INSTITUT DE FRANCE
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Comptes Rendus Mathématique

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Volume 361 (2023), p. 257-264

https://doi.org/10.5802/crmath.403

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 257-264
https://doi.org/10.5802/crmath.403

Geometry and Topology / *Géométrie et Topologie*

Addendum to the paper: Compact embedded minimal surfaces in the Berger sphere

Addendum à l'article: Surfaces minimales compactes intégrées dans la sphère Berger

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Abstract. We construct a two discrete parameter family of compact minimal surfaces embedded in the Berger sphere which may be considered as the analogue of the helicoidal Karcher-Scherk surfaces.

Résumé. Nous construisons une famille à deux paramètres discrets de surfaces minimales compactes plongées dans la sphère de Berger qui peut être considérée comme l'analogue de l'hélicoïde de Karcher-Scherk.

Funding. Heayong Shin was supported by NRF 2019R1F1A1046327, Sung-Eun Koh by NRF S202003S00080, the Konkuk University's research support program for its faculty on sabbatical leave in 2019 and Seong-Deog Yang by NRF2017R1E1A1A03070929 and NRF 2020R1F1A1A01074585. The last author would like to thank the Department of Mathematics at Brown University for its hospitality during his sabatical year.

Manuscript received 9 April 2021, revised 16 February 2022, accepted 29 June 2022.

1. Introduction

Not too many examples of embedded minimal surfaces are known in the Berger sphere, which is a very interesting homogeneous three manifold with an isometry group of dimension 4. In this article, we apply again the method of constructing compact or complete embedded minimal surfaces by repeatedly reflecting the Plateau solution bounded by an appropriate geodesic polygon. The surfaces we construct are analogues of the helicoidal-Karcher-Scherk surface [1], which may

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be obtained by applying a screw motion, at least in the boundary level, to the compact embedded minimal surfaces the authors constructed previously in [4]. See Figure 1.

The construction in this article is basically the same as in [4]. A noticeable difference is that we use geodesic polygons consisting of both horizontal *and* vertical geodesics, while in [4] we used only horizontal geodesics. Note that in the standard three sphere $\mathbb{S}^3$, the vertical geodesics and the horizontal geodesics are congruent each other, while they are not in the Berger sphere.

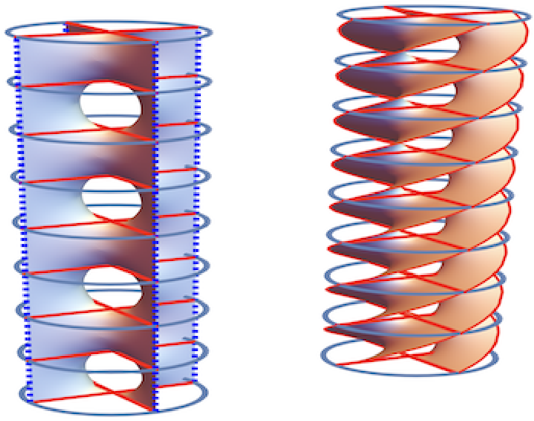

Figure 1. On the left is a half of the minimal surface we construct in this article and on the right is half of that in [3] with $k = l = 2$. Each solid cylinder represents a half of the Berger sphere.

Note that a Karcher-Scherk tower in $\mathbb{E}^3$ admits a continuous deformation, but the surface in the Berger sphere we construct in this article does not. It is an isolated one because the values of lengths and angles of the polygonal boundary can lie in a discrete set. This discrepancy is caused by the geometry of the Berger sphere. At any tangent plane of the Clifford torus which is represented as the cylinder in this article, there are only one horizontal direction and also only one vertical direction up to sign. See Figure 2. It is said in [1] "it is still an open question whether these screw motion invariant Karcher-Scherk surfaces together with the helicoid are the only screw motion invariant embedded minimal surfaces with genus 0 in the quotient." We wonder if the embedded minimal surfaces in [3] and in this article may have similar characterizations but we have no clue on this at this moment.

The Figure 1 would look different if we have used the parametrization of $\mathbb{S}^3$ used in [4]. In particular, the surfaces in [4] in the coordinates of [4] would look like the one on the left of Figure 1.

2. The Berger Sphere

We recall some facts on the Berger spheres.

