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Comptes Rendus
Mathématique

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Volume 358, issue 2 (2020), p. 227-235

Published online: 15 June 2020

https://doi.org/10.5802/crmath.40

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

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2020, 358, nº 2, p. 227-235
https://doi.org/10.5802/crmath.40

Probability Theory / *Probabilités*

On the uniqueness of solutions to quadratic BSDEs with non-convex generators and unbounded terminal conditions

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Abstract. We prove a uniqueness result of the unbounded solution for a quadratic backward stochastic differential equation whose terminal condition is unbounded and whose generator g may be non-Lipschitz continuous in the state variable y and non-convex (non-concave) in the state variable z, and instead satisfies a strictly quadratic condition and an additional assumption. The key observation is that if the generator is strictly quadratic, then the quadratic variation of the first component of the solution admits an exponential moment. Typically, a Lipschitz perturbation of some convex (concave) function satisfies the additional assumption mentioned above. This generalizes some results obtained in [1] and [2].

2020 Mathematics Subject Classification. 60H10.

Funding. Shengjun Fan is supported by the State Scholarship Fund from the China Scholarship Council (No. 201806425013). Ying Hu is partially supported by Lebesgue center of mathematics "Investissements d'avenir" program-ANR-11-LABX-0020-01, by CAESARS-ANR-15-CE05-0024 and by MFG-ANR-16-CE40-0015-01. Shanjian Tang is partially supported by National Science Foundation of China (No. 11631004) and Science and Technology Commission of Shanghai Municipality (No. 14XD1400400).

Manuscript received 28th May 2019, revised and accepted 17th March 2020.

1. Introduction

Since the seminal paper [7] on nonlinear backward stochastic differential equations (BSDEs in short), a lot of efforts have been made to study the well posedness, and many applications have been found in various fields such as mathematical finance, stochastic control and PDEs. In particular, quadratic BSDEs were first investigated in [6] for bounded terminal conditions, which have attracted much attention and are the subject of this article.

We consider the following quadratic BSDE:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)\mathrm{d}s - \int_t^T Z_s \cdot \mathrm{d}B_s, \quad t \in [0, T], \tag{1}$$

where the terminal value ξ is an unbounded random variable, and the generator g has a quadratic growth in the variable z. In [1], the authors obtained the first existence result for this kind of BSDEs, when the terminal value has a certain exponential moment. The uniqueness results were established in [2], [4] and [5] when the generator g is Lipschitz continuous in y, and either convex or concave in z. The case of a non-convex generator g was tackled in [8] and [3], but more assumptions are required on the terminal value ξ than the exponential integrability. In this paper, we prove a uniqueness result for the unbounded solution of quadratic BSDEs, where the generator g may be non-Lipschitz and has a general growth in y, and non-convex (non-concave) in z. Rather than imposing any additional assumption on the terminal value, we suppose that the generator g satisfies an additional assumption which holds typically for a (locally) Lipschitz perturbation of some convex (concave) function (see (H4) and Proposition 3 together with (H4′) and Remark 7 for details), and is strictly quadratic, i.e., either

$$g(\omega, t, y, z) \geq \frac{\overline{\gamma}}{2}|z|^2 - \beta|y| - \alpha_t(\omega),$$

or

$$g(\omega, t, y, z) \leq -\frac{\overline{\gamma}}{2}|z|^2 + \beta|y| + \alpha_t(\omega)$$

holds for two constants $\overline{\gamma} > 0, \beta \geq 0$ and a nonnegative process $\alpha_\cdot$. Under this condition, we can prove that if $(Y_\cdot, Z_\cdot)$ is a solution satisfying $\mathbb{E}[\sup_{t\in[0,T]} \exp(p|Y_t| + p\int_0^t \alpha_s \mathrm{d}s)] < +\infty$ for some $p > 0$, then there exists a constant $\varepsilon > 0$ such that $\mathbb{E}[\exp(\varepsilon \int_0^T |Z_t|^2 \mathrm{d}t)] < +\infty$. See Proposition 2 for details.

Let us close this introduction by introducing some notations that will be used later. Fix the terminal time $T > 0$ and a positive integer d, and let $x \cdot y$ denote the Euclidean inner product for $x, y \in \mathbb{R}^d$. Suppose that $(B_t)_{t\in[0,T]}$ is a d-dimensional standard Brownian motion defined on some complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Let $(\mathscr{F}_t)_{t\in[0,T]}$ be the natural filtration generated by $B_\cdot$ and augmented by all $\mathbb{P}$-null sets of $\mathscr{F}$. All the processes are assumed to be $(\mathscr{F}_t)$-adapted.

