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Trace Operator's Range Characterization for Sobolev Spaces on Lipschitz Domains of $\mathbb{R}^2$

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Abstract. We give, first, two new applications related to the range characterization of the range of trace operator in $H^2(\Omega)$. After this, we characterize the range of trace operator in the Sobolev spaces $W^{3,p}(\Omega)$ when Ω is a connected bounded domain $\mathbb{R}^2$ with Lipschitz-continuous boundary.

Résumé. On donne, d'abord, deux nouvelles applications relatives à la caractérisation de l'image de l'opérateur trace dans $H^2(\Omega)$. Après cela, on caractérise l'image de l'opérateur trace dans les espaces de Sobolev $W^{3,p}(\Omega)$, Ω étant un domaine borné, connexe de $\mathbb{R}^2$ de frontière lipschitzienne.

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1. Introduction

Let Ω be a connected and Lipschitz subset of $\mathbb{R}^2$ whose bounded and orientable boundary is denoted by Γ. For $1 < p < \infty$ and m integer, $W^{m,p}(\Omega)$ denotes the Sobolev space of functions of $L^p(\Omega)$ whose distributional derivatives up to the order m also belong to $L^p(\Omega)$. A famous result of E. Gagliardo [9] gives, for $m = 1$, the characterization of the range of the restriction $\gamma_0(u) = u_{|\Gamma}$ to Γ. More precisely, Gagliardo proves that the operator γ_0 is linear and continuous from $W^{1,p}(\Omega)$ into $W^{1-\frac{1}{p},p}(\Gamma)$ for $1 \leq p < \infty$ and has a continuous right inverse for $p > 1$.

When $u \in W^{2,p}(\Omega)$, then
$$\frac{\partial u}{\partial x_j} \in W^{1,p}(\Omega) \quad \text{for } j = 1, \dots, N.$$
Therefore the normal derivative $\gamma_1(u) = \nabla u \cdot \boldsymbol{n} \in L^p(\Gamma)$ since $\boldsymbol{n} = (n_1,, \cdots,, n_N)$ is defined almost everywhere and belongs to $(L^\infty(\Gamma))^N$. J. Nečas [15] proves that $\gamma_0(u) \in W^{1,p}(\Gamma)$ and that the

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linear mapping $u \to (\gamma_0(u), \gamma_1(u))$ is continuous from $W^{2,p}(\Omega)$ into $W^{1,p}(\Gamma) \times L^p(\Gamma)$. A natural question is to characterize the range of the mapping (γ_0, γ_1). A first answer has been obtained for polygonal-type domains of $\mathbb{R}^2$ by Kondrat'ev and Grisvard (see e.g. [12] for full references) in terms of compatibility conditions at the corners and then the results have been extended to polyhedral-type domains ($N = 3$). These characterizations have been extensively used in order to give regularity results for different types of boundary-value problems.

For general Lipschitz domains a first characterization of the range of (γ_0, γ_1) has been obtained for $N = 2$ in [11] if and $p = 2$ and extended in [8] for the general case $1 < p < \infty$. This result reads as follows: The range of $(\gamma_0, \gamma_1))$ is the set of $(g_0, g_1) \in W^{1,p}(\Gamma) \times L^p(\Gamma)$ such that:

$$\frac{\partial g_0}{\partial \boldsymbol{t}} \boldsymbol{t} + g_1 \boldsymbol{n} \in \boldsymbol{W}^{1-\frac{1}{p},p}(\Gamma). \tag{1}$$

Let us mention, also, that the generalization for the case $N = 3$ and $1 < p < \infty$ was obtained by Buffa et al (see [5]).

The more general characterization of the image of the trace operators in $W^{m,p}(\Omega)$, where Ω is a domain in $\mathbb{R}^N$ with Lipschitz boundary, has been obtained for arbitrary m and N, by Maz'ya, Mitrea and Shaposhnikova [13, see Theorem 7.8 and Corollary 7.11]. These authors used an analytical method based on Taylor expansions in Besov and weighted Sobolev spaces.

In this paper, first of all, we will give two applications of the result of Geymonat and Krasucki [11] to solve a boundary value problem for the bi-laplacian equation. The first application concerns a regularity result for the solution to a non homogeneous Dirichlet problem for the homogeneous Bi-Laplacian equation in a lipschitzian domain. This result improve the one obtained in [7]. Up to our knowledge it is the first time that this result is stated in this form. The second application relies on the existence of very weak solution, in Lipschitz domains, to Dirichlet problem for the Bi-Laplacian equation. It is a first time that one can obtain very weak solution in Lipshitz domains.

Next, due to a new representation of the Hessian in $\mathbb{R}^3$, we characterize the range of the trace operator in $W^{3,p}(\Omega)$, more precisely, we would like to characterize the range of the application $(\gamma_0, \gamma_1, \gamma_2)$ defined on $W^{3,p}(\Omega)$ where

$$\begin{aligned} \gamma_2 : W^{3,p}(\Omega) &\to L^p(\Gamma) \\ u &\to \gamma_2(u) = \left[\left(\nabla^2 u\right) \boldsymbol{n}\right] \cdot \boldsymbol{n}. \end{aligned}$$

Necessary conditions are obtained by Geymonat [10].

Even if this result is a particular case of the obtained in [13], our proof is completely new and different from their. Our proof relies on potential matrices which are similar to potential vectors introduced in [3]. We hope that we can extend our proof to $W^{m,p}(\Omega)$ where Ω is a Lipschitz domain.

The outline of the paper is the following. In Section 2, we fix some notations. In Section 3, we give two applications of the result obtained by Geymonat and Krasucki [11] to solve a Dirichlet boundary value problem for the bi-laplacian equation. In Section 4, we will state a new characterization of the Hessian which allows us to state and prove in Section 5, the characterization of the range of the trace operator is obtained in $W^{m,p}(\Omega)$ where Ω is a Lipschitz domain.

