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Skew graded (A_∞) hypersurface singularities

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Abstract. For a skew version of a graded (A_∞) hypersurface singularity A, we study the stable category of graded maximal Cohen-Macaulay modules over A. As a consequence, we see that A has countably infinite Cohen–Macaulay representation type and is not a noncommutative graded isolated singularity.

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1. Introduction

Representation theory of (graded) maximal Cohen–Macaulay modules is a very active and fruitful area of research [12]. One of the fundamental subjects is to determine the Cohen–Macaulay representation type of (graded) rings [14, 20]. Let k be an algebraically closed field of characteristic different from 2, and let $R = \bigoplus_{i\in\mathbb{N}} R_i$ be an $\mathbb{N}$-graded commutative Gorenstein ring with $R_0 = k$. Then R is said to have *finite Cohen-Macaulay representation type* (respectively, *countable Cohen-Macaulay representation type*) if it has only finitely (respectively, countably) many indecomposable graded maximal Cohen–Macaulay modules up to isomorphism and shift of the grading. The following two results are well-known.

Theorem 1 ([7]). *Let $R = k[x_1,\dots,x_n]/(x_1^2+\cdots+x_n^2)$ be a graded (A_1) hypersurface singularity with* $\deg x_i = 1$. *Then R has finite Cohen–Macaulay representation type.*

Theorem 2 ([5, Theorem B], [2, Propositions 8 and 9]). *Let $R = k[x_1,\dots,x_n]/(x_1^2+\cdots+x_{n-1}^2)$ be a graded (A_∞) hypersurface singularity with $n \geq 2$ and* $\deg x_i = 1$. *Then R has countably infinite Cohen–Macaulay representation type.*

Graded (A_1) and (A_∞) hypersurface singularities play an essential role in the study of higher-dimensional standard graded Gorenstein rings of countable Cohen–Macaulay representation type; see, for example, [18, Section 5].

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From here, we turn our attention to rings that are not necessarily commutative. In [11], Higashitani and the author computed the stable category of graded maximal Cohen–Macaulay modules over a skew version of a graded (A_1) hypersurface singularity by combinatorial methods developed by Mori and the author [15].

Definition 3. *A* (± 1)-skew polynomial algebra *in n variables is defined to be an algebra*

$$k_\varepsilon[x_1, \dots, x_n] := k\langle x_1, \dots, x_n\rangle / \left(x_i x_j - \varepsilon_{ij} x_j x_i \,\middle|\, 1 \le i, j \le n\right),$$

where $\varepsilon = (\varepsilon_{ij}) \in M_n(k)$ is a symmetric matrix such that $\varepsilon_{ii} = 1$ for all $1 \le i \le n$ and $\varepsilon_{ij} = \varepsilon_{ji} \in \{1, -1\}$ for all $1 \le i < j \le n$.

It is easy to see that both $x_1^2 + \cdots + x_n^2$ and $x_1^2 + \cdots + x_{n-1}^2$ are regular central elements of a (± 1)-skew polynomial algebra $k_\varepsilon[x_1, \dots, x_n]$.

Definition 4.

(1) *A* skew graded (A_1) hypersurface singularity *is defined to be a graded algebra*

$$k_\varepsilon[x_1, \dots, x_n] \big/ \left(x_1^2 + \cdots + x_n^2\right),$$

where $\deg x_i = 1$ *for all* $1 \le i \le n$.

(2) *A* skew graded (A_∞) hypersurface singularity *is defined to be a graded algebra*

$$k_\varepsilon[x_1, \dots, x_n] \big/ \left(x_1^2 + \cdots + x_{n-1}^2\right),$$

where $n \ge 2$ *and* $\deg x_i = 1$ *for all* $1 \le i \le n$.

For a skew graded (A_1) or (A_∞) hypersurface singularity A, we write $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ for the stable category of graded maximal Cohen–Macaulay modules over A.

Definition 5. *For a (± 1)-skew polynomial algebra $k_\varepsilon[x_1, \dots, x_n]$, the* graph G_ε associated to *$k_\varepsilon[x_1, \dots, x_n]$ is defined to be the graph with vertex set $V(G_\varepsilon) = \{1, \dots, n\}$ and edge set $E(G_\varepsilon) = \{ij \mid \varepsilon_{ij} = \varepsilon_{ji} = 1, i \ne j\}$.*

Let $\mathbb{F}_2$ denote the field with two elements 0 and 1. For a matrix M with entries in $\mathbb{F}_2$, let $\operatorname{rank}_{\mathbb{F}_2}(M)$ (respectively, $\operatorname{null}_{\mathbb{F}_2}(M)$) denote the rank (respectively, the nullity) of M over $\mathbb{F}_2$.