Denote by $\mathbb{1}_A(\cdot)$ the indicator of set A, and $\mathrm{sgn}(x) := \mathbb{1}_{x>0} - \mathbb{1}_{x\leq 0}$. Let $a \wedge b$ be the minimum of a and b, $a^- := -(a \wedge 0)$ and $a^+ := (-a)^-$. For any real $p \geq 1$, let $\mathscr{S}^p$ be the set of all progressively measurable and continuous real-valued processes $(Y_t)_{t\in[0,T]}$ such that

$$\|Y\|_{\mathscr{S}^p} := \left(\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^p\right]\right)^{1/p} < +\infty,$$

and $\mathscr{M}^p$ the set of all progressively measurable $\mathbb{R}^d$-valued processes $(Z_t)_{t\in[0,T]}$ such that

$$\|Z\|_{\mathscr{M}^p} := \left\{\mathbb{E}\left[\left(\int_0^T |Z_t|^2 \mathrm{d}t\right)^{p/2}\right]\right\}^{1/p} < +\infty.$$

As mentioned before, we will study BSDEs of type (1). The terminal condition ξ is real-valued and $\mathscr{F}_T$-measurable, and the process $g(\cdot,\cdot, y, z) : \Omega \times [0, T] \to \mathbb{R}$ is progressively measurable for each pair (y, z) and continuous in (y, z). By a solution to (1), we mean a pair of progressively measurable processes $(Y_t, Z_t)_{t\in[0,T]}$, taking values in $\mathbb{R} \times \mathbb{R}^d$ such that $\mathbb{P} - a.s.$, the function $t \mapsto Y_t$ is continuous, $t \mapsto Z_t$ is square-integrable, $t \mapsto g(t, Y_t, Z_t)$ is integrable, and verifies (1). And, we usually denote by BSDE (ξ, g) the BSDE whose terminal condition is ξ and whose generator is g.

Finally, we recall that a process $(X_t)_{t\in[0,T]}$ belongs to class (D) if the family of random variables $\{X_\tau : \tau$ is any stopping time taking values in $[0, T]\}$ is uniformly integrable.

2. Main result

We define for any non-negative integrable function $f(\cdot):[0,T]\to[0,+\infty)$ and any constants $\kappa\ge 0$ and $\lambda>0$ the following function:

$$\psi(s,x;f,\kappa,\lambda)=\exp\left(\lambda e^{\kappa s}x+\lambda\int_0^s f(r)e^{\kappa r}\,\mathrm{d}r\right),\ \ (s,x)\in[0,T]\times[0,+\infty). \tag{2}$$

It is easy to verify that for each $x\in[0,+\infty)$, $\mathrm{d}s-a.e.$ on $[0,T]$, it holds that

$$-\psi_x(s,x;f,\kappa,\lambda)(f(s)+\kappa x)+\psi_s(s,x;f,\kappa,\lambda)=0 \tag{3}$$

and

$$-\lambda\psi_x(s,x;f,\kappa,\lambda)+\psi_{xx}(s,x;f,\kappa,\lambda)\ge 0, \tag{4}$$

where and hereafter, $\psi_s(\cdot,\cdot;f,\kappa,\lambda)$ denotes the first-order partial derivative with respect to time, and $\psi_x(\cdot,\cdot;f,\kappa,\lambda)$ and $\psi_{xx}(\cdot,\cdot;f,\kappa,\lambda)$ are the first-order and second-order partial derivatives with respect to space of the time-space function $\psi(\cdot,\cdot;f,\kappa,\lambda)$.

In the whole paper, we always fix a progressively measurable non-negative process $(\alpha_t)_{t\in[0,T]}$ and several real constants $\beta\ge 0$, $0<\overline{\gamma}\le\gamma$, $k\ge 0$, $\overline{k}\ge 0$ and $\delta\in[0,1)$. Let us first introduce the following two assumptions on the generator g.

(H1) $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, it holds that

$$\operatorname{sgn}(y)g(\omega,t,y,z)\le\alpha_t(\omega)+\beta|y|+\frac{\gamma}{2}|z|^2;$$

(H2) There exists a deterministic nondecreasing continuous function $\phi(\cdot):[0,+\infty)\to[0,+\infty)$ with $\phi(0)=0$ such that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$,

$$|g(\omega,t,y,z)|\le\alpha_t(\omega)+\phi(|y|)+\frac{\gamma}{2}|z|^2.$$

The following proposition gives a slight generalization of the existence result of [2] for quadratic BSDEs with unbounded terminal conditions.

Proposition 1. *Suppose that the function ψ is defined in* (2) *and that ξ is a terminal condition and g is a generator which is continuous in (y,z) and satisfies assumptions* (H1) *and* (H2).