2. Notations and preliminaries

In the following, the vectors, the vector functions (or distributions), the matrix functions (or distributions) and the spaces of vector-valued functions are represented by bold symbols.

We use the following differential operators throughout the paper: the divergence operator $\operatorname{div} : \boldsymbol{D}'(\Omega) \to \mathcal{D}'(\Omega)$ is defined by

$$\operatorname{div} \boldsymbol{v} = \nabla \cdot \boldsymbol{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \quad \text{for any} \quad \boldsymbol{v} = (v_i) \in \boldsymbol{\mathcal{D}}'(\Omega).$$

The scalar rotational operator $\operatorname{curl} : \boldsymbol{\mathcal{D}}'(\Omega) \to \mathcal{D}'(\Omega)$ is defined by

$$\operatorname{curl} \boldsymbol{v} = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \quad \text{for any} \quad \boldsymbol{v} \in \boldsymbol{\mathcal{D}}'(\Omega)$$

The vector rotational operator $\mathbf{curl} : \mathcal{D}'(\Omega) \to \boldsymbol{\mathcal{D}}'(\Omega)$ is defined by

$$\mathbf{curl}\, \varphi = \begin{pmatrix} -\dfrac{\partial \varphi}{\partial x_2} \\[2ex] \dfrac{\partial \varphi}{\partial x_1} \end{pmatrix} \quad \text{for any} \quad \varphi \in \mathcal{D}'(\Omega).$$

The Hessian matrix operator $\operatorname{Hess} : \mathcal{D}'(\Omega) \to \mathbb{D}_s(\Omega)$ is defined by

$$\operatorname{Hess} \varphi = \begin{pmatrix} \dfrac{\partial^2 \varphi}{\partial x_1^2} & \dfrac{\partial^2 \varphi}{\partial x_1 \partial x_2} \\[2ex] \dfrac{\partial^2 \varphi}{\partial x_2 \partial x_1} & \dfrac{\partial^2 \varphi}{\partial x_2^2} \end{pmatrix} \quad \text{for any} \quad \varphi \in \mathcal{D}'(\Omega).$$

For any matrix field

$$\boldsymbol{S} = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$$

we define $\boldsymbol{S}^\star$ by

$$\boldsymbol{S}^\star = \begin{pmatrix} s_{22} & -s_{21} \\ -s_{12} & s_{11} \end{pmatrix}$$

Observe that $(\boldsymbol{S}^\star)^\star = \boldsymbol{S}$ and if the matrix $\boldsymbol{S}$ is symmetric, then

$$\mathbf{div}\, \boldsymbol{S}^\star = \mathbf{0} \Longleftrightarrow \mathbf{curl}\, \boldsymbol{S} = \mathbf{0}, \tag{2}$$

where $\mathbf{curl} \boldsymbol{S}$ is the vector field

$$\begin{pmatrix} \operatorname{curl}\ \boldsymbol{S}_1 \\[1ex] \operatorname{curl}\ \boldsymbol{S}_2 \end{pmatrix}$$

with $\boldsymbol{S}_i = \binom{s_{i1}}{s_{i2}}$ is the vector given by the i^{th} line of the matrix $\boldsymbol{S}$.

We define the spaces of rigid displacements by

$$\boldsymbol{R}(\Omega) = \left\{ \boldsymbol{v} = \boldsymbol{a} + \boldsymbol{b} \times \boldsymbol{x},\, \boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^2 \right\}.$$

We define the functional space $\boldsymbol{L}_0^p(\Omega)$ by

$$\boldsymbol{L}_0^p(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^p(\Omega),\, \int_\Omega \boldsymbol{v} \cdot \boldsymbol{r}\, dx = 0,\, \forall\, \boldsymbol{r} \in \boldsymbol{R}(\Omega) \right\}, \tag{3}$$

and $\mathbb{V}_s^{1,p}(\Omega)$ by

$$\mathbb{V}_s^{1,p}(\Omega) = \left\{ \boldsymbol{S} \in \mathbb{W}_{0,s}^{1,p}(\Omega),\, \mathbf{div}\, \boldsymbol{S} = \mathbf{0} \text{ in } \Omega \right\}. \tag{4}$$

If $\boldsymbol{x} = (x_1, x_2)$ is a generic point of $\mathbb{R}^2$, we denotes by $\boldsymbol{x}^\perp = \binom{-x_2}{x_1}$ and by $\boldsymbol{t}$ the tangent vector $\boldsymbol{n}^\perp$. So if w is a function defined on the boundary of Ω, we can write the normal derivative and the tangential derivative of w on Γ as follow:

$$\partial_{\boldsymbol{n}} w = \nabla w \cdot \boldsymbol{n} \quad \text{and} \quad \partial_{\boldsymbol{t}} w = \nabla w \cdot \boldsymbol{t}.$$

$$\left(\mathbf{grad}\, w\right)_{|\Gamma} = (\partial_{\boldsymbol{n}} w)\, \boldsymbol{n} + (\partial_{\boldsymbol{t}} w)\, \boldsymbol{t}.$$

The operator $\partial_{\boldsymbol{t}}$ defined in $W^{1,p}(\Gamma) = \{w \in L^p(\Gamma), \partial_{\boldsymbol{t}} w \in L^p(\Gamma)\}$ can be extended from the space

$$W^{1-\frac{1}{p},p}(\Gamma) = \left\{ w \in L^p(\Gamma), \int_\Gamma \int_\Gamma \frac{|w(\alpha) - w(\beta)|^p}{|\alpha - \beta|^p} d\alpha\, d\beta < \infty \right\}$$

into $W^{-\frac{1}{p},p}(\Gamma)$ (for more details see [11]). Also by using the same argument of proof of [4, Corollary 3.7], the following results holds true:

Proposition 1. *The following linear operator*

$$\partial_{\boldsymbol{t}} : W^{1-\frac{1}{p},p}(\Gamma) \to W^{-\frac{1}{p},p}(\Gamma),$$

is continuous and

$$\operatorname{Ker} \partial_{\boldsymbol{t}} = \mathbb{R}.$$

3. Homogeneous Bi-Laplacian problem

In this section, we will consider the following homogeneous Bi-Laplacian problem:

$$(\mathcal{P}_B) \qquad \begin{cases} \Delta^2 u \;=\; 0 & \text{in } \Omega, \\ u \;=\; g_0 & \text{on } \Gamma, \\ \frac{\partial u}{\partial \boldsymbol{n}} \;=\; g_1 & \text{on } \Gamma. \end{cases} \tag{5}$$

Recall the following result (see [7]).

