INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

Pascal Fong and Sokratis Zikas

Connected algebraic subgroups of groups of birational transformations not contained in a maximal one

Volume 361 (2023), p. 313-322

https://doi.org/10.5802/crmath.406

CENTRE MERSENNE

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 313-322
https://doi.org/10.5802/crmath.406

Algebraic geometry / *Géométrie algébrique*

Connected algebraic subgroups of groups of birational transformations not contained in a maximal one

Pascal Fong[*, a] **and Sokratis Zikas**[a]

[a] Universität Basel, Departement Mathematik und Informatik, Spiegelgasse 1, CH–4051 Basel, Switzerland

E-mails: pascal.fong@unibas.ch, sokratis.zikas@unibas.ch

Abstract. We prove that for each $n \geq 2$, there exist a ruled variety X of dimension n and a connected algebraic subgroup of $\operatorname{Bir}(X)$ which is not contained in a maximal one.

Manuscript received 11 February 2022, revised 13 April 2022 and 11 July 2022, accepted 11 July 2022.

1. Introduction

Let $\mathbf{k}$ be an algebraically closed field. The classification of algebraic subgroups of groups of birational transformations was initiated in [8], where Enriques shows that each connected algebraic subgroup of $\operatorname{Bir}(\mathbb{P}^2)$ is conjugate to an algebraic subgroup of $\operatorname{Aut}^\circ(S)$, with S isomorphic to $\mathbb{P}^2$ or to the n-th Hirzebruch surface $\mathbb{F}_n$ for $n \neq 1$; and these are all maximal, with respect to the inclusion, among the connected algebraic subgroups of $\operatorname{Bir}(\mathbb{P}^2)$. The connected algebraic subgroups of $\operatorname{Bir}(\mathbb{P}^3)$ have been classified over $\mathbf{k} = \mathbb{C}$ by Umemura in a series of four papers [21–24] and it follows again from his classification that each connected algebraic subgroup of $\operatorname{Bir}(\mathbb{P}^3)$ is contained in a maximal one (see also [2,3] for a modern approach). However, it is an open problem whether every connected algebraic subgroup of $\operatorname{Bir}(\mathbb{P}^n)$ is contained in a maximal one when $n \geq 4$.

On the other hand, it is proven in [11, Theorem C] that there exist connected algebraic subgroups of $\operatorname{Bir}(C \times \mathbb{P}^1)$ not contained in a maximal one when C is a smooth curve of positive genus. The proof of this result is based on the existence of infinite increasing sequences of connected algebraic subgroups of $\operatorname{Bir}(C \times \mathbb{P}^1)$ (see [11, Theorem A]), and on the fact that the dimension of a maximal connected algebraic subgroup of $\operatorname{Bir}(C \times \mathbb{P}^1)$ is bounded by 4 (see [11, Theorem B] and [20, Theorem 3]). Our main result in this note is a higher dimensional analogue of [11, Theorem C]:

* Corresponding author.

Theorem A. *Let* $\mathbf{k}$ *be an algebraically closed field of characteristic* 0*. Let* $n \geq 1$ *and* C *be a smooth curve of positive genus. Then there exists a connected algebraic subgroup of* $\mathrm{Bir}(C \times \mathbb{P}^n)$ *which is not contained in a maximal one.*

The idea of the proof is to consider the connected algebraic subgroup $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$, where S is a ruled surface such that $\mathrm{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S)$, and to show that it cannot be contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$. Since $\mathrm{Aut}^\circ(S \times \mathbb{P}^n) \simeq \mathrm{Aut}^\circ(S) \times \mathrm{PGL}_{n+1}(\mathbf{k})$ by [6, Corollary 4.2.7], the existence of infinite increasing sequences of connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ is an immediate consequence of [11, Theorem A]. From this alone, it is nonetheless insufficient to deduce that one of the connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ appearing in the infinite increasing sequences is not contained in a maximal one (see Remark 8), and classifying all connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^{n+1})$ seems out of reach at the moment.

This article is organized as follows. Section 2 contains two results, namely Lemmas 6 and 7, which are important for the proof of the higher dimensional case. As a consequence of these two lemmas, we also get a new and short proof of the dimension two case (see Proposition 9), without using the classification of the maximal connected algebraic subgroups of $\mathrm{Bir}(C \times \mathbb{P}^1)$ ([11, Theorem B]). In Section 3, we prove the higher dimensional case under the extra assumption that $\mathrm{char}(\mathbf{k}) = 0$, in view of using the machinery of the MMP and the G-Sarkisov program. The latter has been developped by Floris in [9], building upon results of Hacon and McKernan in [13]. More precisely, if G is a connected algebraic group, then every G-equivariant birational map between Mori fibre spaces decomposes into G-Sarkisov links (see [9, Theorem 1.2]). We study the possible links in Lemmas 13 and 14. Combining Proposition 9 and Theorem 15, we get Theorem A.

It is very natural to also ask whether for all $n \geq 2$, there exists a variety X of dimension n such that $\mathrm{Bir}(X)$ contains algebraic subgroups which are not lying in a maximal one, without the connectedness assumption. If $n = 2$, the answer is also affirmative (see [10, Lemma 3.1, Corollary B]), and the proof is analogous to that of the connected case. Since the G-Sarkisov program is known only for connected algebraic groups, it is not clear if the proof presented in this article could be adapted for the non-connected case in higher dimension.

Acknowledgments

We are thankful to Jérémy Blanc, Enrica Floris, Ronan Terpereau, Susanna Zimmermann for interesting discussions and remarks. Special thanks to Enrica Floris for pointing out a mistake in the proof of a preliminary version. We are also grateful to the anonymous referees for their careful reading and useful comments. The first and second authors respectively acknowledge support by the Swiss National Science Foundation Grants "Geometrically ruled surfaces" 200020-192217 and "Birational transformations of threefolds" 200020-178807.

