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Hölder estimate for the 3 point-vortex problem with alpha-models

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Abstract. In this article we study quasi-geostrophic point-vortex systems in a general setting called *alpha point-vortex*. We study a particular case of vortex collapses called *mono-scale collapses* and this study gives the Hölder regularity for the 3-vortex problem. This result implies in particular that the trajectories of the vortices are convergent even in the case of a collapse.

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Introduction

We study in this article the dynamical system called the point-vortex system in the general setting of α-models. This consists in $N \geq 1$ points $x_1, \ldots, x_N$ in the plane $\mathbb{R}^2$ that evolve in time according to the following evolution equation ($\alpha \geq 0$):

$$\frac{d}{dt} x_i(t) := \sum_{\substack{j=1 \\ j \neq i}}^{N} a_j \frac{(x_i - x_j)^{\perp}}{\left|x_i - x_j\right|^{\alpha+1}}. \tag{1}$$

In the equation above, the notation $\perp$ stands for the counter clockwise rotation of angle $\pi/2$. The quantity $a_i \neq 0$ is called the *intensity* (or the *circulation*) of the i^{th} vortex.

This dynamical system arises in several models for inviscid planar fluid mechanics. It is a standard model introduced by Helmholtz [12] and still have numerous open problems. The most studied case is $\alpha = 1$, which corresponds to the point-vortex system associated to the bi-dimensional Euler equations. In this case, the point-vortex system gives account of the natural situation where the vorticity ω is sharply concentrated around a finite number of points

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$x_1, \dots, x_N$. This models the intuitive notion of "center of whirlpools" at the surface of a fluid or inside a planar flow. In such a situation, the vorticity of the fluid writes as a pondered sum of Dirac masses and the velocity of the fluid is computed using the Biot–Savart law. The case $\alpha = 2$ corresponds to the point-vortex system for the surface quasi-geostrophic equations that are widely used in geophysical sciences for atmospheric dynamics and weather forecast. In this situation, the point-vortex system is a simplified model for the dynamics of atmospheric vortices. Other values of α also have a physical meaning. For a more extensive presentation of the point-vortex systems, we refer to [8, 9, 15, 16]. See also references therein.

This dynamical system (1) is well-defined as a consequence of the Cauchy–Lipschitz theorem (also called Picard–Lindelöf theorem) provided that the vortices does not show a collapse behavior. In other words, if $T > 0$ is the maximal time of existence for the system, then

$$\liminf_{t \to T^-} \min_{i \neq j} \left| x_i(t) - x_j(t) \right| = 0. \tag{2}$$

The existence of collapses for the 3 vortex problem for the Euler equations ($\alpha = 1$) was obtained independently by [1, 10, 17]. Other works for a better description of 3 vortex collapses when $\alpha = 1$ can be found at [2, 11, 13]. Concerning the surface quasi-geostrophic case ($\alpha = 2$), the collapses for the 3 vortex system have been studied by [4, 18]. The collapses of 3 vortices for other values of $\alpha \geq 0$ has been investigated only recently [6, 19]. It is also worth mentioning a series of results concerning the *improbability of collapses*. More precisely, these results states that the point-vortex system is globally well-defined for almost every initial datum under standard non-degeneracy hypothesis on the intensities [5, 7, 9, 14, 15].

The aim of the present work is to study a particular type of collapse called the *mono-scale collapse* which consists in the hypothesis that the distances between all the vortices involved in a collapse have the same scale, meaning that they are comparable. This hypothesis may seem quite restrictive at first sight, but we are able to prove that this is always verified for the 3-vortex problem, in the general setting $\alpha \geq 1$. In particular, we are able to obtain a Hölder estimate on the trajectories of the vortices for the 3 vortex problem in presence of a collapse. This implies in particular that the position of the 3 vortices admits a limit as $t \to T$, where T is the time of collapse. In the preprint [6] written a few time later, we recover this result with different arguments.