Theorem 2. *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class of $\mathcal{C}^{0,1}$, with $N \geq 2$ and let*

$$g_0 \in H^1(\Gamma) \quad \textit{and} \quad g_1 \in L^2(\Gamma). \tag{6}$$

Then there exists a unique $u \in H^{3/2}(\Omega)$ solution to Problem $(\mathcal{P}_B)$ with the estimate

$$\|u\|_{H^{3/2}(\Omega)} \leq C\Big(\|g_0\|_{H^1(\Gamma)} + \|g_1\|_{L^2(\Gamma)}\Big). \tag{7}$$

On the other hand, we know that if Ω is a bounded open subset of $\mathbb{R}^N$ of class of $\mathcal{C}^{0,1}$ and $f \in L^2(\Omega)$, then there exists a unique solution $u \in H_0^2(\Omega)$ satisfying $\Delta^2 u = f$ in Ω with the estimate

$$\|u\|_{H^2(\Omega)} \leq C\,\|f\|_{L^2(\Omega)}. \tag{8}$$

We know that if $g_0 \in H^1(\Gamma)$ and $g_1 \in L^2(\Gamma)$ verify the condition (1) with $p = 2$, then there exists a function $u \in H^2(\Omega)$ satisfying $u = g_0$ and $\frac{\partial u}{\partial \boldsymbol{n}} = g_1$ on Γ with the estimate

$$\|u\|_{H^2(\Omega)} \leq C \left\| \frac{\partial g_0}{\partial \boldsymbol{t}} \boldsymbol{t} + g_1 \boldsymbol{n} \right\|_{\boldsymbol{H}^{1/2}(\Gamma)}. \tag{9}$$

The question that interests us here is to find such a function u in addition biharmonic in Ω.

Theorem 3. *Let Ω be a bounded open subset of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, with $N \geq 2$. Let g_0 and g_1 be satisfy the conditions* (6) *and* (1)*. Then there exists a unique biharmonic function $u \in H^2(\Omega)$ satisfying $u = g_0$ and $\frac{\partial u}{\partial \boldsymbol{n}} = g_1$ on Γ with the estimate* (9).

Proof. Let $w \in H^2(\Omega)$ such that $w = g_0$ and $\frac{\partial w}{\partial \boldsymbol{n}} = g_1$ on Γ. We know that there exists a unique solution $z \in H_0^2(\Omega)$ satisfying $\Delta^2 z = \Delta^2 w$ in Ω. The required function is given by $u = w - z$. □

Remark 4. Let us introduce the following Hilbert space

$$\boldsymbol{H}_T^{1/2}(\Gamma) = \left\{ \boldsymbol{v} \in \boldsymbol{H}^{1/2}(\Gamma);\ \boldsymbol{v}_\tau = \boldsymbol{0} \right\}.$$

Clearly

$$\boldsymbol{v} \in \boldsymbol{H}_T^{1/2}(\Gamma) \Longleftrightarrow \boldsymbol{v} = g\boldsymbol{n} \quad \text{with} \quad g \in L^2(\Gamma) \quad \text{and} \quad g\boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma)$$

The above result asserts that for any

$$g \in L^2(\Gamma) \quad \text{such that} \quad g\boldsymbol{n} \in \boldsymbol{H}^{1/2}(\Gamma)$$

there exists a function $u \in H^2(\Omega)\cap H^1_0(\Omega)$ such that $\frac{\partial u}{\partial \boldsymbol{n}} = g$ on Γ. Moreover among all functions satisfying these conditions, there is one that is biharmonic.

We will see now an interested consequence of this result which will allow us to establish the existence of very weak solutions in domains which are only Lipschitz. Before that, recall that if Ω is of class $\mathcal{C}^{1,1}$ and $g \in H^{-1/2}(\Gamma)$, then there exists a unique harmonic function $u \in L^2(\Omega)$ satisfying $u = g$ on Γ. When Ω is not sufficiently regular, there is not possible, in general, to define the trace of harmonic function $u \in L^2(\Omega)$ in $H^{-s}(\Gamma)$ for some $s > 0$. So, let us introduce the following Hilbert space:

$$M(\Omega) = \left\{v \in L^2(\Omega);\ \Delta v \in L^2(\Omega)\right\}.$$

We denote its norm by

$$\|v\|_{M(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\Delta v\|^2_{L^2(\Omega)}\right)^{1/2}.$$

It is easy to prove that $\mathcal{D}(\overline{\Omega})$ is dense in $M(\Omega)$.

As a consequence of this density result and of Theorem 3, we can prove the following lemma.