(i) *Let $(Y_\cdot,Z_\cdot)$ be a solution to BSDE (ξ,g) such that $(\psi^p(t,|Y_t|;\alpha_\cdot,\beta,\gamma))_{t\in[0,T]}$ belongs to class (D) for some real $p\ge 1$. Then, $\mathbb{P}-a.s.$, for each $t\in[0,T]$,*

$$p\gamma|Y_t|\le\psi^p(t,|Y_t|;\alpha_\cdot,\beta,\gamma)+\frac{1}{2}p(p-1)\gamma^2\mathbb{E}\left[\int_t^T|Z_s|^2\mathrm{d}s\middle|\mathscr{F}_t\right]\le\mathbb{E}[\psi^p(T,|\xi|;\alpha_\cdot,\beta,\gamma)|\mathscr{F}_t]. \tag{5}$$

(ii) *If $\mathbb{E}[\psi^p(T,|\xi|;\alpha_\cdot,\beta,\gamma)]<+\infty$ for some real $p\ge 1$, then BSDE (ξ,g) admits a solution $(Y_\cdot,Z_\cdot)$ such that $(\psi^p(t,|Y_t|;\alpha_\cdot,\beta,\gamma))_{t\in[0,T]}$ belongs to class (D). Moreover, if $p>1$, then there exists a constant $C>0$ depending only on p such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\psi^p\left(t,|Y_t|;\alpha_\cdot,\beta,\gamma\right)\right]\le C\mathbb{E}[\psi^p(T,|\xi|;\alpha_\cdot,\beta,\gamma)] \tag{6}$$

and $Z_\cdot\in\mathscr{M}^2$. And, if $p>2$, then $Z_\cdot\in\mathscr{M}^p$.

Proof.

(i). Let $L_\cdot$ denote the local time of $Y_\cdot$ at 0. Itô-Tanaka's formula applied to $\psi(s,|Y_s|;\alpha_\cdot,\beta,p\gamma) = \psi(s,|Y_s|;0,\beta,p\gamma)\psi(s,0;\alpha_\cdot,\beta,p\gamma)$ gives, in view of assumption (H1),

$$\begin{aligned}
&\mathrm{d}\psi(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\\
&\quad= -\psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\,\mathrm{sgn}(Y_s)g(s,Y_s,Z_s)\mathrm{d}s + \psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\,\mathrm{sgn}(Y_s)Z_s\cdot\mathrm{d}B_s\\
&\qquad + \psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\mathrm{d}L_s + \frac{1}{2}\psi_{xx}(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)|Z_s|^2\mathrm{d}s + \psi_s(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\mathrm{d}s\\
&\quad\geq \left[-\psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)(\alpha_s+\beta|Y_s|) + \psi_s(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\right]\mathrm{d}s\\
&\qquad + \frac{1}{2}\left[-\gamma\psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)|Z_s|^2 + \psi_{xx}(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)|Z_s|^2\right]\mathrm{d}s\\
&\qquad + \psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\,\mathrm{sgn}(Y_s)Z_s\cdot\mathrm{d}B_s, \qquad \mathrm{d}\mathbb{P}\times\mathrm{d}s - a.e.
\end{aligned}$$

Then, by virtue of (3) and (4) together with the fact that $\psi_x(\cdot,\cdot;f_\cdot,\kappa,\lambda)\geq\lambda$, we have

$$\mathrm{d}\psi(s,|Y_s|;\alpha_\cdot,\beta,p\gamma) \geq \frac{1}{2}p(p-1)\gamma^2|Z_s|^2\mathrm{d}s + \psi_x(s,|Y_s|;\alpha_\cdot,\beta,p\gamma)\,\mathrm{sgn}(Y_s)Z_s\cdot\mathrm{d}B_s,\ \ \mathrm{d}\mathbb{P}\times\mathrm{d}s - a.e. \quad (7)$$

Let us denote, for each $t\in[0,T]$ and each integer $m\geq 1$, the following stopping time

$$\tau_m^t := \inf\left\{s\in[t,T] : \int_t^s \left(\psi_x(r,|Y_r|;\alpha_\cdot,\beta,p\gamma)\right)^2|Z_r|^2\mathrm{d}r \geq m\right\}\wedge T$$

with the convention $\inf\varnothing = +\infty$. It follows from (7) and the definition of τ_m^t that for each $m\geq 1$,

$$\psi(t,|Y_t|;\alpha_\cdot,\beta,p\gamma) + \frac{1}{2}p(p-1)\gamma^2\mathbb{E}\left[\int_t^{\tau_m^t}|Z_s|^2\mathrm{d}s\,\middle|\,\mathscr{F}_t\right] \leq \mathbb{E}\left[\psi(\tau_m^t,|Y_{\tau_m^t}|;\alpha_\cdot,\beta,p\gamma)\,\middle|\,\mathscr{F}_t\right],\quad t\in[0,T].$$

Thus, since $(\psi(s,|Y_s|;\alpha_\cdot,\beta,p\gamma))_{s\in[0,T]}$ belongs to class (D), the desired inequality (5) follows by letting $m\to\infty$ and using Fatou's lemma in the last inequality.

