alpha}(s)$ decreases from $(0,\infty)$ to $(1,A_{\mu,\nu}(\alpha))$ if $\mu>\nu\ge 0$. Based on formula (16), we have

$$L^*_{\mu,\nu,\alpha}(s)=\frac{T_{\mu,\alpha,1}(s)}{T_{\nu,\alpha,1}(s)}=\frac{\int_{2s}^{\infty}\int_0^{\infty}e^{-t\cosh(y)}\cosh ł(2\mu y)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}{\int_{2s}^{\infty}\int_0^{\infty}e^{-t\cosh(y)}\cosh(2\nu y)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}.$$

Let

$$l(y)=\cosh(2\mu y),\qquad r(y)=\cosh(2\nu y),$$

$$w(s,t)=\begin{cases}\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}=\mathscr{M}_{\alpha,1}(s,t) & (s,t)\in D_2,\\ 0 & (s,t)\in(0,\infty)\times(0,\infty)-D_2,\end{cases}$$

and

$$h(t,y)=e^{-t\cosh(y)},$$

where $\mathscr{M}_{\alpha,\beta}(s,t)$ is defined by (23) and $D_2=\{(s,t):t>2s,s>0\}$. By using Lemma 23, we get

$$\frac{\partial}{\partial t}\left(\frac{\frac{\partial}{\partial s}w(s,t)}{w(s,t)}\right)\geq 0,\qquad \frac{\partial}{\partial y}\left(\frac{\frac{\partial}{\partial t}h(t,y)}{h(t,y)}\right)=-\sinh(y)\leq 0. \tag{28}$$

Since $\mu(e^{4y\mu}-1)/(e^{4y\mu}+1)$ is monotonic in μ, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{l(y)}{r(y)}\right)&=2\operatorname{sech}(2\nu y)\left(\mu\sinh(2\mu y)-\nu\cosh(2\mu y)\tanh(2\nu y)\right)\\&=\frac{2e^{2y(\nu-\mu)}\left(\mu\frac{e^{4y\mu}-1}{e^{4y\mu}+1}-\nu\frac{e^{4y\nu}-1}{e^{4y\nu}+1}\right)\left(e^{4y\mu}+1\right)}{\left(e^{4y\nu}+1\right)}\geq 0.\end{aligned}$$

Thus, we get $T_{\mu,\alpha,1}(s)/T_{\nu,\alpha,1}(s)$ is decreasing in $s\in(0,\infty)$ by using Corollary 9 immediately. Moreover, by employing formulas (17) and (18), we have

$$\begin{aligned}\lim_{s\to 0}\frac{T_{\mu,\alpha,1}(s)}{T_{\nu,\alpha,1}(s)}&=\begin{cases}\lim_{s\to 0}\frac{\frac{2^{2\mu-2}}{s^{2\mu}}\Gamma^2(\mu)-\frac{2^{|\mu-\alpha|-1}}{s^{|\mu-\alpha|}}\frac{2^{|\mu+\alpha|-1}}{s^{|\mu+\alpha|}}\Gamma(|\mu-\alpha|)\Gamma(|\mu+\alpha|)}{\frac{2^{2\nu-2}}{s^{2\nu}}\Gamma^2(\nu)-\frac{2^{|\nu-\alpha|-1}}{s^{|\nu-\alpha|}}\frac{2^{|\nu+\alpha|-1}}{s^{|\nu+\alpha|}}\Gamma(|\nu-\alpha|)\Gamma(|\nu+\alpha|)} & \text{if } \alpha\neq\mu,\nu\\ \lim_{s\to 0}\frac{T_{\mu,\mu,1}(s)}{T_{\nu,\mu,1}(s)} & \text{if } \alpha=\mu\\ \lim_{s\to 0}\frac{T_{\mu,\nu,1}(s)}{T_{\nu,\nu,1}(s)} & \text{if } \alpha=\nu\end{cases}\\&=\begin{cases}+\infty & \text{if } \alpha\leq\mu,\\ \frac{\Gamma(\alpha-\mu)\Gamma(\alpha+\mu)}{\Gamma(\alpha-\nu)\Gamma(\alpha+\nu)} & \text{if } \mu<\alpha,\end{cases}\end{aligned} \tag{29}$$

and

$$\lim_{s\to\infty}\frac{T_{\mu,\alpha,1}(s)}{T_{\nu,\alpha,1}(s)}=\lim_{s\to\infty}\frac{-\frac{1}{s}+\frac{1-4\mu^2}{4s^2}+O\left(\frac{1}{s}\right)^3}{-\frac{1}{s}+\frac{1-4\nu^2}{4s^2}+O\left(\frac{1}{s}\right)^3}=1. \tag{30}$$