Lemma 5. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, with $N \geq 2$. The linear mapping $v \longmapsto (v\boldsymbol{n})|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping*

$$M(\Omega) \longrightarrow \left[\boldsymbol{H}^{1/2}_T(\Gamma)\right]'.$$

Moreover, we have the Green formula: For all $v \in M(\Omega)$ and $\varphi \in H^2(\Omega)\cap H^1_0(\Omega)$,

$$\int_\Omega v\Delta\varphi\,\mathrm{d}x - \int_\Omega \varphi\Delta v\,\mathrm{d}x = \left\langle (v\boldsymbol{n})_\Gamma, \nabla\varphi\right\rangle. \tag{10}$$

Remark 6. When Ω is of class $\mathcal{C}^{1,1}$, then the linear mapping $v \longmapsto v|_\Gamma$ defined on $\mathcal{D}(\overline{\Omega})$ can be extended to a linear continuous mapping

$$M(\Omega) \longrightarrow H^{-1/2}(\Gamma)$$

and we have the Green formula: For all $v \in M(\Omega)$ and $\varphi \in H^2(\Omega)\cap H^1_0(\Omega)$,

$$\int_\Omega v\Delta\varphi\,\mathrm{d}x - \int_\Omega \varphi\Delta v\,\mathrm{d}x = \left\langle v, \frac{\partial\varphi}{\partial\boldsymbol{n}}\right\rangle. \tag{11}$$

We now can solve the Laplace equation with singular boundary condition. For that we need to introduce the following spaces:

$$\mathcal{H}_{L^2_0(\Omega)} = \left\{v \in L^2(\Omega);\ \Delta v = 0 \text{ in } \Omega \quad \text{and} \quad v = 0 \text{ on } \Gamma\right\}$$

and

$$L^2(\Omega) \perp \mathcal{H}_{L^2_0(\Omega)} = \left\{z \in L^2(\Omega);\ \forall\, h \in \mathcal{H}_{L^2_0(\Omega)},\ \int_\Omega zh = 0\right\}.$$

Theorem 7. *Let Ω be a bounded open set of $\mathbb{R}^N$ of class $\mathcal{C}^{0,1}$, with $N \geq 2$. For any*

$$g \in H^{-1/2}(\Gamma) \quad \textit{such that} \quad g\boldsymbol{n} \in \left[\boldsymbol{H}^{1/2}_T(\Gamma)\right]'$$

there exists a unique function $u \in L^2(\Omega)$, unique up to an addictive function of $\mathcal{H}_{L^2_0(\Omega)}$, solution to the problem

$$\Delta u = 0 \textit{ in } \Omega \quad \textit{and} \quad u\boldsymbol{n} = g\boldsymbol{n} \quad \textit{on } \Gamma, \tag{12}$$

with the estimate

$$\|u\|_{L^2(\Omega)} \leq C\|g\boldsymbol{n}\|_{[\boldsymbol{H}^{1/2}_T(\Gamma)]'}.$$

Proof. Thanks to the Green formula (10), it is easy to verify that $u \in L^2(\Omega)$ is solution to Problem (12) is equivalent to the following variational formulation: Find $u \in L^2(\Omega)$ such that for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\int_\Omega u \Delta\varphi \, \mathrm{d}x = \langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}^{1/2}(\Gamma)}. \tag{13}$$

Indeed, let $u \in L^2(\Omega)$ be a solution to (12). Then, the Green formula (10) yields (13).

Conversely, let $u \in L^2(\Omega)$ be a solution to (13). Taking φ in $\mathcal{D}(\Omega)$, we obtain $\Delta u = 0$ in Ω and $u \in M(\Omega)$. Using this last relation and again the Green formula (10), we deduce that for all $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$,

$$\langle u\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}^{1/2}(\Gamma)} = \langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)}.$$

Let $\boldsymbol{\mu} \in \boldsymbol{H}_T^{1/2}(\Gamma)$. By Remark 4, we know that there exits $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ such that $\boldsymbol{\mu} = \nabla\varphi$ on Γ. Thus,

$$\langle u\boldsymbol{n}, \boldsymbol{\mu} \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} = \langle u\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} = \langle g\boldsymbol{n}, \boldsymbol{\mu} \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)}.$$

and $u\boldsymbol{n} = g\boldsymbol{n}$ on Γ.

Let's then solve Problem (13). We know that for all $F \in L^2(\Omega) \perp \mathcal{H}_{L_0^2(\Omega)}$, there exists a unique $\varphi \in H^2(\Omega) \cap H_0^1(\Omega)$ satisfying $-\Delta\varphi = F$ in Ω, with the estimate

$$\|v\|_{H^2(\Omega)} \le C \|F\|_{L^2(\Omega)}.$$

Using estimate (8) we get

$$\left| \langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)} \right| \le \|g\boldsymbol{n}\|_{[\boldsymbol{H}_T^{1/2}(\Gamma)]'} \|\nabla\varphi\|_{\boldsymbol{H}^{1/2}(\Gamma)} \le C \, \|g\boldsymbol{n}\|_{[\boldsymbol{H}_T^{1/2}(\Gamma)]'} \|F\|_{L^2(\Omega)}.$$

In other words, we can say that the linear mapping

$$T : F \longmapsto \langle g\boldsymbol{n}, \nabla\varphi \rangle_{[\boldsymbol{H}_T^{1/2}(\Gamma)]' \times \boldsymbol{H}_T^{1/2}(\Gamma)}$$

is continuous on $L^2(\Omega) \perp \mathcal{H}_{L_0^2(\Omega)}$, and according to the Riesz representation theorem, there exists a function $u \in L^2(\Omega)$, unique up to an addictive element of $\mathcal{H}_{L_0^2(\Omega)}$, such that

$$\forall\, F \in L^2(\Omega), \; T(F) = \int_\Omega u F,$$

i.e u is solution of Problem (13). □

4. An Hessian representation for L^p-symmetric matrix fields

In this section, we will present a new characterization of the L^p-symmetric by using the Hessian matrix. For that, we need some results, the first one is the following vector potential theorem which have been presented by Duràn and Muschietti in [8]:

Lemma 8. *Let $1 < p < \infty$, $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$ with $\operatorname{div} \boldsymbol{v} = 0$ in Ω and satisfying the compatibility condition*

$$\langle \boldsymbol{v} \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 \quad \textit{for} \quad j = 0, \cdots, J.$$

Then there exists a function $\psi \in W^{1,p}(\Omega)$ such that $\mathbf{curl}\, \psi = \boldsymbol{v}$ in Ω.