(ii). Thanks to (i), proceeding as in the proof of Proposition 3 in [2] with a localization argument, we conclude that if $\mathbb{E}[\psi^p(T,|\xi|;\alpha_\cdot,\beta,\gamma)] < +\infty$ for some real $p\geq 1$, then BSDE (ξ,g) has a solution $(Y_\cdot,Z_\cdot)$ such that $(\psi^p(t,|Y_t|;\alpha_\cdot,\beta,\gamma))_{t\in[0,T]}$ belongs to class (D) and (5) holds. Moreover, for $p>1$, it is clear from (5) that $Z_\cdot\in\mathscr{M}^2$. Since (5) holds for $p=1$, we apply Doob's maximal inequality to get (6). Finally, the conclusion that $Z_\cdot\in\mathscr{M}^p$ for $p>2$ has been given in Corollary 4 of [2]. □

To obtain a stronger integrability with respect to the process $Z_\cdot$, we need the following assumption, called hereafter the strictly (positive) quadratic condition:

(H3) $\mathrm{d}\mathbb{P}\times\mathrm{d}t - a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, it holds that

$$g(\omega,t,y,z) \geq \frac{\overline{\gamma}}{2}|z|^2 - \beta|y| - \alpha_t(\omega). \quad (8)$$

Proposition 2. *Let ψ be defined in* (2)*, ξ be a terminal condition, g be a generator satisfing* (H3)*, and $(Y_\cdot,Z_\cdot)$ be a solution to BSDE (ξ,g). If $\mathbb{E}[\sup_{t\in[0,T]}\psi(t,|Y_t|;\alpha_\cdot,0,p_0)] < +\infty$ for some real $p_0>0$, then for each real $\varepsilon\in(0,\varepsilon_0]$ with $\varepsilon_0 := \frac{\overline{\gamma}^2}{18}\wedge\frac{p_0\overline{\gamma}}{12+6\beta T}$, we have*

$$\mathbb{E}\left[\exp\left(\varepsilon\int_0^T|Z_s|^2\mathrm{d}s\right)\right] < +\infty. \quad (9)$$

In particular, for each $p>0$ and $\delta\in[0,1)$, $\mathbb{E}[\exp(p\int_0^T|Z_s|^{1+\delta}\mathrm{d}s)] < +\infty$.

Proof. Since $(Y_\cdot,Z_\cdot)$ is a solution to BSDE (ξ,g) and (8) holds, we have for each $n\geq 1$,

$$\frac{\overline{\gamma}}{2}\int_0^{\sigma_n}|Z_s|^2\mathrm{d}s \leq Y_0 - Y_{\sigma_n} + \int_0^{\sigma_n}(\alpha_s+\beta|Y_s|)\mathrm{d}s + \int_0^{\sigma_n}Z_s\cdot\mathrm{d}B_s \leq X + \int_0^{\sigma_n}Z_s\cdot\mathrm{d}B_s,$$

where $X := (2+\beta T)\sup_{t\in[0,T]}|Y_t| + \int_0^T \alpha_s \mathrm{d}s$, and

$$\sigma_n := \inf\left\{s \in [0,T] : \int_0^s |Z_r|^2 \mathrm{d}r \geq n\right\} \wedge T.$$

Then, for each $\varepsilon > 0$ such that $3\varepsilon(2+\beta T) \leq p_0$, we have

$$\exp\left(\frac{\overline{\gamma}}{2}\varepsilon \int_0^{\sigma_n} |Z_s|^2 \mathrm{d}s\right) \leq \exp(\varepsilon X)\exp\left(\varepsilon \int_0^{\sigma_n} Z_s \cdot \mathrm{d}B_s - \frac{3}{2}\varepsilon^2 \int_0^{\sigma_n} |Z_s|^2 \mathrm{d}s\right)\exp\left(\frac{3}{2}\varepsilon^2 \int_0^{\sigma_n} |Z_s|^2 \mathrm{d}s\right).$$

Observe that the process

$$H(t) := \exp\left(3\varepsilon \int_0^{t\wedge\sigma_n} Z_s \cdot \mathrm{d}B_s - \frac{9}{2}\varepsilon^2 \int_0^{t\wedge\sigma_n} |Z_s|^2 \mathrm{d}s\right)$$

is a positive martingale with $H(0) = 1$. Taking expectation in both sides of the last inequality and applying Hölder's inequality, we obtain

$$\mathbb{E}\left[\exp\left(\frac{\overline{\gamma}}{2}\varepsilon \int_0^{\sigma_n} |Z_s|^2 \mathrm{d}s\right)\right] \leq \left(\mathbb{E}\left[\exp(3\varepsilon X)\right]\right)^{\frac{1}{3}} \left(\mathbb{E}\left[\exp\left(\frac{9}{2}\varepsilon^2 \int_0^{\sigma_n} |Z_s|^2 \mathrm{d}s\right)\right]\right)^{\frac{1}{3}}.$$