Combining the formulas (29) and (30) as well as the fact that $T_{\mu,\alpha,1}(s)/T_{\nu,\alpha,1}(s)$ is decreasing in $s\in(0,\infty)$, we deduce that the function $L^*_{\mu,\nu,\alpha}(s)$ decreases from $(0,\infty)$ to $(1,A_{\mu,\nu}(\alpha))$ if $\mu>\nu\geq 0$.

Taking the reciprocal of the function $L^*_{\mu,\nu,\alpha}(s)$, we obtain that the function $L^*_{\mu,\nu,\alpha}(s)$ increases from $(0,\infty)$ to $(A^{-1}_{\mu,\nu}(\alpha),1)$ if $\nu>\mu\geq 0$. Lastly, the proof can be completed by utilizing the symmetry of $T_{\nu,\alpha,\beta}(s)$. □

Now we will consider the monotonicity of the function $L^{**}_{\mu,\nu,\alpha}(s)$.

Theorem 25. *The function $L^{**}_{\mu,\nu,\alpha}(s)$ decreases from $(0,\infty)$ to $(1,B_{\mu,\nu}(\alpha))$ if and only if $|\mu|\neq|\nu|$, where*

$$B_{\mu,\nu}(\alpha):=\begin{cases}\infty & \text{if } \alpha\leq\max\{|\mu|,|\nu|\},\\ \frac{\Gamma(\alpha-\nu)\Gamma(\alpha+\nu)+\Gamma(\alpha-\mu)\Gamma(\alpha+\mu)}{2\Gamma\left(\alpha-\frac{\mu+\nu}{2}\right)\Gamma\left(\alpha+\frac{\mu+\nu}{2}\right)} & \text{if } \alpha>\max\{|\mu|,|\nu|\}.\end{cases}$$

Proof.

(1) Assume that $\mu\nu\geq 0$. It is sufficient to prove that the function $L^{**}_{\mu,\nu,\alpha}(s)$ is decreasing for all $\mu,\nu\geq 0$.

Based on formula (16), we obtain

$$\frac{T_{\mu,\alpha,1}(s)+T_{\nu,\alpha,1}(s)}{T_{(\mu+\nu)/2,\alpha,1}(s)}=\frac{\int_{2s}^{\infty}\int_0^{\infty}e^{-t\cosh(y)}\left(\cosh(2\mu y)+\cosh(2\nu y)\right)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}{\int_{2s}^{\infty}\int_0^{\infty}e^{-t\cosh(y)}\cosh\left((\mu+\nu)y\right)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}.$$

It is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{(\cosh(2\mu y)+\cosh(2\nu y))}{\cosh((\mu+\nu)y)}\right)=(\mu-\nu)e^{-(\mu+\nu)y}\left(e^{2\mu y}-e^{2\nu y}\right)\geq 0. \tag{31}$$

Combining (28) and (31), we complete the proof based on Corollary 9.

(2) Assume that $\mu\nu<0$. It is sufficient to prove that the function $\frac{T_{\mu,\alpha,1}(s)+T_{\nu,\alpha,1}(s)}{T_{(\mu-\nu)/2,\alpha,1}(s)}$ is decreasing for all $\mu,\nu>0$.

From

$$\frac{T_{\mu,\alpha,1}(s)+T_{\nu,\alpha,1}(s)}{T_{(\mu-\nu)/2,\alpha,1}(s)}=\frac{\int_{2s}^{\infty}\int_0^{\infty}e^{-t\cosh(y)}\left(\cosh(2\mu y)+\cosh(2\nu y)\right)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}{\int_{2s}^{\infty}\int_0^{\infty}e^{-t\cosh(y)}\cosh\left((\mu-\nu)y\right)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t},$$

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{(\cosh(2\mu y)+\cosh(2\nu y))}{\cosh((\mu-\nu)y)}\right)=(\mu+\nu)e^{-(\mu+\nu)y}\left(e^{2(\mu+\nu)y}-1\right)\geq 0,$$

and (28), we complete the proof.