2. Some preliminaries and the case of dimension two

From now on, C will always denote a smooth curve of genus g over a field $\mathbf{k}$. In this section, $\mathbf{k}$ is an algebraically closed field of arbitrary characteristic. The following invariant was used by Maruyama in [19, 20] for his classification of ruled surfaces and their automorphisms.

Definition 1. *Let* V *be a rank-*2 *vector bundle over* C *and* $\tau \colon S = \mathbb{P}(V) \to C$ *be a ruled surface. We say that* τ *is* decomposable *if* V *is the direct sum of two line bundles over* C*. Otherwise, we say that* τ *is* indecomposable*. We define the* Segre invariant *of* S *as*

$$\mathfrak{S}(S) = \min\{\sigma^2, \sigma \text{ section of } \tau\}.$$

Remark 2. Let $\tau\colon S \to C$ be a ruled surface.

(1) Let $p \in S$ and σ be a section of τ. Recall that the blow-up of S at p followed by the contraction of the strict transform of the fibre passing through p, yields a ruled surface $\tau'\colon S' \to C$ and a birational map $\epsilon\colon S \dashrightarrow S'$ called the *elementary transformation of S centered at p* (see e.g. [14, V. Example 5.7.1]). Let σ' be the strict transform of σ by ϵ. If $p \in \sigma$, then $\sigma'^2 = \sigma^2 - 1$. Else, $\sigma'^2 = \sigma^2 + 1$.

(2) As S is obtained by finitely many elementary transformations from $C \times \mathbb{P}^1$ (see e.g. [14, V. Exercise 5.5]) and $\mathfrak{S}(C \times \mathbb{P}^1) = 0$ (see e.g. [11, Lemma 2.14]), it follows that $\mathfrak{S}(S) > -\infty$. If moreover $\mathfrak{S}(S) < 0$, then there exists a unique section with negative self-intersection number (see e.g. [10, Lemma 2.10 (1)]).

(3) The Segre invariant $\mathfrak{S}(S)$ equals $-e$, where e is the invariant defined in [14, V. Proposition 2.8]. If τ is indecomposable, then by [14, V. Theorem 2.12 (b)], we get $\mathfrak{S}(S) \geq 2 - 2g = -\deg(K_C)$. In particular, if $\mathfrak{S}(S) < -\deg(K_C)$, then τ is decomposable.

We recall the statement of Blanchard's lemma and its corollary (see [6, Proposition 4.2.1, Corollary 4.2.6]):

Proposition 3. *Let $f\colon X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$, and let G be a connected group scheme acting on X. Then there exists a unique action of G on Y such that f is G-equivariant.*

Corollary 4. *Let $f\colon X \to Y$ be a proper morphism of schemes such that $f_*(\mathcal{O}_X) = \mathcal{O}_Y$. Then f induces a homomorphism of group schemes $f_*\colon \mathrm{Aut}^\circ(X) \to \mathrm{Aut}^\circ(Y)$.*

Remark 5. Let $\tau\colon S \to C$ be a decomposable ruled surface. Assume that C has genus $g = 1$ and $\mathfrak{S}(S) \neq 0$, or that $g \geq 2$. Then by [20, Lemma 7], the morphism induced by Blanchard's lemma $\tau_*\colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial.

In the next two lemmas, we compute $\mathrm{Aut}^\circ(S)$ and its orbits for a ruled surface $\tau\colon S \to C$ with $\mathfrak{S}(S) < -(1 + \deg(K_C))$ (which is decomposable by Remark 2 3).

Lemma 6. *Let C be a curve of genus $g \geq 1$. Let $\tau\colon S = \mathbb{P}(V) \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Let σ be the minimal section of τ and $L(\sigma)$ be the line subbundle of V associated to σ. We choose trivializations of τ such that σ is the infinity section. Then the following hold:*

(1) *The group $\mathrm{Aut}^\circ(S)$ is isomorphic to $\mathbb{G}_m \ltimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$, where $\det(V)$ denotes the determinant line bundle of V. This isomorphism associates $\alpha \in \mathbb{G}_m$ and $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$, to the element $\mu_{\alpha,\gamma} \in \mathrm{Aut}^\circ(S)$ obtained by gluing the automorphisms:*

$$\begin{aligned} U_i \times \mathbb{P}^1 &\to U_i \times \mathbb{P}^1 \\ (x, [u : v]) &\mapsto \big(x, [\alpha u + \gamma_{|U_i}(x) v : v]\big). \end{aligned}$$

(2) *The $\mathrm{Aut}^\circ(S)$-orbits in S are $\{p\}$ and $\tau^{-1}(\tau(p)) \setminus \{p\}$ for $p \in \sigma$.*

Proof. (1). The proof follows from the computation made in [20, case (b) p. 92]. For the sake of self-containess, we recall it below. Since τ is decomposable, we can write its transition maps as $t_{ij}\colon U_j \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, $(x, [u : v]) \mapsto \big(x, [a_{ij}(x) u : b_{ij}(x) v]\big)$, where $[u : v]$ denotes the coordinates of $\mathbb{P}^1$, $a_{ij} \in \mathcal{O}_C(U_i \cap U_j)^*$ denotes the transition maps of the line bundle $L(\sigma)$ and $b_{ij} \in \mathcal{O}_C(U_i \cap U_j)^*$. Let $\mu \in \mathrm{Aut}^\circ(S)$. The morphism induced by Blanchard's lemma $\tau_*\colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial (Remark 5). Moreover, σ is fixed by $\mathrm{Aut}^\circ(S)$ as it is the unique minimal section. Therefore, for each trivializing open subset $U_i \subset C$, μ induces an automorphism $\mu_i\colon U_i \times \mathbb{P}^1 \to U_i \times \mathbb{P}^1$, given by $(x, [u : v]) \mapsto \big(x, [\alpha_i(x) u + \gamma_i(x) v : v]\big)$, where $\alpha_i \in \mathcal{O}_C(U_i)^*$ and $\gamma_i \in \mathcal{O}_C(U_i)$. The condition $\mu_i t_{ij} = t_{ij} \mu_j$ implies that $\alpha_i = \alpha_j = \alpha \in \mathbb{G}_m$ and $\gamma_i = b_{ij}^{-1} a_{ij} \gamma_j$. Since $a_{ij} b_{ij}$ are the transition maps of the line bundle $\det(V)$, and a_{ij} denote the transition maps of $L(\sigma)$, it implies that $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$. The data of $\alpha \in \mathbb{G}_m$ and $\gamma \in \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$ determine uniquely the automorphism μ, this proves that we have an embedding $\mathrm{Aut}^\circ(S) \hookrightarrow \mathbb{G}_m \ltimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$. Conversely, one can check that the automorphisms defined in the statement commute with the transition maps, hence their gluing defines an automorphism of S. Because $\mathbb{G}_m \ltimes \Gamma(C, \det(V)^\vee \otimes L(\sigma)^{\otimes 2})$ is also connected, we get that it is isomorphic to $\mathrm{Aut}^\circ(S)$.