Consequently, for $\varepsilon \leq \overline{\gamma}/9$, we have

$$\left(\mathbb{E}\left[\exp\left(\frac{\overline{\gamma}}{2}\varepsilon \int_0^{\sigma_n} |Z_s|^2 \mathrm{d}s\right)\right]\right)^{\frac{2}{3}} \leq \left(\mathbb{E}\left[\exp(3\varepsilon X)\right]\right)^{\frac{1}{3}} < +\infty,$$

which yields the inequality (9) immediately from Fatou's lemma. Finally, for each $p > 0$, $\delta \in [0,1)$, $x \geq 0$ and $\varepsilon \in (0,\varepsilon_0]$, it follows from Young's inequality that

$$px^{1+\delta} = \left(\frac{2}{1+\delta}\varepsilon x^2\right)^{\frac{1+\delta}{2}} \left(p^{\frac{2}{1-\delta}}\left(\frac{2}{1+\delta}\varepsilon\right)^{-\frac{1+\delta}{1-\delta}}\right)^{\frac{1-\delta}{2}} \leq \varepsilon x^2 + \frac{1-\delta}{2} p^{\frac{2}{1-\delta}} \left(\frac{1+\delta}{2\varepsilon}\right)^{\frac{1+\delta}{1-\delta}}.$$

Thus, the last desired assertion follows from (9). The proof is complete. □

In what follows, the following assumption on the generator g will be used.

(H4) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, for each $(y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d$, $i = 1,2$ and each $\theta \in (0,1)$, it holds that

$$\mathbb{1}_{\{y_1-\theta y_2>0\}}\left(g(\omega,t,y_1,z_1) - \theta g(\omega,t,y_2,z_2)\right) \leq (1-\theta)\left(\beta\left|\delta_\theta y\right| + \gamma|\delta_\theta z|^2 + h(\omega,t,y_1,y_2,z_1,z_2,\delta)\right), \quad (10)$$

where

$$\delta_\theta y := \frac{y_1-\theta y_2}{1-\theta}, \quad \delta_\theta z := \frac{z_1-\theta z_2}{1-\theta},$$

and

$$h(\omega,t,y_1,y_2,z_1,z_2,\delta) := \alpha_t(\omega) + k(|y_1|+|y_2|) + \overline{k}\left(|z_1|^{1+\delta} + |z_2|^{1+\delta}\right).$$

One typical example of (H4) is

$$g(\omega,t,y,z) := g_1(z) + g_2(z),$$

where $g_1 : \mathbb{R}^d \to \mathbb{R}$ is convex with quadratic growth, and $g_2 : \mathbb{R}^d \to \mathbb{R}$ is Lipschitz continuous, i.e., g is a Lipschitz perturbation of some convex function. More generally, we have

Proposition 3. *Assumption* (H4) *holds for the generator* g *as soon as it is continuous in* (y,z) *and satisfies* (H1) *together with anyone of the following conditions:*

(i) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, $g(\omega,t,\cdot,\cdot)$ *is convex;*

(ii) g *is Lipschitz in the variable* y *and* δ*-locally Lipschitz in the variable* z*, i.e.,* $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, *for each* $(y_i, z_i) \in \mathbb{R} \times \mathbb{R}^d$, $i = 1,2$, *we have*

$$|g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)| \leq \beta|y_1-y_2| + \gamma(1+|z_1|^\delta + |z_2|^\delta)|z_1-z_2|; \quad (11)$$

(iii) $\mathrm{d}\mathbb{P} \times \mathrm{d}t - a.e.$, *for each* $(y,z) \in \mathbb{R} \times \mathbb{R}^d$, $g(\omega,t,\cdot,z)$ *is Lipschitz, and* $g(\omega,t,y,\cdot)$ *is convex;*

(iv) $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, *for each* $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $g(\omega,t,\cdot,z)$ *is convex, and* $g(\omega,t,y,\cdot)$ *is* δ*-locally Lipschitz, i.e.,* (11) *holds with* $y_1=y_2=y$.

Proof. Given $(y_i,z_i)\in\mathbb{R}\times\mathbb{R}^d$, $i=1,2$ and $\theta\in(0,1)$.

(i)**.** Assume that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, $g(\omega,t,\cdot,\cdot)$ is convex. In view of (H1), if $\delta_\theta y>0$, then

$$\begin{aligned}g(\omega,t,y_1,z_1)&=g\left(\omega,t,\theta y_2+(1-\theta)\delta_\theta y,\theta z_2+(1-\theta)\delta_\theta z\right)\\&\le\theta g(\omega,t,y_2,z_2)+(1-\theta)g\left(\omega,t,\delta_\theta y,\delta_\theta z\right)\\&\le\theta g(\omega,t,y_2,z_2)+(1-\theta)\left(\alpha_t(\omega)+\beta|\delta_\theta y|+\frac{\gamma}{2}|\delta_\theta z|^2\right).\end{aligned}$$

Thus, the inequality (10) holds with $(\gamma,k,\overline{k})$ being replaced with $(\gamma/2,0,0)$.