1. Presentation of the problem and main result

In the study of the point-vortex system associated to the Euler equation made by [14, § 4], the authors focus on systems of vortices which circulations satisfy the following hypothesis:

$$\forall \ A \subseteq \{1 \cdots N\} \ s.t. \ A \neq \emptyset, \qquad \sum_{i \in A} a_i \neq 0. \tag{3}$$

In this article, the hypothesis above will be referred as the "*non-neutral clusters hypothesis*". This hypothesis exactly means that any subset of vortices is non-neutral: the sum of the intensities is nonzero. This term of "*neutral*" is coming from the study of coulomb interactions where the same hypothesis exist. In [9], under the non-neutral clusters hypothesis, a uniform bound was obtained. This bound writes as follows, provided that the dynamics is well-defined on $[0, T)$:

$$\forall \ t \in [0, T), \quad \forall \ i = 1 \dots N, \qquad |x_i(t) - x_i(0)| \leq C. \tag{4}$$

Moreover it is proved that the constant C is independent on the initial datum and on $\alpha \geq 0$. Nevertheless, such a bound (4) fails to describe the collapse in itself. In particular, it does not says whether or not the positions of the vortices are convergent as t goes to the time of collapse. What makes this problem non trivial comes from the fact that the velocity of the vorticies (1) blows up in presence of a collapse (2).

The main idea guiding this work consists in the following simple observation. Consider in dimension 1 a dynamical system $x : t \mapsto x(t) \in \mathbb{R}$, with $x(t) > 0$, governed by the following evolution equation.

$$\forall\ t, \qquad \frac{d}{dt}x(t) = -\frac{1}{x(t)^{\alpha}}. \tag{5}$$

This simple evolution system can be seen as a toy problem for the more general and complicated α-point vortex model (1). It is a direct computation to check that the solution to this toy-system is

$$x(t) = \Big[x(0)^{\alpha+1} - (\alpha+1)\,t\Big]^{\frac{1}{\alpha+1}}. \tag{6}$$

Such a solution "collapses" with the value 0 and shows a $1/(\alpha+1)$ Hölder regularity. This toy problem is expected to be a model of collapses for the α vortices under the non-degeneracy hypothesis (3).

For this article, we restrict the analysis to the particular case of the *mono-scale clusters*. A system of vortices from a given initial datum $X \in \mathbb{R}^{2N}$ supposed well-defined on the interval of time $[0,T)$ is said to satisfy the *mono-scale clusters* condition if their exists two positive constants C_1, C_2 and a partition $(P_1 \cdots P_K)$ of the set $\{1 \cdots N\}$ such that the two following conditions are satisfied:

$$\forall\ k \in \{1\cdots K\}, \quad \forall\ t \in [0;T), \qquad \max_{\substack{i,j \in P_k \\ i \neq j}} |x_i(t) - x_j(t)| \leq C_1 \min_{\substack{i,j \in P_k \\ i \neq j}} \big|x_i(t) - x_j(t)\big|, \tag{7}$$

$$\forall\ k \neq l \in \{1\cdots K\}, \quad \forall\ t \in [0;T), \qquad \min_{i \in P_k} \min_{j \in P_l} \big|x_i(t) - x_j(t)\big| \geq C_2. \tag{8}$$

The subsets $P_k \subseteq \{1 \cdots N\}$ are called the *clusters of vortices* since they give the indices of the sets of vortices that stay "together" during the dynamics. Condition (7) means that the vortices belonging to a same cluster have all their mutual distance that stay of same order, or comparable, even in case of collapse. This property justifies the name "*mono-scale clusters*".

Proposition 1 (Mono-scale Hölder estimate). *Let $T > 0$ be a final time and let $\alpha > 0$. Let $X \in \mathbb{R}^{2N}$ an initial datum. Consider the point-vortex dynamic (1). Assume that the dynamics is well-defined on $[0,T)$ and that the mono-scale hypothesis (7)-(8) holds. Assume also that these mono-scale clusters are non-neutral:*

$$\forall\ k = 1, \ldots, K, \qquad \sum_{i \in P_k} a_k \neq 0.$$

Then there exists a constant $C > 0$ such that for all $i = 1 \ldots N$,

$$\forall\ t_1, t_2 \in [0,T), \qquad |x_i(t_2) - x_i(t_1)| \leq C\,|t_2 - t_1|^{\frac{1}{\alpha+1}}.$$

Moreover, the constant C is independent of the initial condition.

In particular, the position of the vortices converges as $t \to T^-$. In other words, if the system shows a "*pathological behavior*" (meaning : non-convergent) in the presence of a collapse (2) then it has necessarily a multi-scale structure.