(2). Since the morphism induced by Blanchard's lemma $\tau_*\colon \mathrm{Aut}^\circ(S) \to \mathrm{Aut}^\circ(C)$ is trivial (Remark 5), each $\mathrm{Aut}^\circ(S)$-orbit is contained in a fibre of τ. As σ is the unique section with negative self-intersection number, it is fixed pointwise by $\mathrm{Aut}^\circ(S)$. It remains to see that $\mathrm{Aut}^\circ(S)$ acts transitively on $\tau^{-1}(\tau(p)) \setminus \{p\}$ for each p lying on σ.

Let $L = \det(V)^\vee \otimes L(\sigma)^{\otimes 2}$. It follows from [11, Proposition 2.15] that $\deg(L) = -\mathfrak{S}(S) > 1 + \deg(K_C)$. Let $p \in \sigma$ and let $\tau(p) = z$. We get by Serre duality that

$$h^1(C, L) = h^0(C, K_C \otimes L^\vee) = 0,$$

where the last equality follows from the fact that $\deg(K_C \otimes L^\vee) < -1$. Similarly we get the equality $h^1(C, L \otimes \mathcal{O}_C(z)^\vee) = 0$. By Riemann–Roch, $h^0(C, L \otimes \mathcal{O}_C(z)^\vee) = \deg(L) - g < \deg(L) - g + 1 = h^0(C, L)$. Therefore, z is not a base point of the complete linear system $|L|$, i.e. there exists $\gamma \in H^0(C, L)$ such that $\gamma(z) \neq 0$, and the subgroup $\mathbb{G}_a \simeq \{\mu_{1,\lambda\gamma}; \lambda \in \mathbf{k}\}$ acts transitively on $\tau^{-1}(z) \setminus \{p\}$ (see 1 for the definition of $\mu_{1,\lambda\gamma}$). □

Let S be a ruled surface as in Lemma 6, and $\phi\colon S \dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map. In the following lemma, we compute the fixed points of the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S'.

Lemma 7. *Let C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. If $\tau'\colon S' \to C$ is a ruled surface and there exists an $\mathrm{Aut}^\circ(S)$-equivariant birational map $\phi\colon S \dashrightarrow S'$ which is not an isomorphism, then $\mathfrak{S}(S') < \mathfrak{S}(S)$ and $\phi\mathrm{Aut}^\circ(S)\phi^{-1} \subsetneq \mathrm{Aut}^\circ(S')$. The fixed points of the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S' are the points lying on the minimal section of τ' and the base points of ϕ^{-1}. Moreover, we can write ϕ as a product of $\mathrm{Aut}^\circ(S)$-equivariant elementary transformations centered on the minimal sections.*

Proof. By [7, Theorem 7.7], we can write $\phi = \phi_n \cdots \phi_1$ where each ϕ_i is an $\mathrm{Aut}^\circ(S)$-equivariant elementary transformation. Without loss of generality, we can assume that this decomposition is minimal (i.e. the number of elementary transformations n is minimal among all possible factorizations), and we prove the statement by induction on $n \geq 1$.

Let σ be the minimal section of τ. By Lemma 6 2, the algebraic group $\mathrm{Aut}^\circ(S)$ acts transitively on $\tau^{-1}(\tau(p)) \setminus \{p\}$ for every $p \in \sigma$. Since ϕ_1 is $\mathrm{Aut}^\circ(S)$-equivariant, it follows that $\phi_1\colon S \dashrightarrow S_1$ is an elementary transformation centered on a point $p_1 \in \sigma$. The strict transform of σ by ϕ_1 is the minimal section σ_1 of the ruled surface $\tau_1\colon S_1 \to C$, and so $\mathfrak{S}(S_1) = \mathfrak{S}(S) - 1$. Since the base point q_1 of ϕ_1^{-1} does not lie on the minimal section σ_1 of τ_1, it follows by Lemma 6 2 that q_1 is not fixed by $\mathrm{Aut}^\circ(S_1)$. Since q_1 is fixed by $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$, we have the strict inequality $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1} \subsetneq \mathrm{Aut}^\circ(S_1)$. In the complement of the fibres $f_{p_1} \subset S$ and $f_{q_1} \subset S_1$ containing the points p_1 and q_1 respectively, ϕ_1 is an isomorphism. Therefore, by Lemma 6, the only fixed points of $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1}$ that lie in the complement of f_{q_1} are the points on the minimal section σ_1. It remains to check that the only fixed points on f_{q_1} are the point $q_1' \in \sigma_1$ and the base point q_1 of ϕ^{-1}. Let U be a trivializing open subset of τ with $\tau(p_1) \in U$, and let $f \in \mathcal{O}_C(U)$ such that $\mathrm{div}(f)_{|U} = \tau(p_1)$. We also choose trivializations of τ such that σ is the infinity section. Up to isomorphisms at the source and the target, $\phi_{1|U}$ equals $(x,[u:v]) \mapsto (x,[f(x)u:v])$. By Lemma 6 1, there is an action of $\mathbb{G}_m$ on S given locally by $(x,[u:v]) \mapsto (x,[\alpha u:v])$. It implies that there is an action of $\phi_1\mathbb{G}_m\phi_1^{-1}$ on S_1, given locally by $(x,[u:v]) \mapsto (x,[\alpha f(x)u : f(x)v]) = (x,[\alpha u:v])$. Therefore, $\phi_1\mathbb{G}_m\phi_1^{-1} \subset \mathrm{Aut}^\circ(S')$ acts transitively on $f_{q_1} \setminus \{q_1, q_1'\}$. Since $\phi_1\mathrm{Aut}^\circ(S)\phi_1^{-1} \subset \mathrm{Aut}^\circ(S')$