The previous lemma is the key to generalize the Airy's function theorem in $\mathbb{L}^p(\Omega)$. In fact it suffices to follow the same steps of proof of [11, Theorem 2] to obtain the following result:

Lemma 9. *Given $\boldsymbol{S} = (s_{ij})_{i,j=1,2} \in \mathbb{L}_s^p(\Omega)$, then $\boldsymbol{S}$ fulfills the following statements:*

$$\mathbf{div}\, \boldsymbol{S} = \mathbf{0} \;\; \textit{in} \;\; \Omega, \tag{14}$$

$$\langle \boldsymbol{S}_i \cdot \boldsymbol{n}, 1 \rangle_{\Gamma_j} = 0 \quad \textit{for} \quad i = 1, 2 \quad \textit{and} \quad j = 0, \cdots, J, \tag{15}$$

$$\langle \boldsymbol{S}_1 \cdot \boldsymbol{n}, x_2 \rangle_{\Gamma_j} = \langle \boldsymbol{S}_2 \cdot \boldsymbol{n}, x_1 \rangle_{\Gamma_j} \quad \textit{for} \quad j = 0, \cdots, J, \tag{16}$$

if and only if there exists an Airy's function $w \in W^{2,p}(\Omega)$ *such that*

$$s_{11} = \frac{\partial^2 w}{\partial x_2^2}, \quad s_{12} = -\frac{\partial^2 w}{\partial x_1 \partial x_2} \quad \textit{and} \quad s_{22} = \frac{\partial^2 w}{\partial x_1^2}. \tag{17}$$

We are now in position to give a characterization of L^p-symmetric matrix field as a Hessian of a scalar field belonging to $W^{2,p}(\Omega)$.

Theorem 10. *Given* $\boldsymbol{S} \in \mathbb{L}_s^p(\Omega)$, *then* $\boldsymbol{S}$ *fulfills the following statements:*

$$\operatorname{curl} \boldsymbol{S}_i = 0 \quad in \quad \Omega, \quad for \quad i = 1, 2 \tag{18}$$

$$\left\langle \boldsymbol{S}\boldsymbol{t}, \boldsymbol{e}^i \right\rangle_{\Gamma_j} = 0 \qquad for \quad i = 1, 2 \quad and \quad j = 0, \cdots, J, \tag{19}$$

$$\langle \boldsymbol{S}\boldsymbol{t}, \boldsymbol{x} \rangle_{\Gamma_j} = 0 \qquad for \quad j = 0, \cdots, J, \tag{20}$$

if and only if there exists $w \in W^{2,p}(\Omega)$ *such that*

$$\boldsymbol{S} = \mathbf{Hess}\, w \quad in \quad \Omega. \tag{21}$$

Proof.

(i) First, let $\boldsymbol{S} = \mathbf{Hess}\, w$ with $w \in W^{2,p}(\Omega)$. It is clear that $\boldsymbol{S}$ belongs to $\mathbb{L}_s^p(\Omega)$ and satisfies (14). It remains to show that $\boldsymbol{S}$ satisfies the compatibility conditions (19) and (20). Lemma 9 implies that the following compatibility conditions hold true

$$\left\langle \boldsymbol{S}^\star \boldsymbol{n}, e^i \right\rangle_{\Gamma_j} = 0 \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad j = 0, \cdots, J, \tag{22}$$

$$\left\langle \boldsymbol{S}_1^\star \cdot \boldsymbol{n}, x_2 \right\rangle_{\Gamma_j} = \left\langle \boldsymbol{S}_2^\star \cdot \boldsymbol{n}, x_1 \right\rangle_{\Gamma_j} \quad \text{for} \quad j = 0, \cdots, J. \tag{23}$$

Let us observe the following equalities

$$\boldsymbol{S}_1^\star \cdot \boldsymbol{n} = \boldsymbol{S}_2 \cdot \boldsymbol{t} \quad \text{and} \quad \boldsymbol{S}_2^\star \cdot \boldsymbol{n} = -\boldsymbol{S}_1 \cdot \boldsymbol{t}.$$

So, we have the relations (19) and (20).

(ii) Conversely, let $\boldsymbol{S} \in \mathbb{L}_s^p(\Omega)$ satisfies the compatibility conditions (14)-(20). Then, the matrix $\boldsymbol{S}^\star \in \mathbb{L}_s^p(\Omega$ satisfies (22) and (23). Moreover, as $\mathbf{curl}\, \boldsymbol{S} = \mathbf{0}$ in Ω, then $\mathbf{div}\, \boldsymbol{S}^\star = \mathbf{0}$ in Ω. Due to Lemma 9, there exists $w \in W^{2,p}(\Omega)$ such that

$$\boldsymbol{S}^\star = \begin{pmatrix} \dfrac{\partial^2 w}{\partial x_2^2} & -\dfrac{\partial^2 w}{\partial x_1 \partial x_2} \\[2ex] -\dfrac{\partial^2 w}{\partial x_2 \partial x_1} & \dfrac{\partial^2 w}{\partial x_1^2} \end{pmatrix}.$$

Consequently,

$$\boldsymbol{S} = \mathbf{Hess}\, w \quad \text{in} \quad \Omega.$$

□

5. The range of the traces of $W^{3,p}(\Omega)$

Geymonat [10] proved that if Ω is a Lipschitz domain of $\mathbb{R}^2$ and $(g_0, g_1, g_2) \in W^{1,p}(\Gamma) \times L^p(\Gamma) \times L^p(\Gamma)$ belongs to the range of the operator $(\gamma_0, \gamma_1, \gamma_2)$, then it must satisfy the following conditions

$$\boldsymbol{q} := \frac{\partial g_0}{\partial \boldsymbol{t}} \boldsymbol{t} + g_1 \boldsymbol{n} \in \boldsymbol{W}^{1,p}(\Gamma), \tag{24}$$

and

$$\boldsymbol{H} := \left[\left(\nabla \boldsymbol{q}\, \boldsymbol{t}\right) \cdot \boldsymbol{t}\right] \boldsymbol{t} \otimes \boldsymbol{t} + \left[\left(\nabla \boldsymbol{q}\, \boldsymbol{t}\right) \cdot \boldsymbol{n}\right] (\boldsymbol{t} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{t}) + g_2\, \boldsymbol{n} \otimes \boldsymbol{n} \in \mathbb{W}^{1-\frac{1}{p},p}(\Gamma). \tag{25}$$

In this section, we will show that the necessary conditions (24) and (25) are sufficient. First, we will show the following results.