(ii)**.** Let the inequality (11) holds. Note by (H1) that $|g(\omega,t,0,0)|\le\alpha_t(\omega)$. Then, in view of the fact that $2\delta<1+\delta$, using Young's inequality, we deduce that for each $\varepsilon>0$,

$$\begin{aligned}&g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2)\\&\quad\le|g(\omega,t,y_1,z_1)-g(\omega,t,y_2,z_2)|+(1-\theta)|g(\omega,t,y_2,z_2)|\\&\quad\le\beta|y_1-y_2|+\gamma\left(1+|z_1|^\delta+|z_2|^\delta\right)|z_1-z_2|+(1-\theta)\left(|g(\omega,t,0,0)|+\beta|y_2|+\gamma\left(1+|z_2|^\delta\right)|z_2|\right)\\&\quad\le\beta(|y_1-\theta y_2|+(1-\theta)|y_2|)+\gamma\left(1+|z_1|^\delta+|z_2|^\delta\right)(|z_1-\theta z_2|+(1-\theta)|z_2|)\\&\qquad\quad+(1-\theta)\left(\alpha_t(\omega)+\beta|y_2|+\gamma\left(1+|z_2|^\delta\right)|z_2|\right)\\&\quad\le(1-\theta)\left[\beta|\delta_\theta y|+2\beta|y_2|+\alpha_t(\omega)+\varepsilon|\delta_\theta z|^2+c\left(1+|z_1|^{1+\delta}+|z_2|^{1+\delta}\right)\right],\end{aligned}$$

where c is a constant depending only on $(\gamma,\delta,\varepsilon)$. Thus, the inequality (10) holds with $(\gamma,\alpha_\cdot,k,\overline{k})$ being replaced with $(\varepsilon,\alpha_\cdot+c,2\beta,c)$.

(iii)**.** Assume that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $g(\omega,t,\cdot,z)$ is Lipschitz, and $g(\omega,t,y,\cdot)$ is convex. Then, noticing by (H1) that $|g(\omega,t,0,z)|\le\alpha_t+\gamma|z|^2/2$, we have

$$\begin{aligned}&g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2)\\&\quad\le|g(\omega,t,y_1,z_1)-g(\omega,t,y_2,z_1)|+g(\omega,t,y_2,z_1)-\theta g(\omega,t,y_2,z_2)\\&\quad\le\beta|y_1-y_2|+g(\omega,t,y_2,\theta z_2+(1-\theta)\delta_\theta z)-\theta g(\omega,t,y_2,z_2)\\&\quad\le\beta(|y_1-\theta y_2|+(1-\theta)|y_2|)+(1-\theta)\left(|g(\omega,t,y_2,\delta_\theta z)-g(\omega,t,0,\delta_\theta z)|+|g(\omega,t,0,\delta_\theta z)|\right)\\&\quad\le(1-\theta)\left(\beta|\delta_\theta y|+2\beta|y_2|+\alpha_t(\omega)+\frac{\gamma}{2}|\delta_\theta z|^2\right),\end{aligned}$$

Thus, (10) holds with $(\gamma,k,\overline{k})$ being $(\gamma/2,2\beta,0)$.

(iv)**.** Assume that $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, $g(\omega,t,\cdot,z)$ is convex, and $g(\omega,t,y,\cdot)$ is δ-locally Lipschitz. In view of (H1) and the fact that $2\delta<1+\delta$, we can apply Young's inequality to get that if $\delta_\theta y>0$, then for each $\varepsilon>0$,

$$\begin{aligned}&g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_2)\\&\qquad\le g(\omega,t,y_1,z_1)-\theta g(\omega,t,y_2,z_1)+\theta|g(\omega,t,y_2,z_1)-g(\omega,t,y_2,z_2)|\\&\qquad\le g(\omega,t,\theta y_2+(1-\theta)\delta_\theta y,z_1)-\theta g(\omega,t,y_2,z_1)+\theta\gamma\left(1+|z_1|^\delta+|z_2|^\delta\right)|z_1-z_2|\\&\qquad\le(1-\theta)\left(|g(\omega,t,\delta_\theta y,z_1)-g(\omega,t,\delta_\theta y,0)|+g(\omega,t,\delta_\theta y,0)\right)\\&\qquad\qquad+\gamma\left(1+|z_1|^\delta+|z_2|^\delta\right)(|z_1-\theta z_2|+(1-\theta)|z_2|)\\&\qquad\le(1-\theta)\left[\alpha_t(\omega)+\beta|\delta_\theta y|+\varepsilon|\delta_\theta z|^2+c\left(1+|z_1|^{1+\delta}+|z_2|^{1+\delta}\right)\right],\end{aligned}$$

where c is a constant depending only on $(\gamma,\delta,\varepsilon)$. Thus, the inequality (10) holds with $(\gamma,\alpha_\cdot,k,\overline{k}) := (\varepsilon,\alpha_\cdot+c,\beta,c)$. The proposition is then proved. □

Remark 4. It is easy to verify that the sum of generators satisfying assumption (H4) still satisfies (H4). Hence, in view of Proposition 3, the generator g satisfying assumption (H4) is neither necessarily convex nor Lipschitz in the variables (y,z), and it may have a general growth in the variable y.

The main result of this paper is stated as follows.