2.1. $\mathbb{S}^3$ *as a special unitary group*

Let us identify the unit sphere $\mathbb{S}^3 = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}$ and the special unitary group SU(2) by the map

$$(z, w) \mapsto \begin{bmatrix} z & w \\ -\overline{w} & \overline{z} \end{bmatrix}.$$

The Lie algebra $\mathfrak{su}(2)$ is spanned by

$$X_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

which generate the left-invariant vector fields:

$$X_1(z,w) = \begin{bmatrix} -w & z \\ -\bar{z} & -\bar{w} \end{bmatrix}, \quad X_2(z,w) = \begin{bmatrix} iw & iz \\ i\bar{z} & -i\bar{w} \end{bmatrix}, \quad X_3(z,w) = \begin{bmatrix} iz & -iw \\ -i\bar{w} & -i\bar{z} \end{bmatrix}.$$

Viewed as tangent vector fields on $\mathbb{S}^3$,

$$X_1(z,w) = (-w,z), \quad X_2(z,w) = (iw,iz), \quad X_3(z,w) = (iz,-iw).$$

2.2. *A circle action*

The orbits of the right circle action on $\mathbb{S}^3$

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \quad \text{or} \quad (z,w) \mapsto (e^{i\theta}z, e^{-i\theta}w) \tag{1}$$

are the fibers of the Hopf fibration $H : \mathbb{S}^3 \to \mathbb{S}^2(1/2)$ given by

$$H(z,w) = \left(zw, \frac{1}{2}(|z|^2 - |w|^2)\right).$$

The fiber over the general point (z,w) is the circle $\{(e^{i\theta}z, e^{-i\theta}w)\}$ to which the vector field $X_3(z,w) = (iz,-iw)$ is tangent.

2.3. *The Berger metric*

Now consider the left invariant Riemannian metrics g_δ on $\mathbb{S}^3 = \mathrm{SU}(2)$ given in terms of the left invariant vector fields X_1, X_2, X_3 by

$$\begin{aligned} &X_i \cdot X_j = 0, \ i \neq j, \\ &X_1 \cdot X_1 = X_2 \cdot X_2 = \delta^2, \quad X_3 \cdot X_3 = 1. \end{aligned}$$

The Berger sphere is the Riemannian manifold $(\mathbb{S}^3, g_\delta)$. When $\delta = 1$, the Berger sphere is the standard round sphere $\mathbb{S}^3$.

2.4. *Isometries*

Since it is a left-invariant metric, one can see that the left multiplication of SU(2)

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix} \mapsto \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} \begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}, \quad a\bar{a} + b\bar{b} = 1,$$

that is,

$$(z,w) \mapsto (az - b\bar{w}, b\bar{z} + aw)$$

is an isometry. Note that the circle action (1) is also an isometry. Then the maps

$$(z,w) \mapsto (e^{i\theta}z, w), \quad (z,w) \mapsto (z, e^{i\theta}w) \tag{2}$$

are isometries as well.

2.5. *Geodesics and ruled minimal surfaces*

A geodesic in the Berger sphere is called *horizontal* if it is orthogonal to the Hopf fibers everywhere and is called *vertical* if it is tangent to the Hopf fibers everywhere. One can see that the curve

$$t \mapsto (e^{it}z, e^{-it}w)$$

is a vertical geodesic passing through the point (z,w) whose whole image is the Hopf fiber through (z,w) and that the curve

$$t \mapsto (z\cos t - e^{-i\theta} w \sin t, e^{i\theta} z \sin t + w\cos t)$$

which is the integral curve of the vector field $\cos\theta X_1 + \sin\theta X_2$ passing through the point (z,w) is a horizontal geodesic.

The following was shown in [3]:

Proposition 1. *A surface in the Berger sphere is a ruled minimal surface if and only if it is congruent to the parametric surface*

$$X(s,t) = (e^{i\alpha s}\cos t, e^{is}\sin t), \quad \alpha \in \mathbb{R}.$$

3. A parametrization

Let $P = \{(s,t,\theta) : 0 \le s \le 2\pi, 0 \le t \le \pi/2, 0 < \theta \le 2\pi\}$ and consider the map $\Phi : P \to \mathbb{S}^3$

$$\Phi(s,t,\theta) = (e^{is}\cos t, e^{i(\theta-s)}\sin t),$$

which is one-to-one on P', the interior of P, and yields coordinates on $\Phi(P')$.

In [4], we used the parametrization

$$\Psi : P \to \mathbb{S}^3, \quad \Psi(s,t,\theta) = (e^{is}\cos t, e^{i(s-\theta)}\sin t).$$

Switching the parametrization from Ψ to Φ enables us to adopt most of the formulae and figures in [4] without modification and the same construction procedure. However, we change the construction procedure following that of [5], which we think is a little simpler.