Lemma 11. *The operator*

$$\mathbf{div}: \mathbb{W}^{1,p}_{0,s} \to \boldsymbol{L}^p_0(\Omega), \tag{26}$$

is onto. Consequently, for each vector field $\boldsymbol{v} \in \boldsymbol{L}^p_0(\Omega)$, there exists a symmetric matrix field $\boldsymbol{S}$ in $\mathbb{W}^{1,p}_{0,s}(\Omega)$ such that

$$\mathbf{div}\,\boldsymbol{S} = \boldsymbol{v} \quad \textit{in} \quad \Omega,$$

and there exists a constant C depending only on p and Ω such that

$$\|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{v}\|_{\boldsymbol{L}^p(\Omega)}.$$

Proof. The proof is based on [6, Theorem 3] and it is composed on three steps.

Step 1. We show a vector version of J. L. Lions lemma. Here, we follow the same steps of proof of [1, Theorem 3.1]. Let $\boldsymbol{v} \in \mathcal{D}'(\Omega)$ be such that $\nabla_s \boldsymbol{v} \in \mathbb{W}^{-1,p}_s(\Omega)$. The identity

$$\partial_j(\partial_k v_i) = \partial_j(\nabla_s v)_{ik} + \partial_k(\nabla_s v)_{ij} - \partial_i(\nabla_s v)_{jk}$$

implies that for any $k, i = 1,2$, the distribution $\partial_k v_i$ has a gradient in $\boldsymbol{W}^{-2,p}(\Omega)$. Then [2, Proposition 2.1] implies that $\partial_k v_i$ is in $W^{-1,p}(\Omega)$. In other words, ∇v_i belongs to $\boldsymbol{W}^{-1,p}(\Omega)$ for each $i = 1,2$. Again [2, Proposition 2.1] implies that $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$.

Step 2. We show an extension of Donati's theorem. Let p' be the conjugate of p and $\boldsymbol{S} \in \mathbb{W}^{-1,p'}_s(\Omega)$ be such that

$$_{\mathbb{W}^{-1,p'}(\Omega)}\langle \boldsymbol{S}, \boldsymbol{e}\rangle_{\mathbb{W}^{1,p}_0(\Omega)} = 0 \quad \text{for all } \boldsymbol{e} \in \mathbb{V}^{1,p}_{0,s}(\Omega). \tag{27}$$

Moreau's theorem [14] implies that there exists $\boldsymbol{v} \in \mathcal{D}'(\Omega)$ such that $\nabla_s \boldsymbol{v} = \boldsymbol{S}$ in Ω. By Step 1, we get $\boldsymbol{v} \in \boldsymbol{L}^p(\Omega)$.

Step 3. We show that the operator (26) **is onto.** As consequence of Step 2, we deduce that the following operator

$$\nabla_s : \boldsymbol{L}^{p'}(\Omega)/\boldsymbol{R}(\Omega) \to \left[\mathbb{V}^{1,p}_s(\Omega)\right]^\circ. \tag{28}$$

is an isomorphism. Above the polar set is defined as follows:

$$\left[\mathbb{V}^{1,p}_s(\Omega)\right]^\circ = \left\{\mathbb{S} \in \mathbb{W}^{-1,p'}_s(\Omega) \quad \text{satisfying (27)}\right\}.$$

So, the dual operator

$$\mathbf{div}: \mathbb{W}^{1,p}_{0,s}(\Omega)/\mathbb{V}^{1,p}_s(\Omega) \to \boldsymbol{L}^p_0(\Omega), \tag{29}$$

is an isomorphism.

□

Lemma 12. *Let $\boldsymbol{A} \in \mathbb{W}^{1-\frac{1}{p},p}_s(\Gamma)$ satisfies the compatibility conditions* (19) *and* (20) *of Theorem 10. Then, there exists $\boldsymbol{S} \in \mathbb{W}^{1,p}_s(\Omega)$ such that*

$$\mathbf{curl}\,\boldsymbol{S} = \mathbf{0} \quad \textit{in} \quad \Omega \quad \textit{and} \quad \boldsymbol{S} = \boldsymbol{A} \quad \textit{on}\ \Gamma. \tag{30}$$

Moreover, there exists a constant C depending only on p and Ω such that

$$\|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}. \tag{31}$$

Proof. Let $\boldsymbol{A}$ be as in the statement of Lemma 12 and $\boldsymbol{M} \in \mathbb{W}^{1,p}_s(\Omega)$ be such that $\boldsymbol{M}_{|\Gamma} = \boldsymbol{A}$ on Γ and satisfies the estimate

$$\|\boldsymbol{M}\|_{\boldsymbol{W}^{1,p}(\Omega)} \leq C\|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}. \tag{32}$$

Let us observe that

$$\operatorname{div} \boldsymbol{M}^\star = \begin{pmatrix} \dfrac{\partial m_{22}}{\partial x_1} - \dfrac{\partial m_{21}}{\partial x_2} \\ -\dfrac{\partial m_{12}}{\partial x_1} + \dfrac{\partial m_{11}}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \operatorname{curl} \boldsymbol{M}_2 \\ -\operatorname{curl} \boldsymbol{M}_1 \end{pmatrix}.$$

Now, setting $\boldsymbol{v} = \operatorname{\mathbf{div}} \boldsymbol{M}^\star$. We search $\boldsymbol{R} \in \mathbb{W}^{1,p}_{0,s}(\Omega)$ such that $\operatorname{\mathbf{div}} \boldsymbol{R} = \boldsymbol{v}$ in Ω. By using (19), we get

$$\int_\Omega \boldsymbol{v} \cdot \boldsymbol{e}^1 \, dx = \int_\Omega \left(\frac{\partial m_{22}}{\partial x_1} - \frac{\partial m_{21}}{\partial x_2} \right) dx = \int_\Gamma \boldsymbol{M}_2 \cdot \boldsymbol{t} \, d\sigma = 0.$$