Theorem 5. *Suppose that the function ψ is defined in* (2) *and that ξ is a terminal condition, g is a generator which is continuous in the state variables (y,z) and satisfies assumptions* (H1) *and* (H2), *and $\mathbb{E}[\psi^p(T,|\xi|;\alpha_\cdot,\beta,\gamma)]<+\infty$ for each real $p\ge 1$. Then, we have*

(i) *If g also satisfies assumption* (H4) *with $\overline{k}=0$, then BSDE (ξ,g) admits a unique solution $(Y_\cdot,Z_\cdot)$ such that for each $p\ge 1$, $\mathbb{E}\left[\sup_{t\in[0,T]}\psi^p\left(t,|Y_t|;\alpha_\cdot,\beta,\gamma\right)\right]<+\infty$. Moreover, $Z_\cdot\in\mathcal{M}^p$ for each $p\ge 1$.*

(ii) *If g also satisfies assumptions* (H3) *and* (H4), *then BSDE (ξ,g) admits a unique solution $(Y_\cdot,Z_\cdot)$ such that for each $p\ge 1$, $\mathbb{E}\left[\sup_{t\in[0,T]}\psi^p\left(t,|Y_t|;\alpha_\cdot,\beta,\gamma\right)\right]<+\infty$. Moreover, $\mathbb{E}[\exp(\varepsilon\int_0^T|Z_s|^2\mathrm{d}s)]<+\infty$ for some real $\varepsilon>0$.*

Proof. The existence is a direct consequence of Propositions 1 and 2. We now show the uniqueness part. Let us assume that (H4) holds.

Let both $(Y_\cdot,Z_\cdot)$ and $(Y'_\cdot,Z'_\cdot)$ be solutions to BSDE (ξ,g) such that for each $p\ge 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\psi^p\left(t,|Y_t|;\alpha_\cdot,\beta,\gamma\right)\right]<+\infty \quad\text{and}\quad \mathbb{E}\left[\sup_{t\in[0,T]}\psi^p\left(t,|Y'_t|;\alpha_\cdot,\beta,\gamma\right)\right]<+\infty. \tag{12}$$

We use the θ-difference technique developed in [2]. For each fixed $\theta\in(0,1)$, define

$$\delta_\theta U_\cdot := \frac{Y_\cdot-\theta Y'_\cdot}{1-\theta} \quad\text{and}\quad \delta_\theta V_\cdot := \frac{Z_\cdot-\theta Z'_\cdot}{1-\theta}.$$

Then, the pair $(\delta_\theta U_\cdot,\delta_\theta V_\cdot)$ solves the following BSDE:

$$\delta_\theta U_t = \xi+\int_t^T\delta_\theta g(s)\mathrm{d}s-\int_t^T\delta_\theta V_s\cdot\mathrm{d}B_s,\quad t\in[0,T], \tag{13}$$

where

$$\delta_\theta g(s) := \frac{1}{1-\theta}\left[g(s,Y_s,Z_s)-\theta g(s,Y'_s,Z'_s)\right].$$

It follows from (10) that

$$\mathbb{1}_{\{\delta_\theta U_s>0\}}\delta_\theta g(s)\le\overline{\alpha}_s+\beta|\delta_\theta U_s|+\gamma|\delta_\theta V_s|^2, \tag{14}$$

with

$$\overline{\alpha}_s := \alpha_s+k(|Y_s|+|Y'_s|)+\overline{k}\left(|Z_s|^{1+\delta}+|Z'_s|^{1+\delta}\right).$$

(i)**.** Let $\overline{k}=0$. In view of (12), using Hölder's inequality, it is not hard to verify that

$$\mathbb{E}\left[\sup_{t\in[0,T]}\psi\left(t,|\delta_\theta U_t|;\overline{\alpha}_\cdot,\beta,\gamma\right)\right]<+\infty. \tag{15}$$

Thus, in view of (13), (14) and (15), we apply Itô-Tanaka's formula to $\psi(s,\delta_\theta U_s^+;\overline{\alpha}_\cdot,\beta,\gamma)$ and argue as in the proof of Proposition 1 to deduce that for each $t\in[0,T]$,

$$\gamma\delta_\theta U_t^+\le\psi(t,\delta_\theta U_t^+;\overline{\alpha}_\cdot,\beta,\gamma)\le\mathbb{E}\left[\psi(T,\xi^+;\overline{\alpha}_\cdot,\beta,\gamma)|\mathscr{F}_t\right]\le\mathbb{E}\left[\psi(T,|\xi|;\overline{\alpha}_\cdot,\beta,\gamma)|\mathscr{F}_t\right],$$

and then

$$\gamma(Y_t-\theta Y'_t)^+\le(1-\theta)\mathbb{E}\left[\psi(T,|\xi|;\overline{\alpha}_\cdot,\beta,\gamma)|\mathscr{F}_t\right].$$

Letting $\theta \to 1$ in the last inequality yields that $\mathbb{P}-a.s.$, for each $t \in [0,T]$, $Y_t \le Y'_t$. Thus, the desired conclusion follows by interchanging the position of $Y_\cdot$ and $Y'_\cdot$.