One consequence of this proposition is the main result of this article:

Theorem 2 (Hölder estimate for 3 point-vortex without external fields). *Let $T > 0$ be a final time and let $\alpha > 1$. Let $X \in \mathbb{R}^{2N}$ an initial datum. Consider the point-vortex dynamic (1) in the case $N = 3$ and assume that this dynamics is well-defined on the interval of time $[0,T)$. Assume that the system is non-neutral:*

$$a_1 + a_2 + a_3 \neq 0.$$

Then, there exists a constant $C > 0$ such that for all $i \in \{1,2,3\}$:

$$\forall\ t_1, t_2 \in [0,T), \qquad |x_i(t_2) - x_i(t_1)| \leq C\,|t_2 - t_1|^{\frac{1}{\alpha+1}}.$$

The proof of this theorem is divided into two parts. We prove on the one hand Proposition 1 for the mono-scale collapses. On the other hand, we establish that the 3 vortex problem always shows mono-scale collapses if $\alpha > 1$.

This theorem does not solve the problem in the case $0 < \alpha \leq 1$. Nevertheless, in the particular case where $\alpha = 1$, corresponding to the Euler point-vortex, the 3 vortex problem is already widely studied [1, 2, 11, 13] so that this result is already standard in this case. In [11], they study the case where the 3 vortices are submitted to a lipschitz external field and obtain a similar conclusion. We do not know if their arguments can be extend to the case $\alpha > 1$.

2. Monoscale structure of the 3-vortex collapses

2.1. *Hamiltonian of the point-vortex*

One main element to point-out is the Hamiltonian nature of the point-vortex equations (1). This allows us to obtain several conservation laws which eventually provide strong tools to study the equation.

The Hamiltonian of the point-vortex system, when $\alpha > 1$ is given by

$$\begin{aligned} H : \mathbb{R}^{2N} &\longrightarrow \mathbb{R}, \\ X = (x_1 \cdots x_N) &\longmapsto \sum_{i \neq j} \frac{a_i\, a_j}{\left|x_j - x_i\right|^{\alpha - 1}}. \end{aligned} \tag{9}$$

The system (1) is said to be Hamiltonian because it can be rewritten

$$a_i \frac{d}{dt} x_i(t) = \nabla^{\perp}_{x_i} H\big(X\big)$$

The first consequence of this Hamiltonian reformulation is the preservation of the Hamiltonian H along the flow of (1) [3, § 3]. This Hamiltonian formulation allows us to extract the quantities conserved by the flow, one of them being the vorticity vector:

$$M(X) := \sum_{i=1}^{N} a_i\, x_i.$$

When the system is non-neutral, meaning that $\sum_i a_i \neq 0$, this lemma implies the preservation of the *center of vorticity* of the system defined by

$$B(X) := \left(\sum_{i=1}^{N} a_i\right)^{-1} \sum_{i=1}^{N} a_i\, x_i. \tag{10}$$

2.2. *Shortest and second shortest distances*

In this section we study the relative value between the shortest distance and the second shortest distance between vortices. The shortest distance is defined by

$$\zeta_1(t) := \min_{i < j = 1 \cdots N} \left|x_i(t) - x_j(t)\right|. \tag{11}$$

The second shortest distance is defined by

$$\zeta_2(t) := \min_{\substack{i < j = 1 \cdots N \\ (i,j) \neq (i_1, j_1)}} \left|x_i(t) - x_j(t)\right|. \tag{12}$$

where (i_1, j_1) with $i_1 < j_1$ is a pair of indices that realizes the shortest distance:

$$\left|x_{i_1}(t) - x_{j_1}(t)\right| = \zeta_1(t). \tag{13}$$

One of the consequence of the Hamiltonian structure of the equations is that a pair of vortex cannot collapse without the presence of a third one. Otherwise, the Hamiltonian of the system (9) would blow up, which in contradiction with the fact that it is a constant of the movement. More precisely, we are able to prove that these two distances must be of same order.

Lemma 3 (Comparing small distances). *Let $T>0$ be a final time and let $\alpha>1$. Let $X\in\mathbb{R}^{2N}$ an initial datum such that the dynamics for the point-vortex dynamic* (1) *is well-defined on* $[0,T)$.

Then there exists a constant C such that

$$\forall\ t\in[0,T),\qquad \zeta_2(t)\le C\zeta_1(t),$$

with ζ_1 *and* ζ_2 *defined at* (11) *and* (12).