By the same, we get

$$\int_\Omega \boldsymbol{v} \cdot \boldsymbol{e}^2 \, dx = -\int_\Gamma \boldsymbol{M}_1 \cdot \boldsymbol{t} \, d\sigma = 0.$$

And by using (20), we get

$$\int_\Omega \boldsymbol{v} \cdot \boldsymbol{x}^\perp \, dx = -\int_\Omega \boldsymbol{M}^\star : \nabla \boldsymbol{x}^\perp \, dx + \int_\Gamma \left(\boldsymbol{M}^\star \boldsymbol{n}\right) \cdot \boldsymbol{x}^\perp \, d\sigma = -\int_\Gamma (\boldsymbol{M} \boldsymbol{t}) \cdot \boldsymbol{x} \, d\sigma = 0.$$

The second integral above is equal to zero since $\boldsymbol{M}^\star$ is symmetric and also the third on the boundary by using (20). Then, Lemma 11 implies that there exists $\boldsymbol{R} \in W^{1,p}_{0,s}(\Omega)$ such that $\operatorname{\mathbf{div}} \boldsymbol{R} = \operatorname{\mathbf{div}} \boldsymbol{M}^\star$ and satisfies the estimate

$$\|\boldsymbol{R}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}. \tag{33}$$

The symmetric matrix $\boldsymbol{S} = \boldsymbol{M} - \boldsymbol{R}^\star \in \mathbb{W}^{1,p}_s(\Omega)$ satisfies

$$\boldsymbol{S}_{|\Gamma} = \boldsymbol{M}_{|\Gamma} = \boldsymbol{A} \quad \text{with} \quad \|\boldsymbol{S}\|_{\mathbb{W}^{1,p}(\Omega)} \leq C \|\boldsymbol{A}\|_{\mathbb{W}^{1-\frac{1}{p},p}(\Gamma)}.$$

Observe that $\operatorname{\mathbf{div}} \boldsymbol{S}^\star = \boldsymbol{0}$, then $\operatorname{\mathbf{curl}} \boldsymbol{S} = \boldsymbol{0}$. Moreover, (32) and (33) implies that the estimate (31) holds, which ends the proof. □

Lemma 13. *Let $g_0 \in \boldsymbol{W}^{1,p}(\Gamma)$, g_1, g_2 in $L^p(\Gamma)$ be such that the vector field $\boldsymbol{q} = \frac{\partial g_0}{\partial \boldsymbol{t}} \boldsymbol{t} + g_1 \boldsymbol{n}$ be in $W^{1,p}(\Gamma)$. Then, the matrix field $\boldsymbol{H}$ defined by*

$$\boldsymbol{H} = \left[\left(\nabla \boldsymbol{q}\, \boldsymbol{t}\right) \cdot \boldsymbol{t}\right] \boldsymbol{t} \otimes \boldsymbol{t} + \left[\left(\nabla \boldsymbol{q}\, \boldsymbol{t}\right) \cdot \boldsymbol{n}\right] (\boldsymbol{t} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{t}) + g_2\, \boldsymbol{n} \otimes \boldsymbol{n},$$

satisfies

$$\left\langle \boldsymbol{H}\,\boldsymbol{t}, \boldsymbol{e}^1 \right\rangle_{\Gamma_j} = \left\langle \boldsymbol{H}\,\boldsymbol{t}, \boldsymbol{e}^2 \right\rangle_{\Gamma_j} = \langle \boldsymbol{H}\,\boldsymbol{t}, \boldsymbol{x} \rangle_{\Gamma_j} = 0, \; j = 1, \cdots, J. \tag{34}$$

Proof. As $\boldsymbol{q} \in \boldsymbol{W}^{1,p}(\Gamma)$, there exists $w \in W^{2,p}(\Omega)$ such that $w_{|\Gamma} = g_0$, $\frac{\partial w}{\partial \boldsymbol{n}} = g_1$ and $\boldsymbol{q} = (\nabla w)_{|\Gamma}$ (see [11]). By definition of tangential derivatives, we get

$$\left(\nabla \boldsymbol{q}\right) \boldsymbol{t} = \partial_{\boldsymbol{t}} \boldsymbol{q} = \left(\nabla^2 w\right) \boldsymbol{t}.$$

A simple calculus gives

$$(\boldsymbol{n} \otimes \boldsymbol{n})\, \boldsymbol{t} = \boldsymbol{0}, \; (\boldsymbol{n} \otimes \boldsymbol{t})\, \boldsymbol{t} = \boldsymbol{n}, \; (\boldsymbol{t} \otimes \boldsymbol{n})\, \boldsymbol{t} = \boldsymbol{0}, \; (\boldsymbol{t} \otimes \boldsymbol{t})\, \boldsymbol{t} = \boldsymbol{t}. \tag{35}$$

Then, we get

$$\begin{aligned} \boldsymbol{H}\,\boldsymbol{t} &= \left[\left(\nabla q\, \boldsymbol{t}\right) \cdot \boldsymbol{t}\right] \left[(\boldsymbol{t} \otimes \boldsymbol{t})\, \boldsymbol{t}\right] + \left[\left(\nabla q\, \boldsymbol{t}\right) \cdot \boldsymbol{n}\right] \left[(\boldsymbol{t} \otimes \boldsymbol{n} + \boldsymbol{n} \otimes \boldsymbol{t})\, \boldsymbol{t}\right] + g_2 \left[(\boldsymbol{n} \otimes \boldsymbol{n})\, \boldsymbol{t}\right] \\ &= \left[\left(\nabla q\, \boldsymbol{t}\right) \cdot \boldsymbol{t}\right] \boldsymbol{t} + \left[\left(\nabla q\, \boldsymbol{t}\right) \cdot \boldsymbol{n}\right] \boldsymbol{n} \\ &= \left(\nabla q\right) \boldsymbol{t} = \nabla^2 w\, \boldsymbol{t}. \end{aligned}$$