Since

$$\Phi(s,0,\theta) = (e^{is}, 0), \quad \Phi(s,\pi/2,\theta) = (0, e^{i(\theta-s)}),$$

we have the following:

Proposition 2. *For any θ, the image of the curve $s \mapsto \Phi(s,0,\theta)$ is the Hopf fiber over $(1,0)$ and the image of the curve $s \mapsto \Phi(s,\pi/2,\theta)$ is the Hopf fiber over $(0,1)$.*

Note also that, when $t = \pi/2$, since $\Phi(s,\pi/2,\theta) = \Phi(s',\pi/2,\theta')$ if and only if $s+\theta = s'+\theta' \mod 2\pi$, those points $s+\theta = s'+\theta' \mod 2\pi$ correspond to a single point on the fiber $\{(0,e^{is})\}$ over $(0,1)$.

3.1. *The Berger metric in terms of the parametrization* Φ

Now let

$$\partial_s := \frac{\partial\Phi}{\partial s}(s,t,\theta), \quad \partial_t := \frac{\partial\Phi}{\partial t}(s,t,\theta), \quad \partial_\theta := \frac{\partial\Phi}{\partial\theta}(s,t,\theta).$$

Computations give

$$\begin{aligned}\partial_s &= X_3,\\ \partial_t &= \cos(\theta-2s)X_1 + \sin(\theta-2s)X_2,\\ \tan t\,\partial_s + (1+\tan t)\partial_\theta &= -\sin(\theta-2s)X_1 + \cos(\theta-2s)X_2.\end{aligned}$$

Then we have the following:

Proposition 3. *For any s_0, θ_0,*

(i) *the t-curve $H_{s_0,\theta_0}(t) := \Phi(s_0, t, \theta_0)$ is a horizontal geodesic.*
(ii) *the s-curve $V_{\theta_0}(s) := \Phi(s, \pi/4, \theta_0)$ is a vertical geodesic.*

It was shown in [6] that the reflection across a horizontal or a vertical geodesic is an isometry of the Berger sphere. In particular, we have the following:

Proposition 4. *The following two maps*

$$\Phi(s,t,\theta) \mapsto \Phi(2s_0 - s, t, 2\theta_0 - \theta),$$
$$\Phi(s,t,\theta) \mapsto \Phi(s - \theta + \theta_0, \pi/2 - t, 2\theta_0 - \theta)$$

which represent reflections along the geodesics $H_{s_0,\theta_0}(t)$ and $V_{\theta_0}(s)$, respectively, are isometries.

One can also see that the rotation

$$\mathrm{Rot}_{\theta_0} : \Phi(s,t,\theta) \mapsto \Phi(s,t,\theta+\theta_0)$$

with respect to the Hopf fiber over the point $(1,0)$ and the translation

$$T_{s_0} : \Phi(s,t,\theta) \mapsto \Phi(s+s_0, t, \theta)$$

along the Hopf fiber are also isometries. Note that T_{s_0} is the circle action in Section 2.2.

Then Proposition 1 and (1), (2) give the following:

Proposition 5. *The following surfaces are ruled minimal surfaces.*

$$\begin{aligned}
(t,\theta) &\mapsto \Phi(s_0, t, \theta) = (e^{is_0}\cos t, e^{i(\theta - s_0)}\sin t),\\
(s,t) &\mapsto \Phi(s, t, \theta_0) = (e^{is}\cos t, e^{i(\theta_0 - s)}\sin t),\\
(s,\theta) &\mapsto \Phi(s, \pi/4, \theta) = \frac{1}{\sqrt{2}}(e^{is}, e^{i(\theta - s)}).
\end{aligned}$$

4. Construction

Note that the surface

$$\mathscr{T} := \Phi(s,\pi/4,\theta) = \frac{1}{\sqrt{2}}(e^{is}, e^{i(\theta-s)})$$

divides the Berger sphere into two regions $\{\Phi(s,t,\theta) : 0 \le t \le \pi/4)\}$ and $\{\Phi(s,t,\theta) : \pi/4 \le t \le \pi/2\}$ and that the two regions are congruent; in fact, the reflection along any vertical geodesic V_{θ_0} gives the congruence.