Proof. Suppose for the sake of contradiction that there exists a sequence of time (t_n) such that $\zeta_1(t_n)/\zeta_2(t_n)\to 0$ as $n\to+\infty$. Recall the definition of the indices (i_1,j_1) at (13). These indices depends on n the minimal distance is not always realized by the same pair of vortices as n grows. Nevertheless and since the number of vortices is finite, it is always possible up to an omitted extraction to assume that (i_1,j_1) does not depend on $n\in\mathbb{N}$. Indeed at least one pair of vortices realizes the minimal distance infinitely many times as $n\to+\infty$. After such an extraction, the contradiction hypothesis implies

$$\forall\ i<j\in\{1\cdots N\},\quad (i,j)\neq\big(i_1,j_1\big),\quad \frac{\big|x_{i_1}(t_n)-x_{j_1}(t_n)\big|}{\big|x_i(t_n)-x_j(t_n)\big|}\longrightarrow 0,\quad \text{as } n\to+\infty. \tag{14}$$

In particular (14) gives

$$\big|x_{i_1}(t_n)-x_{j_1}(t_n)\big|\longrightarrow 0\quad \text{as } n\to+\infty.$$

Using the definition of the Hamiltonian (9),

$$a_{i_1}a_{j_1}=\frac{H(X)}{2\big|x_{i_1}(t_n)-x_{j_1}(t_n)\big|^{1-\alpha}}-\sum_{\substack{i<j=1\\(i,j)\neq(i_1,j_1)}}^{N}a_i\,a_j\,\frac{\big|x_i(t)-x_j(t)\big|^{1-\alpha}}{\big|x_{i_1}(t_n)-x_{j_1}(t_n)\big|^{1-\alpha}}. \tag{15}$$

With the preservation of the Hamiltonian, the limit $|x_{i_1}(t_n)-x_{j_1}(t_n)|\to 0$ implies

$$\frac{H(X)}{2\big|x_{i_1}(t_n)-x_{j_1}(t_n)\big|^{1-\alpha}}\longrightarrow 0,\quad \text{as } n\to+\infty.$$

Using now (14) gives, as $n\to+\infty$,

$$\forall\ i<j\in\{1\cdots N\},\quad (i,j)\neq(i_1,j_1),\quad \frac{\big|x_{i_1}(t_n)-x_{j_1}(t_n)\big|^{1-\alpha}}{\big|x_i(t_n)-x_j(t_n)\big|^{1-\alpha}}\longrightarrow+\infty.$$

Therefore, passing to the limit $n\to+\infty$ in (15)

$$a_{i_1}a_{j_1}=0.$$

This is in contradiction with $a_i\neq 0$ for all i. □

Corollary 4. *The point-vortex system* (1) *with $\alpha>1$ when the number of vortices is $N=3$ satisfy the mono-scale property* (7)(8) *with $K=1$.*

Proof. Since we have only 3 vortices, the triangular inequality writes:

$$\max_{i,j\in\{1,2,3\}}\big|x_i(t)-x_j(t)\big|\ \le\ \zeta_1(t)+\zeta_2(t).$$

Combining this with Lemma 3 eventually gives (7) with $K=1$. Concerning (8), this condition is void when $K=1$. □

3. Proof of Proposition 1

For the rest of the article, the notation C stands for some constant (independent of the time) which value has no real importance and that may change from one line to another.

3.1. *Comparison with the toy model*

The first step of the proof consists in giving a rigorous meaning to the idea adumbrated at (5) and (6) which consists in obtaining the Hölder regularity by comparing with the dynamics of the toy model. For that purpose we study this toy model in the general setting given by the following lemma.