acts fibrewise (Remark 5) and is connected, we get that q_1 and q_1' are the fixed points of the action of $\phi_1 \operatorname{Aut}^\circ(S)\phi_1^{-1}$ on f_{q_1}.

Assume the statement holds for the birational map $\psi = \phi_i \cdots \phi_1 \colon S \dashrightarrow S_i$, for some $i \geq 1$, and where $\tau_i \colon S_i \to C$ is a ruled surface with a minimal section σ_i. We now prove that the statement is then true for $\phi_{i+1}\psi$. By induction, the fixed points of $\psi \operatorname{Aut}^\circ(S)\psi^{-1}$ on S_i are the points lying on the minimal section σ_i and the base points of ψ^{-1}.

Assume that ϕ_{i+1} is centered on a base point of ψ^{-1}, which is (the image of) the base point of the inverse of a previous elementary transformation ϕ_j. A local calculation yields that we may cancel both ϕ_j and ϕ_{i+1}, which contradicts the minimality of the factorization of ϕ. So ϕ_{i+1} is centered on a point lying on the minimal section σ_i. Hence $\mathfrak{S}(S_{i+1}) = \mathfrak{S}(S_i) - 1 < \mathfrak{S}(S)$ by induction, and $\phi_{i+1}(\psi \operatorname{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1} \subset \operatorname{Aut}^\circ(S_{i+1})$. The base point of ϕ_{i+1} is fixed by $\phi_{i+1}(\psi \operatorname{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1}$, but is not fixed by $\operatorname{Aut}^\circ(S_i)$ (by Lemma 6). Thus, we get the strict inclusion $\phi_{i+1}(\psi \operatorname{Aut}^\circ(S)\psi^{-1})\phi_{i+1}^{-1} \subsetneq \operatorname{Aut}^\circ(S_{i+1})$. □

The infinite increasing sequences of automorphism groups given in [11, Theorem A] can be obtained from Lemma 7, but they do not imply that $\operatorname{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup. As it is explained below, we can get an infinite increasing sequence of connected algebraic subgroups, where each of them is included in a maximal one, which a fortiori cannot be the same for all of them.

Remark 8. Let $n \geq d \geq 2$. Define the connected algebraic groups

$$G_d = \{\mathbb{A}^2 \to \mathbb{A}^2, (x,y) \mapsto (x, y + p(x)), p \in \mathbf{k}[x]_{\leq d}\},$$

acting regularly on $\mathbb{A}^2$, and then birationally on $\mathbb{P}^2$ via any embedding $\mathbb{A}^2 \hookrightarrow \mathbb{P}^2$. Then $G_d \subsetneq G_{d+1}$ for all d. On the other hand, using an explicit description of $\operatorname{Aut}^\circ(\mathbb{F}_n)$ from [1, §4.2], we get for all $n \geq d$ that G_d is a subgroup of $\operatorname{Aut}^\circ(\mathbb{F}_n)$, which is a maximal connected algebraic subgroup of $\operatorname{Bir}(\mathbb{P}^2)$.

Notice that for any variety X, using Remark 8, we may produce an infinite increasing sequence of algebraic subgroups of $\operatorname{Bir}(X \times \mathbb{P}^2)$. In particular, for $n \geq 2$ and C a curve of positive genus, the same is true for $\operatorname{Bir}(C \times \mathbb{P}^n) \simeq \operatorname{Bir}(C \times \mathbb{P}^{n-2} \times \mathbb{P}^2)$.

We reprove below partially [11, Theorem C], without using [11, Theorem B].

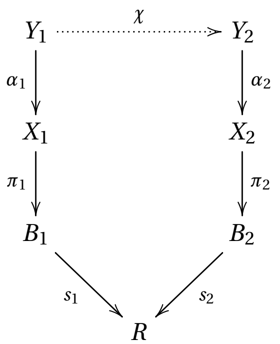
Proposition 9. *Let C be a curve of genus $g \geq 1$ and let $\tau \colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Then $\operatorname{Aut}^\circ(S)$ is not contained in a maximal connected algebraic subgroup of $\operatorname{Bir}(S)$.*

Proof. Assume that $\operatorname{Aut}^\circ(S)$ is contained in a maximal connected algebraic subgroup G of $\operatorname{Bir}(S)$. Then G acts regularly on a variety Y by Weil regularization theorem (see [25], or [17, 26] for a modern proof). By [5, Corollary 3], we can choose Y to be normal and projective. Using an equivariant resolution of singularities (see [18, Remark B, p. 155]), we can also assume Y to be smooth. Then by Blanchard's lemma (see Proposition 3), the successive contractions of the (-1)-curves gives rise to a ruled surface S' such that the induced birational morphism $Y \to S'$ is G-equivariant. Since G is maximal and connected, it follows that $G \simeq \operatorname{Aut}^\circ(S')$. The induced birational map $\phi \colon S \dashrightarrow S'$ is $\operatorname{Aut}^\circ(S)$-equivariant. If ϕ is an isomorphism, then $\mathfrak{S}(S) = \mathfrak{S}(S')$. Else ϕ factorises as product of $\operatorname{Aut}^\circ(S)$-equivariant elementary transformations centered on the minimal sections and $\mathfrak{S}(S') < \mathfrak{S}(S)$ (by Lemma 7). In both cases, we have $\mathfrak{S}(S') \leq \mathfrak{S}(S)$. Let $\epsilon \colon S' \dashrightarrow S''$ be an elementary transformation centered on the minimal section of $\tau' \colon S' \to C$. Then again by Lemma 7, it follows that $\epsilon \operatorname{Aut}^\circ(S')\epsilon^{-1} \subsetneq \operatorname{Aut}^\circ(S'')$, which contradicts the maximality of G as a connected algebraic subgroup of $\operatorname{Bir}(S)$. □