We first construct a minimal surface $\mathscr{D}_-$ embedded in $\{\Phi(s,t,\theta) : 0 \le t \le \pi/4\}$ whose boundary components consist of vertical geodesics, and then reflect the surface $\mathscr{D}_-$ with respect to a vertical geodesic in the boundary to get the minimal surface $\mathscr{D}_+$ embedded in $\{\Phi(s,t,\theta) : \pi/4 \le t \le \pi/2\}$. Then we show that the embedded minimal surface $\mathscr{D} := \mathscr{D}_- \cup \mathscr{D}_+$ is smooth without boundary.

Let m and l be positive integers and

$$k = 2ml.$$

As the basic block of our construction, we consider the pentahedral region

$$P_0 := \{\Phi(s,t,\theta) : 0 \le s \le \pi/k,\ 0 \le t \le \pi/4,\ 0 \le \theta \le \pi/m\}$$

bounded by five ruled minimal surfaces

$$\mathscr{H}_0 := \Phi(0,t,\theta), \quad \mathscr{H}_1 := \Phi(\pi/k, t, \theta), \quad \mathscr{V}_0 := \Phi(s,t,0), \quad \mathscr{V}_1 := \Phi(s,t,\pi/m)$$

and $\mathscr{T}$, see Figure 2.

For notational convenience, let

$$H_{p,q} := H_{\pi p/k, \pi q/m}, \quad V_q := V_{\pi q/m}$$

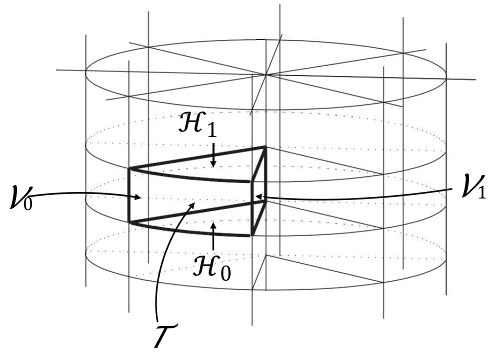

Figure 2. P_0

and let $R_{p,q}$ denote the reflection across the horizontal geodesic $H_{p,q}$ and let R^q denote the reflection across the vertical geodesic V_q.

Let $\Gamma \subset \partial P_0$ be the piecewise geodesic polygon of six segments of horizontal geodesics $H_{0,0}, H_{1,0}, H_{1,1}, H_{0,1}$ and vertical geodesics V_0, V_1, which is a subset of the 1-skeleton of P_0, see Fig.3.

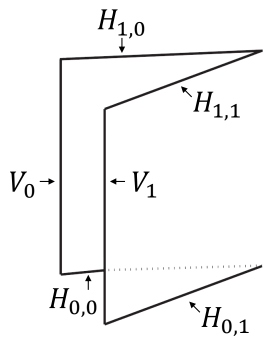

Figure 3. Γ

Then, since the region P_0 is mean-convex by [2], Γ spans an embedded minimal disk D_0 which lies inside of P_0. Now D_0 can be analytically extended across the boundary segments $H_{1,1}$ by reflection to get a minimal surface

$$D_{0,0} := D_0 \cup R_{1,1}(D_0)$$

which is embedded in the region

$$P_{0,0} := \{\Phi(s,t,\theta) : 0 \le s \le 2\pi/k,\ 0 \le t \le \pi/4,\ 0 \le \theta \le 2\pi/m\}.$$

The boundary of $D_{0,0}$ consists of the geodesic segments

$$H_{0,0}(t), H_{1,0}(t), H_{0,1}(t), H_{1,2}(t), H_{2,1}(t), H_{2,2}(t)\ ;\ 0 \le t \le \pi/4$$

and

$$V_0(s);\ 0 \le s \le \pi/k, \quad V_1(s);\ 0 \le s \le 2\pi/k, \quad V_2(s);\ \pi/k \le s \le 2\pi/k.$$