Lemma 5 (Hölder estimate for the generalized toy problem). *Consider a general dynamical system $X : t \in [0,T) \mapsto X(t) \in \mathbb{R}^d$ supposed weakly differentiable. Assume that this system satisfy*

$$\forall\ t \in [0,T), \qquad \left|\frac{d}{dt}X(t)\right| \le \frac{1}{|X(t)-X_0|^\alpha} + 1, \tag{16}$$

for some $X_0 \in \mathbb{R}^d$ (with $\alpha \ge 0$). Then, there exists a constant C such that

$$\forall\ t_1, t_2 \in [0,T), \quad |X(t_2)-X(t_1)| \le C\,|t_2-t_1|^{\frac{1}{\alpha+1}}.$$

Proof. We first prove this lemma in the case $d=1$. Define $x(t) := X(t)-X_0 \in \mathbb{R}$. Equation (16) gives

$$\forall\ t \in [0,T), \qquad \left|\frac{d}{dt}x(t)\right| \le \frac{1}{|x(t)|^\alpha} + 1. \tag{17}$$

We now observe that the derivative of x is bounded by 2 whenever $|x| \ge 1$. Then two cases occurs. Either we have $|x(t)| \ge 1$ for all times (and the dynamics is uniformly Lipschitz so that there is nothing to prove) or there exists a time t_0 such that $|x(t)| \le 1$. In the second situation, the bound on the derivative of $x(t)$ when $|x| \ge 1$ implies $|x(t)| \le 1+2T$ for all $t \in [0,T)$. Therefore, and with such a bound at hand, it is direct from (17) to check that

$$\forall\ t \in [0,T), \qquad \left|\frac{d}{dt}x(t)|x(t)|^\alpha\right| \le C.$$

Thus, if one integrates this inequality in time,

$$\forall\ t_1, t_2 \in [0,T), \qquad \Big|\,x(t_2)|x(t_2)|^\alpha - x(t_1)|x(t_1)|^\alpha\,\Big| \le C\,|t_2-t_1|. \tag{18}$$

In the case where $x(t_2)$ and $x(t_1)$ have the same sign, the convexity inequality $|a^{\alpha+1}-b^{\alpha+1}| \ge |a-b|^{\alpha+1}$ (true for $a,b \ge 0$) gives

$$\Big|\,x(t_2)|x(t_2)|^\alpha - x(t_1)|x(t_1)|^\alpha\,\Big| \ge \big|x(t_2)-x(t_1)\big|^{\alpha+1}.$$

In the other case, using the convexity inequality $(a+b)^{\alpha+1} \le 2^\alpha(a^{\alpha+1}+b^{\alpha+1})$,

$$\Big|\,x(t_2)|x(t_2)|^\alpha - x(t_1)|x(t_1)|^\alpha\,\Big| \ge \frac{1}{2^\alpha}\big|x(t_2)-x(t_1)\big|^{\alpha+1}.$$

In both cases, Equation (18) leads to

$$\forall\ t_1, t_2 \in [0,T), \quad \big|x(t_2)-x(t_1)\big| \le C\,|t_2-t_1|^{\frac{1}{\alpha+1}}.$$

This eventually conclude the proof in the case $d=1$.

We now consider $d \ge 2$. For $i = 1\dots N$, by simply observing that $|X_i(t)-X_{0,i}| \le |X(t)-X_0|$, we infer from Condition (16) that (17) holds for $x(t) := X_i(t)-X_{0,i}$. We have reduced the problem to the case $d=1$ already studied and this concludes the proof. □

3.2. *Study of the relative dynamics inside a cluster*

Now that we have a good understanding of the toy model in its generality, there remain to study the dynamics of the vortices in comparison with this toy model.

Lemma 6. *Consider the point-vortex dynamics* (1) *under the mono-scale assumption* (7)-(8). *Consider the partition* $(P_1 \cdots P_k)$ *given by the mono-scale assumption and let* $k \in \{1 \cdots K\}$. *Let* $i, j \in P_k$. *Then we have*

$$\forall \;\; t_1, t_2 \in [0, T), \quad \Big| y_{ij}(t_2) - y_{ij}(t_1) \Big| \;\le\; C \big| t_2 - t_1 \big|^{\frac{1}{\alpha+1}},$$

where $y_{ij}(t) := x_i(t) - x_j(t)$.