3. Higher dimensional case

In what follows, we would like to utilize the machinery of the G-Sarkisov program for a connected algebraic group G. Thus from now on, we furthermore assume that $\mathrm{char}(\mathbf{k}) = 0$. The G-Sarkisov program is a non-deterministic algorithm that decomposes every G-equivariant birational map between two G-Mori fibre spaces as a product of simpler maps called G-Sarkisov links. Its non-equivariant version was proven by Hacon and McKernan in [13] and, building on their result, Floris proved the G-equivariant version in [9]. We follow the strategy of the proof of Proposition 9, and in view of using G-Sarkisov program, we recall first the definition:

Definition 10. *Let G be a connected algebraic group. A G-*Mori fibre space *is a Mori fibre space with a regular action of G. Let $\pi_1: X_1 \to B_1$ and $\pi_2: X_2 \to B_2$ be two birational G-Mori fibre spaces. A G-*Sarkisov diagram *between X_1/B_1 and X_2/B_2 is a commutative diagram of the form*



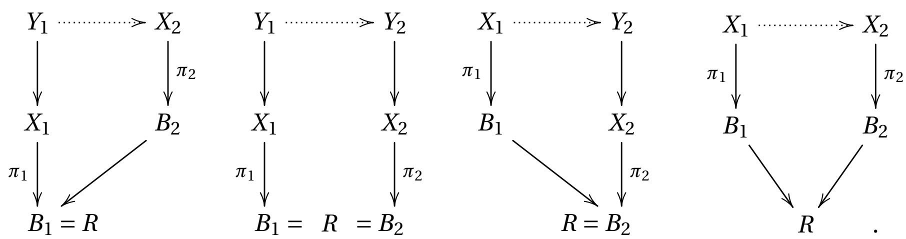
which satisfies the following properties:

(1) *all morphisms appearing in the diagram are either isomorphisms or outputs of some G-equivariant MMP on a $\mathbb{Q}$-factorial klt G-pair (Z,Φ) (recall that a G-pair is a pair (Z,Φ) such that G acts regularly on Z and there is an induced regular action on Φ),*
(2) *maximal dimensional varieties have $\mathbb{Q}$-factorial and terminal singularities,*
(3) *α_1 and α_2 are G-equivariant divisorial contractions or isomorphisms,*
(4) *s_1 and s_2 are G-equivariant extremal contractions or isomorphisms,*
(5) *χ is an isomorphism or a composition of G-equivariant anti-flips/flop/flips (in that order),*
(6) *the relative Picard rank $\rho(Z/R)$ of any variety Z in the diagram is at most 2.*

We call R the base *of the diagram.*

Property 6 implies that α_1 is a divisorial contraction if and only if s_1 is an isomorphism. A similar statement holds for the right hand side of the diagram. Depending whether s_1 or s_2 is an isomorphism, we get four types of Sarkisov diagrams:

Type I Type II Type III Type IV



*The birational map $\psi = \alpha_2 \chi \alpha_1^{-1}$ between X_1 and X_2 is called a G-*Sarkisov link.

Remark 11. Property 2 does not follow directly from the original definition of a (G-)Sarkisov diagram of [13] and [9]. For a proof, see [4, Proposition 4.25].

In subsequent proofs we are going to make heavy use of the following elementary but useful observation:

Remark 12. Let Z be one of the varieties appearing in a G-Sarkisov diagram, such that the relative Picard rank $\rho(Z/R)$ is 2. Then the G-Sarkisov diagram is uniquely determined by the datum of $Z \to R$, by a process known as the 2*-ray game* (see [4, §2.F]).

More specifically, the 2-ray game is a deterministic process that assigns to any such $Z \to R$ a G-Sarkisov diagram. Moreover any G-Sarkisov diagram can be recovered by the 2-ray game on any of its relative Picard rank 2 morphisms. Thus, up to orientation of the diagram, there is a unique G-Sarkisov diagram that contains $Z \to R$.

Lemma 13. *Let $n \geq 1$ and C be a curve of genus $g \geq 1$. Let $\tau\colon S \to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S) < -(1+\deg(K_C))$ with minimal section σ and let $\phi\colon S \dashrightarrow S'$ be an* $\mathrm{Aut}^\circ(S)$*-equivariant birational map (possibly the identity) to a $\mathbb{P}^1$-bundle $\tau'\colon S' \to C$. Let $\pi' = \tau' \times \mathrm{id}_{\mathbb{P}^n}\colon S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$ and $\pi'_1\colon S' \times \mathbb{P}^n \to S'$ be the projection to the first factor. Then the following hold:*

(1) *The only non-trivial* $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$*-Sarkisov diagrams, where $\pi'\colon S' \times \mathbb{P}^n \to C \times \mathbb{P}^n$ is the LHS Mori fibre space, are the following ones:*

$$\begin{array}{ccc} T\times\mathbb{P}^n & = & T\times\mathbb{P}^n \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ S'\times\mathbb{P}^n & & S''\times\mathbb{P}^n \\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi''} \\ C\times\mathbb{P}^n & = & C\times\mathbb{P}^n \end{array} \qquad \begin{array}{ccc} S'\times\mathbb{P}^n & = & S'\times\mathbb{P}^n \\ \downarrow{\scriptstyle\pi'} & & \downarrow{\scriptstyle\pi'_1} \\ C\times\mathbb{P}^n & & S' \\ & {\scriptstyle p_1}\searrow \quad \swarrow{\scriptstyle\tau'} & \\ & C & \end{array} .$$

In the first case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S'' \times \mathbb{P}^n$ is equal to $\psi \times \mathrm{id}_{\mathbb{P}^n}$, where $\psi\colon S' \dashrightarrow S''$ is an elementary transformation of $\mathbb{P}^1$-bundles whose center p is a point fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, and T is the blow-up of S' at p. In the second case, the induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ is equal to $\mathrm{id}_{S'\times\mathbb{P}^n}$.