Finally, Theorem 10 implies that

$$\left\langle \boldsymbol{H}\,\boldsymbol{t}, \boldsymbol{e}^i \right\rangle_{\Gamma_j} = \left\langle \nabla^2 w\, \boldsymbol{t}, \boldsymbol{e}^i \right\rangle_{\Gamma_j} = 0, \; i = 1,2, \; j = 1, \cdots, J,$$

$$\langle \boldsymbol{H}\,\boldsymbol{t}, \boldsymbol{x} \rangle_{\Gamma_j} = \left\langle \nabla^2 w\, \boldsymbol{t}, \boldsymbol{x} \right\rangle_{\Gamma_j} = 0, \; j = 1, \cdots, J.$$

□

We are now in position to characterize the range of the trace operator in $W^{3,p}(\Omega)$.

Theorem 14. *Let $g_0 \in W^{1,p}(\Gamma)$, $g_1, g_2 \in L^p(\Gamma)$ be given. Then, there exists $w \in W^{3,p}(\Omega)$ such that*

$$w = g_0, \quad \frac{\partial w}{\partial \boldsymbol{n}} = g_1 \quad \textit{and} \quad \frac{\partial^2 w}{\partial \boldsymbol{n}^2} = g_2 \quad \textit{on } \Gamma, \tag{36}$$

if and only if g_0, g_1 and g_2 satisfy the conditions (24) *and* (25).

Proof.

(i) First, let $w \in W^{3,p}(\Omega)$, $g_0 = w_{|\Gamma}$, $g_1 = \frac{\partial w}{\partial \boldsymbol{n}}$ and $g_2 = \frac{\partial^2 w}{\partial \boldsymbol{n}^2}$. By definition of tangential derivatives, then the vector field $\boldsymbol{q}$ and the matrix field $\boldsymbol{H}$ defined in Lemma 13 satisfy the conditions (24) and (25):

$$\boldsymbol{q} = (\nabla w)_{|\Gamma} \in \boldsymbol{W}^{1,p}(\Gamma) \quad \text{and} \quad \boldsymbol{H} = \left(\nabla^2 w\right)_{|\Gamma} \in \mathbb{W}_s^{1-\frac{1}{p},p}(\Gamma).$$

(ii) Conversely, Lemma 13 implies that $\boldsymbol{H}$ satisfies the compatibility conditions (19) and (20), then Lemma 12 implies that there exists $\boldsymbol{S} \in \mathbb{W}_s^{1,p}(\Omega)$ such that $\operatorname{curl} \boldsymbol{S} = \boldsymbol{0}$ in Ω and $\boldsymbol{S} = \boldsymbol{H}$ on Γ. As the matrix $\boldsymbol{S}$ satisfies the conditions (14)-(20), then there exists $w_0 \in W^{2,p}(\Omega)$ such that $\nabla^2 w_0 = \boldsymbol{S}$ in Ω. Consequently, $w_0 \in W^{3,p}(\Omega)$ and $(\nabla^2 w_0)_{|\Gamma} = \boldsymbol{H}$. A simple calculus gives

$$(\boldsymbol{n} \otimes \boldsymbol{n})\, \boldsymbol{n} = \boldsymbol{n}, \ (\boldsymbol{n} \otimes \boldsymbol{t})\, \boldsymbol{n} = \boldsymbol{0}, \ (\boldsymbol{t} \otimes \boldsymbol{n})\, \boldsymbol{n} = \boldsymbol{t}, \ (\boldsymbol{t} \otimes \boldsymbol{t})\, \boldsymbol{n} = \boldsymbol{0}.$$

Then,

$$(\boldsymbol{H}\boldsymbol{n}) \cdot \boldsymbol{n} = \left(\left(\nabla^2 w_0\right) \boldsymbol{n}\right) \cdot \boldsymbol{n} = \frac{\partial^2 w_0}{\partial \boldsymbol{n}^2} = g_2 \quad \text{on } \Gamma. \tag{37}$$

Also, using (35), we get

$$\left(\nabla^2 w_0\right) \boldsymbol{t} = \boldsymbol{H}\boldsymbol{t} = (\nabla \boldsymbol{q})\, \boldsymbol{t} \quad \text{on } \Gamma.$$

Hence, Proposition 1 implies that $\boldsymbol{q} = (\nabla w_0)_{|\Gamma} + \boldsymbol{c_0}$ where $\boldsymbol{c_0} \in \mathbb{R}^2$. Let us observe that the following fonction $w_1 = w_0 + \boldsymbol{c_0} \cdot \boldsymbol{x}$ satisfies

$$\frac{\partial^2 w_1}{\partial \boldsymbol{n}^2} = \frac{\partial^2 w_0}{\partial \boldsymbol{n}^2} = g_2, \quad \boldsymbol{q} = \nabla w_1 \quad \text{and} \quad \frac{\partial w_1}{\partial \boldsymbol{n}} = g_1.$$

Moreover,

$$(\nabla w_1) \cdot \boldsymbol{t} = \boldsymbol{q} \cdot \boldsymbol{t} = \nabla g_0 \cdot \boldsymbol{t}.$$

Again, Proposition 1 implies that $g_0 = (w_1)_{|\Gamma} + c_1$ where $c_1 \in \mathbb{R}$. Finally, the function $w = w_1 + c_1$ answers to our question since

$$w = g_0, \quad \frac{\partial w}{\partial \boldsymbol{n}} = \frac{\partial w_1}{\partial \boldsymbol{n}} = g_1 \quad \text{and} \quad \frac{\partial^2 w}{\partial \boldsymbol{n}^2} = \frac{\partial^2 w_0}{\partial \boldsymbol{n}^2} = g_2$$

which ends the proof of Theorem 14.

□

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