Proof. Consider the partition $(P_1 \cdots P_K)$ given by the mono-scale assumption (7)-(8) and let $k \in \{1 \cdots K\}$. We focus on the case where $\#P_k \ge 2$ and we take $i \ne j \in P_k$. We are interested in the dynamics of

$$y_{ij} : t \in [0, T) \longmapsto x_i(t) - x_j(t)$$

The equations of motion (1) give

$$\frac{d}{dt} y_{ij}(t) = \sum_{\substack{l=1 \\ l \ne i}}^{N} a_l \frac{(x_l(t) - x_i(t))^{\perp}}{|x_l(t) - x_i(t)|^{\alpha+1}} - \sum_{\substack{l=1 \\ l \ne j}}^{N} a_l \frac{\big(x_l(t) - x_j(t)\big)^{\perp}}{\big|x_l(t) - x_j(t)\big|^{\alpha+1}}.$$

Thus,

$$\left| \frac{d}{dt} y_{ij}(t) \right| \le \sum_{\substack{l=1 \\ l \ne i}}^{N} \frac{|a_l|}{\big|y_{il}(t)\big|^{\alpha}} + \sum_{\substack{l=1 \\ l \ne j}}^{N} \frac{|a_l|}{\big|y_{jl}(t)\big|^{\alpha}}.$$

Concerning the indices l that does not belong to the set P_k, it is possible to estimate the associated term in the sums above using the constant C_2 given by mono-scale assumption (8). This gives

$$\left| \frac{d}{dt} y_{ij}(t) \right| \le \sum_{\substack{l \in P_k \\ l \ne i}} \frac{|a_l|}{\big|y_{il}(t)\big|^{\alpha}} + \sum_{\substack{l \in P_k \\ l \ne j}} \frac{|a_l|}{\big|y_{jl}(t)\big|^{\alpha}} + C \le C \left(\frac{1}{\min_{l \ne m \in P_k} \big|y_{lm}(t)\big|^{\alpha}} + 1 \right).$$

For this indices l that belong to the same cluster P_k, they are estimated using the mono-scale assumption (7). We are led to

$$\left| \frac{d}{dt} y_{ij}(t) \right| \le C \left(\frac{1}{\max_{l \ne l \in P_k} \big|y_{lm}(t)\big|^{\alpha}} + 1 \right) \le C \left(\frac{1}{\big|y_{ij}(t)\big|^{\alpha}} + 1 \right).$$

Therefore $t \mapsto y_{ij}(t)$ satisfy the hypothesis of Lemma 5 which eventually gives the announced Hölder regularity. □

3.3. *Proof of Proposition 1 completed*

It is now possible to complete the proof of Proposition 1. The system made only of the vortices of P_k can be seen as an autonomous point-vortex system with an extra term, viewed as an external field, which consists in the interaction with the vortices that does not belong to P_k. Assumption (8) ensures that the interaction between the vortices of different clusters is uniformly Lipschitz. Invoking now the conservation of the center of vorticity (10), we obtain that $B_k := (\sum_{j \in P_k} a_j)^{-1} \sum_{j \in P_k} a_j x_j$, the center of vorticity of the the cluster P_k, is uniformly Lipschitz on the interval of time $[0, T)$. Remark that B_k is well-defined as a consequence of the non-neutral clusters hypothesis (3). Let $i \in P_k$, and let $t_1, t_2 \in [0, T)$, we have by triangular inequality

$$|x_i(t_2) - x_i(t_1)| \le |(x_i(t_2) - B_k(t_2)) - (x_i(t_1) - B_k(t_1))| + |B_k(t_2) - B_k(t_1)|.$$

The second term of the sum above is directly estimated using the fact that B_k is uniformly Lipschitz. To conclude, we need to estimate the first term. With the definition of B_k, we write

$$\big(x_i(t_2)-B_k(t_2)\big)-\big(x_i(t_1)-B_k(t_1)\big) = \frac{\sum_{j\in P_k} a_j\big(x_i(t_2)-x_j(t_2)\big)-a_j\big(x_i(t_1)-x_j(t_1)\big)}{\sum_{j\in P_k} a_j} = \frac{\sum_{j\in P_k} a_j\big(y_{ij}(t_2)-y_{ij}(t_1)\big)}{\sum_{j\in P_k} a_j}.$$

Thus,

$$\Big|\big(x_i(t_2)-B_k(t_2)\big)-\big(x_i(t_1)-B_k(t_1)\big)\Big| \le C\sum_{j\in P_k}\big|y_{ij}(t_2)-y_{ij}(t_1)\big|,$$

and we can conclude the proof of Proposition 1 by using Lemma 6 in this last estimate. □

3.4. *Proof of Theorem 2 completed*

Proof. We can deduce from Lemma 3 that if there is only 3 vortices, then the collapse is mono-scale in the sense given by (7)-(8). Therefore, the conclusion of Theorem 2 follows by direct application of Proposition 1. □

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