(2) *The only non-trivial* $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$*-Sarkisov diagrams, where $\pi'_1\colon S' \times \mathbb{P}^n \to S'$ is the LHS Mori fibre space, are the following ones:*

$$\begin{array}{ccc} T\times\mathbb{P}^n & = & T\times\mathbb{P}^n \\ \downarrow{\scriptstyle\eta\times\mathrm{id}_{\mathbb{P}^n}} & & \downarrow{\scriptstyle\pi''_1} \\ S'\times\mathbb{P}^n & & T \\ \downarrow{\scriptstyle\pi'_1} & \swarrow{\scriptstyle\eta} & \\ S' & & \end{array} \qquad \begin{array}{ccc} S'\times\mathbb{P}^n & = & S'\times\mathbb{P}^n \\ \downarrow{\scriptstyle\pi'_1} & & \downarrow{\scriptstyle\pi'} \\ S' & & C\times\mathbb{P}^n \\ & {\scriptstyle\tau}\searrow \quad \swarrow{\scriptstyle p_1} & \\ & C & \end{array} .$$

The induced Sarkisov link $S' \times \mathbb{P}^n \dashrightarrow T \times \mathbb{P}^n$ is equal to $\eta^{-1} \times \mathrm{id}_{\mathbb{P}^n}$ in the former case and $\mathrm{id}_{S'\times\mathbb{P}^n}$ in the latter, where $\eta\colon T \to S'$ is the blowup of S' at point p fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$.

Proof. **(1).** We distinguish between two cases depending on the base R of the diagram: if $R = C \times \mathbb{P}^n$ then we have a link of Type I or II and so the first step of the link is an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant divisorial contraction $\alpha\colon Y \to S' \times \mathbb{P}^n$. Note that by [6, Corollary 4.2.7], it follows that $(\phi \times \mathrm{id}_{\mathbb{P}^n})\mathrm{Aut}^\circ(S \times \mathbb{P}^n)(\phi \times \mathrm{id}_{\mathbb{P}^n})^{-1} \simeq \phi\mathrm{Aut}^\circ(S)\phi^{-1} \times \mathrm{PGL}_{n+1}(\mathbf{k})$. Let $(q,x) \in S' \times \mathbb{P}^n$ be a point in the center of α. If q is not point fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, then and by Lemma 6 and the description of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, the closure of the orbit of (q,x) is a Cartier divisor and thus α is an isomorphism, contradicting the assumption that α is a divisorial contraction.

Thus we may assume that q is fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$. In that case the orbit of (q,x) is precisely $\{q\}\times\mathbb{P}^n$. Notice that the codimension of $\{q\}\times\mathbb{P}^n$ is 2 and so by [4, Lemma 2.13]

$$\alpha=(\eta\times\mathrm{id}_{\mathbb{P}^n})\colon T\times\mathbb{P}^n\to S'\times\mathbb{P}^n,$$

where $\eta\colon T\to S'$ is the blowup of S' at q. By Remark 12, the unique Sarkisov diagram containing $T\times\mathbb{P}^n\to C\times\mathbb{P}^n$ is the one given in the statement.

We now consider the case when $R\neq C\times\mathbb{P}^n$. Then we have a contraction $C\times\mathbb{P}^n\to R$ of relative Picard rank 1. Since $\rho(C\times\mathbb{P}^n)=2$, the cone of curves $\overline{\mathrm{NE}}(C\times\mathbb{P}^n)$ has two extremal rays and so there are only two such contractions, namely the projections to the two factors: $C\times\mathbb{P}^n\to C$ and $C\times\mathbb{P}^n\to\mathbb{P}^n$. However, by property (1) of Definition 10, $C\times\mathbb{P}^n\to\mathbb{P}^n$ would have to be an output of some MMP on a klt pair (Z,Φ), and thus by [12] its exceptional locus would be rationally connected, a contradiction. Thus $R=C$ and again we conclude by Remark 12 for $S'\times\mathbb{P}^n\to C\times\mathbb{P}^n$.

(2). We again proceed by a similar distinction of cases. If $R=S'$ then, as in the proof of (1), the first step is an $\mathrm{Aut}^\circ(S\times\mathbb{P}^n)$-equivariant divisorial contraction $\eta\times\mathrm{id}_{\mathbb{P}^n}\colon T\times\mathbb{P}^n\to S'\times\mathbb{P}^n$, where $\eta\colon T\to S'$ is the blow-up of a point of S' fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$, and we conclude by Remark 12.