It is clear that

$$R_{1,1}(D_{0.0}) = D_{0,0}.$$

For each $p = 0, 1, \dots, k-1$ and $q = 0, 1, \dots, m-1$, we define

$$D_{p,q} := (T_{2\pi p/k} \circ \mathrm{Rot}_{2\pi q/m})(D_{0,0}).$$

The surface $D_{p,q}$ is embedded minimal surface in the region

$$P_{p,q} := \{\Phi(s,t,\theta) : 2\pi p/k \le s \le 2\pi(p+1)/k,\ 0 \le t \le \pi/4,\ 2\pi q/m \le \theta \le 2\pi(q+1)/m\}$$

with the boundary consisting of the geodesic segments

$$H_{2p,2q}(t), H_{2p+1,2q}(t), H_{2p,2q+1}(t), H_{2p+1,2q+2}(t), H_{2p+2,2q+1}(t), H_{2p+2,2q+2}(t)\ ;\ 0 \le t \le \pi/4$$

and

$$\begin{aligned} &V_{2q}(s),\ 2p\pi/k \le s \le (2p+1)\pi/k,\\ &V_{2q+1}(s),\ 2p\pi/k \le s \le (2p+2)\pi/k,\\ &V_{2q+2}(s),\ (2p+1)\pi/k \le s \le (2p+2)\pi/k. \end{aligned}$$

One can see that the surfaces $D_{p,q}$ are disjoint with each other except along the common boundary segments. More specifically, $D_{p,q}$ has common boundary $H_{2p,2q}$ with $D_{p-1,q-1}$, $H_{2p+1,2q}$ with $D_{p,q-1}$, $H_{2p,2q+1}$ with $D_{p-1,q}$, $H_{2p+2,2q+1}$ with $D_{p+1,q}$, $H_{2p+1,2q+2}$ with $D_{p,q+1}$ and $H_{2p+2,2q+2}$ with $D_{p+1,q+1}$.

Now, let us consider the surface

$$D_{p,q} \cup D_{p-1,q-1}$$

joined along $H_{2p,2q}$. Since $R_{1,1}(D_{0,0}) = D_{0,0}$ and since direct computation shows that

$$R_{p,2q} \circ T_{2\pi p/k} \circ \mathrm{Rot}_{2\pi q/m} \circ R_{1,1} = T_{2\pi(p-1)/k} \circ \mathrm{Rot}_{2\pi(q-1)/m}$$

one can see that

$$\begin{aligned} R_{2p,2q}(D_{p,q}) &= (R_{2p,2q} \circ T_{2\pi p/k} \circ \mathrm{Rot}_{2\pi q/m})(D_{0,0})\\ &= (R_{2p,2q} \circ T_{2\pi p/k} \circ \mathrm{Rot}_{2\pi q/m} \circ R_{1,1})(D_{0,0})\\ &= (T_{2\pi(p-1)/k} \circ \mathrm{Rot}_{2\pi(q-1)/m})(D_{0,0})\\ &= D_{p-1,q-1}. \end{aligned}$$

Therefore, $D_{p,q}$ and $D_{p-1,q-1}$ are joined smoothly across the common boundary $H_{2p,2q}$.

This argument shows that the surface

$$\mathscr{D}_- := \cup_{p,q} D_{p,q}$$

is a smooth embedded minimal surface in the region $\{\Phi(s,t,\theta) : 0 \le t \le \pi/4\}$. By construction, one can see that the surface $\mathscr{D}_-$ is invariant under the transformation $T_{2\pi/k}$ and $\mathrm{Rot}_{2\pi/m}$ and that $\partial\mathscr{D}_-$, the boundary of $\mathscr{D}_-$, is

$$\partial\mathscr{D}_- = \bigcup_{q=0}^{2m-1} V_q.$$

Now let

$$\mathscr{D}_+ := R^0(\mathscr{D}_-)$$

then $\mathscr{D}_+$ is a smooth minimal surface embedded in $\{\Phi(s,t,\theta) : \pi/4 \le t \le \pi/2\}$. Since $R^0(V_q) = V_{2m-q}$, the boundary of $\mathscr{D}_+$ also consists of vertical geodesics $V_0, V_1, \dots, V_{2m-1}$. Since direct computation gives that

$$R^i \circ R^0 = T_{\pi i/m} \circ \mathrm{Rot}_{2\pi i/m}$$

and since $k = 2ml$, we have

$$R^i(\mathscr{D}_+) = (R^i \circ R^0)(\mathscr{D}_-) = (T_{\pi i/m} \circ \mathrm{Rot}_{2\pi i/m})(\mathscr{D}_-) = (T_{2\pi il/k} \circ \mathrm{Rot}_{2\pi i/m})(\mathscr{D}_-) = \mathscr{D}_-.$$

This implies that the surface $\mathscr{D}_+$ is the reflection of $\mathscr{D}_-$ along each common boundary V_i.

Hence the surface

$$\mathscr{D} := \mathscr{D} + - \cup \mathscr{D}_-$$

is a smooth minimal surface without boundary embedded in the Berger sphere. This completes the construction.

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