(ii)**.** Let assumption (H3) hold. Thanks to Proposition 2, we have $\mathbb{E}[\exp(p\int_0^T |Z_s|^{1+\delta}\mathrm{d}s)] < +\infty$ and $\mathbb{E}[\exp(p\int_0^T |Z'_s|^{1+\delta}\mathrm{d}s)] < +\infty$ for each $p \ge 1$. Then, in view of (12), using Hölder's inequality, we can conclude that (15) still holds. Thus, the same computation as above yields the uniqueness result.

The proof is then complete. □

Remark 6. In view of Proposition 3 and Remark 4, Theorem 5 generalizes the uniqueness result for quadratic BSDEs with unbounded terminal conditions obtained in [2]. Indeed, for the uniqueness of solutions to quadratic BSDEs with unbounded terminal conditions, Theorem 5 covers the case of a Lipschitz perturbation of some convex function when the strictly quadratic condition holds.

The following Remark 7 illustrates that some previous results remain true when some other conditions are satisfied. For this, let us introduce the following assumptions on the generator g.

(H3′) $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y,z)\in\mathbb{R}\times\mathbb{R}^d$, it holds that
$$g(\omega,t,y,z) \le -\frac{\overline{\gamma}}{2}|z|^2 + \beta|y| + \alpha_t(\omega).$$

(H4′) $\mathrm{d}\mathbb{P}\times\mathrm{d}t-a.e.$, for each $(y_i,z_i)\in\mathbb{R}\times\mathbb{R}^d$, $i=1,2$ and each $\theta\in(0,1)$, it holds that
$$\begin{aligned}\mathbb{1}_{\{\theta y_1 - y_2 > 0\}}&\left(\theta g(\omega,t,y_1,z_1) - g(\omega,t,y_2,z_2)\right)\\ &\le (1-\theta)\left(\beta\left|\bar\delta_\theta y\right| + \gamma\left|\bar\delta_\theta z\right|^2 + h(\omega,t,y_1,y_2,z_1,z_2,\delta)\right),\end{aligned} \tag{16}$$
where
$$\bar\delta_\theta y := \frac{\theta y_1 - y_2}{1-\theta},\quad \bar\delta_\theta z := \frac{\theta z_1 - z_2}{1-\theta},$$
and
$$h(\omega,t,y_1,y_2,z_1,z_2,\delta) := \alpha_t(\omega) + k(|y_1|+|y_2|) + \overline{k}\left(|z_1|^{1+\delta} + |z_2|^{1+\delta}\right).$$

Remarks 7.

(a) It is easy to check that the conclusions in Proposition 2 still hold if assumption (H3) is replaced with assumption (H3′).

(b) It is not difficult to verify that the assertions in Proposition 3 and Remark 4 still hold if assumption (H4) is replaced with assumption (H4′) and the word "convex" is replaced with "concave". In particular, a (locally) Lipschitz perturbation of some concave function satisfies (H4′).

(c) By virtue of (a) and (b), in the same way as in Theorem 5, we can prove that all the conclusions in Theorem 5 still hold if assumptions (H3) and (H4) are replaced with (H3′) and (H4′), respectively.

Remarks 8.

(a) Letting $y_1 = y_2 = y$ and $z_1 = z_2 = z$ in (10) and (16) respectively yields that
$$\mathbb{1}_{\{y>0\}} g(\omega,t,y,z) \le \beta|y| + \gamma|z|^2 + \alpha_t(\omega) + 2k|y| + 2\overline{k}|z|^{1+\delta}$$
and
$$-\mathbb{1}_{\{y<0\}} g(\omega,t,y,z) \le \beta|y| + \gamma|z|^2 + \alpha_t(\omega) + 2k|y| + 2\overline{k}|z|^{1+\delta},$$
whose combination implies assumption (H1).

(b) Letting first $z_1 = z_2 = z$ in (10) or (16) and then letting $\theta \to 1$ yields that

$$\mathbb{1}_{\{y_1-y_2>0\}}\big(g(\omega,t,y_1,z)-g(\omega,t,y_2,z)\big) \le \beta|y_1-y_2|,$$

which means that g satisfies the monotonicity condition in the state variable y.

Example 9. For each $(\omega,t,y,z) \in \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d$, define

$$g_1(\omega,t,y,z) = |z|^2 - |z|^{\frac{3}{2}} + \sin|z| + y^2\mathbb{1}_{y\le 0} - |y| + |B_t(\omega)|$$

and

$$g_2(\omega,t,y,z) = -|z|^2 + \sin|z|^{\frac{4}{3}} + |z| - y^3\mathbb{1}_{y\ge 0} + \sin|y| + |B_t(\omega)|.$$

By virtue of Proposition 3 together with Remark 7 (b), it is not difficult to verify that g_1 (resp. g_2) is continuous in (y,z) and satisfies assumptions (H1)—(H4) (resp. (H1), (H2), (H3$'$) and (H4$'$)). However, both of them are non-convex (non-concave) in z and non-Lipschitz in y.

Acknowledgements

The authors are grateful to the anonymous referee for their helpful suggestions.

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