If $R\neq S'$, then $S'\to R$ is one of the two morphisms $S'\to C$ or $S'\to\check{S}'$, where the latter is the contraction of the minimal section. Again, by [12] we may exclude the latter case since its exceptional locus is not rationally connected. Finally, Remark 12, once again, guarantees that the Sarkisov diagram is the one in the statement. □

Lemma 14. *Let $n\geq 1$ and C be a curve of genus $g\geq 1$. Let $\tau\colon S\to C$ be a decomposable $\mathbb{P}^1$-bundle such that $\mathfrak{S}(S)<-(1+\deg(K_C))$ with minimal section σ. Let $\phi\colon S\dashrightarrow S'$ be an $\mathrm{Aut}^\circ(S)$-equivariant birational map, with S' being a smooth projective surface which is not minimal. Denote by $\pi_1'\colon S'\times\mathbb{P}^n\to S'$ the projection to the first factor. Then the only non-trivial $\mathrm{Aut}^\circ(S\times\mathbb{P}^n)$-Sarkisov diagrams, where $\pi_1'\colon S'\times\mathbb{P}^n\to S'$ is the LHS Mori fibre space, are the following ones:*

$$\begin{array}{ccc} T\times\mathbb{P}^n & = & T\times\mathbb{P}^n\\ \downarrow{\scriptstyle \eta\times\mathrm{id}_{\mathbb{P}^n}} & & \downarrow{\scriptstyle \pi_1''}\\ S'\times\mathbb{P}^n & & T\\ \downarrow{\scriptstyle \pi_1'} & \swarrow{\scriptstyle \eta} & \\ S' & & \end{array}\qquad\qquad \begin{array}{ccc} S'\times\mathbb{P}^n & = & S'\times\mathbb{P}^n\\ \downarrow{\scriptstyle \pi_1'} & & \downarrow{\scriptstyle \kappa\times\mathrm{id}_{\mathbb{P}^n}}\\ S' & & T\times\mathbb{P}^n\\ & \searrow{\scriptstyle \kappa} & \downarrow{\scriptstyle \pi_1''}\\ & & T. \end{array}$$

In the first case, $\eta\colon T\to S'$ is the blow-up of a point p fixed by $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$. In the second case, $\kappa\colon S'\to T$ is the contraction of a (-1)-curve l. In both cases, π_1'' denotes the projection to the first factor.

Proof. We again distinguish between two cases depending on the base R of the Sarkisov diagram: if $R=S'$ then the first step of the link is an $\mathrm{Aut}^\circ(S\times\mathbb{P}^n)$-equivariant divisorial contraction $\alpha\colon Y\to S'\times\mathbb{P}^n$. We follow the same strategy of the proof of Lemma 13: first by [6, Corollary 4.2.7], $(\phi\times\mathrm{id}_{\mathbb{P}^n})\mathrm{Aut}^\circ(S\times\mathbb{P}^n)(\phi\times\mathrm{id}_{\mathbb{P}^n})^{-1}=\phi\mathrm{Aut}^\circ(S)\phi^{-1}\times\mathrm{PGL}_{n+1}(\mathbf{k})$. This again implies that α has to be an extraction with center of the form $\{q\}\times\mathbb{P}^n$, where q is a point fixed by the action of $\phi\mathrm{Aut}^\circ(S)\phi^{-1}$ on S'. Since the center is of codimension 2, again using [4, Lemma 2.13], we conclude that

$$a=\eta\times\mathrm{id}_{\mathbb{P}^n}\colon T\times\mathbb{P}^n\to S'\times\mathbb{P}^n,$$

where $\eta\colon T\to S'$ is the blow-up of q. By Remark 12, the diagram is the one given in the statement.

If $R\neq S'$,we have a morphism $S'\to R$ of relative Picard rank 1. Since S' is not minimal, its Picard rank is greater or equal to 3 which already implies that $R=T$ is a surface. Again, using Remark 12 we may conclude that the diagram is the one proposed in the statement. Moreover, by property (2) of Definition 10, $T \times \mathbb{P}^n$ has to have terminal singularities. Thus the singular locus of $T \times \mathbb{P}^n$ has codimension at least 3 (see [16, Corollary 5.18]). If $q \in T$ is singular, then $\{q\} \times \mathbb{P}^n$ is singular and has codimension 2 in $T \times \mathbb{P}^n$. This implies that T is smooth and consequently, $S' \to T$ is the contraction of a (-1)-curve. □

We prove below the higher dimensional analog of Proposition 9.

Theorem 15. *Let $n \geq 1$. Let C be a curve of genus $g \geq 1$, let S be a decomposable $\mathbb{P}^1$-bundle over C such that $\mathfrak{S}(S) < -(1 + \deg(K_C))$. Then $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$ is not contained in a maximal connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$.*

Proof. Assume that $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$ is contained in a maximal connected algebraic subgroup $G \subset \mathrm{Bir}(S \times \mathbb{P}^n)$. By [5, Corollary 3], there exists a normal and projective variety Y, G-birationally equivalent to $S \times \mathbb{P}^n$, and on which G acts regularly. Then we use an equivariant resolution of singularities (see [15, Theorem 3.36, Proposition 3.9.1]) to furthermore assume that Y is smooth. Running an MMP, which is G-equivariant by [9, Lemma 2.5], we get an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant birational map $\chi\colon S \times \mathbb{P}^n \dashrightarrow Y$ such that $G \simeq \mathrm{Aut}^\circ(Y)$ and $Y \to B$ is a Mori fibre space. By [9, Theorem 1.2], χ decomposes as a product of $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant Sarkisov links. By Lemmas 13 and 14, it follows that $Y = T \times \mathbb{P}^n$ for some surface T and χ is of the form $\psi \times \mathrm{id}_{\mathbb{P}^n}$, where $\psi\colon S \dashrightarrow T$ is an $\mathrm{Aut}^\circ(S)$-equivariant birational map. Up to possibly performing an extra link of Type IV (namely the RHS link in Lemma 13(1)), we may assume that $B = T$ and θ is given by the projection to the first factor. Contracting successively all (-1)-curves in T yields an $\mathrm{Aut}^\circ(S \times \mathbb{P}^n)$-equivariant birational map $\phi \times \mathrm{id}_{\mathbb{P}^n}\colon S \times \mathbb{P}^n \dashrightarrow S' \times \mathbb{P}^n$ (by Blanchard's lemma, see Proposition 3), where ϕ is $\mathrm{Aut}^\circ(S)$-equivariant and S' is a ruled surface. Two cases arise: either ϕ is an isomorphism and $\mathfrak{S}(S) = \mathfrak{S}(S')$, or ϕ is not an isomorphism and $\mathfrak{S}(S') < \mathfrak{S}(S)$ by Lemma 7. In both cases, $\mathfrak{S}(S') \leq \mathfrak{S}(S)$ and since G is maximal, G is isomorphic to $\mathrm{Aut}^\circ(S' \times \mathbb{P}^n) \simeq \mathrm{Aut}^\circ(S') \times \mathrm{PGL}_{n+1}(\mathbf{k})$ ([6, Corollary 4.2.7]). Let $\phi'\colon S' \dashrightarrow S''$ be an elementary transformation of S' centered at a point on the minimal section. Then $\phi' \mathrm{Aut}^\circ(S') \phi'^{-1} \subsetneq \mathrm{Aut}^\circ(S'')$ by Lemma 6. Thus $(\phi' \times \mathrm{id}_{\mathbb{P}^n}) \mathrm{Aut}^\circ(S' \times \mathbb{P}^n)(\phi' \times \mathrm{id}_{\mathbb{P}^n})^{-1} \subsetneq \mathrm{Aut}^\circ(S'' \times \mathbb{P}^n)$, which contradicts the maximality of G as connected algebraic subgroup of $\mathrm{Bir}(S \times \mathbb{P}^n)$. □