Moreover, we have

$$\lim_{s\to 0}\frac{T_{\mu,\alpha,1}(s)+T_{\nu,\alpha,1}(s)}{T_{(\mu+\nu)/2,\alpha,1}(s)}=\begin{cases}\infty & \text{if } \alpha\leq\max\{|\mu|,|\nu|\},\\ \frac{\Gamma(\alpha-\nu)\Gamma(\alpha+\nu)+\Gamma(\alpha-\mu)\Gamma(\alpha+\mu)}{\Gamma\left(\alpha-\frac{\mu+\nu}{2}\right)\Gamma\left(\alpha+\frac{\mu+\nu}{2}\right)} & \text{if } \alpha>\max\{|\mu|,|\nu|\},\end{cases}$$

and

$$\lim_{s\to\infty}\frac{T_{\mu,\alpha,1}(s)+T_{\nu,\alpha,1}(s)}{T_{(\mu+\nu)/2,\alpha,1}(s)}=\lim_{s\to\infty}\frac{-\frac{2}{s}+O\left(\frac{1}{s}\right)^2}{-\frac{1}{s}+O\left(\frac{1}{s}\right)^2}=2.$$

Then the proof of Theorem 25 is completed. □

Finally, we will consider the monotonicity of the function $W_{\nu,n_1,n_2,\alpha}(s)$, where $n_1,n_2\in\mathbb{N}$.

Theorem 26. *The function $W_{\nu,n_1,n_2,\alpha}(s)$ decreases if and only if $n_1>n_2$, and increases if and only if $n_1<n_2$.*

Proof. Without loss of generality, we prove that $W_{\nu,n_1,n_2,\alpha}(s)$ decreases if n_1 is even and n_2 is odd with $n_1>n_2$.

We have

$$\frac{\frac{\partial^{n_1}T_{\nu,\alpha,1}(s)}{\partial\nu^{n_1}}}{\frac{\partial^{n_2}T_{\nu,\alpha,1}(s)}{\partial\nu^{n_2}}}=\frac{\int_{2s}^{\infty}\int_0^{\infty}y^{n_1}e^{-t\cosh(y)}\cosh(2\nu y)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}{\int_{2s}^{\infty}\int_0^{\infty}y^{n_2}e^{-t\cosh(y)}\sinh(2\nu y)\frac{1-\cosh\left(2\alpha\operatorname{arccosh}\left(\frac{t}{2s}\right)\right)}{\sqrt{t^2-4s^2}}\mathrm{d}y\mathrm{d}t}.$$

Noting that $((n_1-n_2)(e^{8\nu y}-1)-8\nu ye^{4\nu y})|_{y=0}=0$ and

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}y}\left((n_1-n_2)\left(e^{8\nu y}-1\right)-8\nu ye^{4\nu y}\right)\\&=8\nu e^{4\nu y}\left((n_1-n_2)e^{4\nu y}-4\nu y-1\right)\\&\geq 8\nu e^{4\nu y}\left(e^{4\nu y}-4\nu y-1\right)\geq 0,\end{aligned}$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\frac{y^{n_1}\cosh(2\nu y)}{y^{n_2}\sinh(2\nu y)}\right)=\frac{y^{n_1-n_2}\left((n_1-n_2)\left(e^{8\nu y}-1\right)-8\nu ye^{4\nu y}\right)}{\left(e^{4\nu y}-1\right)^2}\geq 0.$$

The proof is completed by using Corollary 9. □

Taking $n_1=n$ and $n_2=0$ in Theorem 26, we have the following corollary.

Corollary 27. *If $n \in \mathbb{N}$ and $n \geq 1$, then the function*

$$s \mapsto \frac{\frac{\partial^n T_{\nu,\alpha,1}(s)}{\partial \nu^n}}{T_{\nu,\alpha,1}(s)}$$

is decreasing on $(0,\infty)$.

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