Proof of Theorem A. Let C be a curve of positive genus and $S \to C$ be a ruled surface. As S is birational to $C \times \mathbb{P}^1$, we get for all $n \geq 1$ that $\mathrm{Bir}(C \times \mathbb{P}^n) \simeq \mathrm{Bir}(S \times \mathbb{P}^{n-1})$. We conclude with Proposition 9 for $n = 1$ and Theorem 15 for $n \geq 2$. □

References

[1] J. Blanc, "Sous-groupes algébriques du groupe de Cremona", *Transform. Groups* **14** (2009), no. 2, p. 249-285.

[2] J. Blanc, A. Fanelli, R. Terpereau, "Automorphisms of $\mathbb{P}^1$-bundles over rational surfaces", 2021, https://arxiv.org/abs/1707.01462.

[3] ———, "Connected Algebraic Groups Acting on three-dimensional Mori Fibrations", *Int. Math. Res. Not.* (2021), article no. rnab293.

[4] J. Blanc, S. Lamy, S. Zimmermann, "Quotients of higher-dimensional Cremona groups", *Acta Math.* **226** (2021), no. 2, p. 211-318.

[5] M. Brion, "Algebraic group actions on normal varieties", *Trans. Mosc. Math. Soc.* **78** (2017), p. 85-107.

[6] M. Brion, P. Samuel, V. Uma, *Lectures on the structure of algebraic groups and geometric applications*, CMI Lecture Series in Mathematics, vol. 1, Hindustan Book Agency; Chennai Mathematical Institute, 2013, viii+120 pages.

[7] I. V. Dolgachev, V. A. Iskovskikh, "Finite subgroups of the plane Cremona group", in *Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. I*, Progress in Mathematics, vol. 269, Birkhäuser, 2009, p. 443-548.

[8] F. Enriques, "Sui gruppi continui di trasformazioni cremoniane nel piano", *Rom. Acc. L. Rend. (5)* **2** (1893), no. 1, p. 468-473.

[9] E. Floris, "A note on the G-Sarkisov program", *Enseign. Math.* **66** (2020), no. 1-2, p. 83-92.

[10] P. Fong, "Algebraic subgroups of the group of birational transformations of ruled surfaces", 2021, https://arxiv.org/abs/2111.09697.

[11] ———, "Connected algebraic groups acting on algebraic surfaces", 2021, https://arxiv.org/abs/2004.05101.
[12] C. D. Hacon, J. McKernan, "On Shokurov's rational connectedness conjecture", *Duke Math. J.* **138** (2007), no. 1, p. 119-136.
[13] ———, "The Sarkisov program", *J. Algebr. Geom.* **22** (2013), no. 2, p. 389-405.
[14] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. 52, Springer, 1977, xvi+496 pages.
[15] J. Kollár, *Lectures on resolution of singularities*, Annals of Mathematics Studies, vol. 166, Princeton University Press, 2007, vi+208 pages.
[16] J. Kollár, S. Mori, *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, 1998, With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original, viii+254 pages.
[17] H. Kraft, "Regularization of Rational Group Actions", 2018, https://arxiv.org/abs/1808.08729.
[18] J. Lipman, "Desingularization of two-dimensional schemes", *Ann. Math.* **107** (1978), no. 1, p. 151-207.
[19] M. Maruyama, *On classification of ruled surfaces*, Lectures in Mathematics, Department of Mathematics, Kyoto University, vol. 3, Kinokuniya Book-Store Co., 1970, iv+75 pages.
[20] ———, "On automorphism groups of ruled surfaces", *J. Math. Kyoto Univ.* **11** (1971), p. 89-112.
[21] H. Umemura, "Sur les sous-groupes algébriques primitifs du groupe de Cremona à trois variables", *Nagoya Math. J.* **79** (1980), p. 47-67.
[22] ———, "Maximal algebraic subgroups of the Cremona group of three variables. Imprimitive algebraic subgroups of exceptional type", *Nagoya Math. J.* **87** (1982), p. 59-78.
[23] ———, "On the maximal connected algebraic subgroups of the Cremona group. I", *Nagoya Math. J.* **88** (1982), p. 213-246.
[24] ———, "On the maximal connected algebraic subgroups of the Cremona group. II", in *Algebraic groups and related topics (Kyoto/Nagoya, 1983)*, Advanced Studies in Pure Mathematics, vol. 6, North-Holland, 1985, p. 349-436.
[25] A. Weil, "On algebraic groups of transformations", *Am. J. Math.* **77** (1955), p. 355-391.
[26] D. Zaitsev, "Regularization of birational group operations in the sense of Weil", *J. Lie Theory* **5** (1995), no. 2, p. 207-224.