Theorem 6 ([11, Theorem 1.3]). *Let $A_\varepsilon = k_\varepsilon[x_1, \dots, x_n]/(x_1^2 + \cdots + x_n^2)$ be a skew graded (A_1) hypersurface singularity with $\deg x_i = 1$, and let G_ε the graph associated to $k_\varepsilon[x_1, \dots, x_n]$. Consider the matrix*

$$\Delta_\varepsilon = \begin{pmatrix} & & & 1 \\ & M(G_\varepsilon) & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \in M_{n+1}(\mathbb{F}_2),$$

where $M(G_\varepsilon)$ is the adjacency matrix of G_ε (with entries in $\mathbb{F}_2$). Then there exists an equivalence of triangulated categories

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod} k^{2^r}\right),$$

where $r = \operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon)$.

It follows from Theorem 6 that A_ε has 2^r indecomposable non-projective graded maximal Cohen–Macaulay modules up to isomorphism and degree shift. Therefore, we have the following result, which is a generalization of Theorem 1.

Corollary 7 ([11, Theorem 1.3]). *Let $A_\varepsilon = k_\varepsilon[x_1, \dots, x_n]/(x_1^2 + \cdots + x_n^2)$ be a skew graded (A_1) hypersurface singularity. Then A_ε has finite Cohen–Macaulay representation type.*

The purpose of this paper is to investigate the stable category of graded maximal Cohen–Macaulay modules over a skew graded (A_∞) hypersurface singularity in a manner analogous to Theorem 6. We prove the following theorem.

Theorem 8. *Let $A_\varepsilon = k_\varepsilon[x_1, \dots, x_n]/(x_1^2 + \cdots + x_{n-1}^2)$ be a skew graded (A_∞) hypersurface singularity with $n \geq 2$ and $\deg x_i = 1$. Let G_ε the graph associated to $k_\varepsilon[x_1, \dots, x_n]$. Consider the matrix*

$$\Delta_\varepsilon = \begin{pmatrix} & & & 1 \\ & M(G_\varepsilon) & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \in M_{n+1}(\mathbb{F}_2),$$

where $M(G_\varepsilon)$ is the adjacency matrix of G_ε (with entries in $\mathbb{F}_2$). Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n+1} \in \mathbb{F}_2^{n+1}$ denote the columns of Δ_ε.

(1) *If $\boldsymbol{v}_n$ can be expressed as a linear combination of the other columns $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_{n+1}$, then there exists an equivalence of triangulated categories*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod} \Lambda^{2^{r-1}}\right),$$

where Λ is the finite-dimensional algebra given by the quiver with relations

$$\begin{tikzcd} 1 \arrow[loop right, "a"] \end{tikzcd} \qquad a^2 = 0, \tag{1}$$

and $r = \operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon)$.

(2) *If $\boldsymbol{v}_n$ cannot be expressed as a linear combination of the other columns $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_{n+1}$, then there exists an equivalence of triangulated categories*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod} \Gamma^{2^{r}}\right),$$

where Γ is the finite-dimensional algebra given by the quiver with relations

$$\begin{tikzcd} 1 \arrow[r, shift left, "a"] & 2 \arrow[l, shift left, "b"] \end{tikzcd} \qquad ab = 0,\ ba = 0, \tag{2}$$

and $r = \operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon)$.

Now consider the case $\varepsilon_{ij} = 1$ for all $1 \leq i < j \leq n$ and $n \geq 2$. In this case, $k_\varepsilon[x_1, \dots, x_n] = k[x_1, \dots, x_n]$ and G_ε is the complete graph K_n, so

$$\Delta_\varepsilon = \begin{pmatrix} & & & 1 \\ & M(K_n) & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix}}_{n+1}.$$

Let $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n+1} \in \mathbb{F}_2^{n+1}$ be the columns of Δ_ε. If n is even, then we have $\boldsymbol{v}_n = \boldsymbol{v}_1 + \cdots + \boldsymbol{v}_{n-1} + \boldsymbol{v}_{n+1}$ and $\operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon) = 1$. If n is odd, then one can check that $\boldsymbol{v}_n$ cannot be written as a linear combination of $\boldsymbol{v}_1, \dots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_{n+1}$ and that $\operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon) = 0$. Hence Theorem 8 contains the following result.

Corollary 9. *Let $R = k[x_1, \dots, x_n]/(x_1^2 + \cdots + x_{n-1}^2)$ be a graded (A_∞) hypersurface singularity with $n \geq 2$ and $\deg x_i = 1$.*

(1) *If n is even, then there exists an equivalence of triangulated categories $\underline{\mathsf{CM}}^{\mathbb{Z}}(R) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} \Lambda)$, where Λ is the finite-dimensional algebra given by the quiver with relations* (1).

(2) *If n is odd, then there exists an equivalence of triangulated categories $\underline{\mathsf{CM}}^{\mathbb{Z}}(R) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} \Gamma)$, where Γ is the finite-dimensional algebra given by the quiver with relations* (2).

Remark 10. Corollary 9 also follows from graded Knörrer's periodicity theorem and the result of Buchweitz, Eisenbud, and Herzog [4, Appendix]. In tilting theory, by Buchweitz, Iyama, and Yamaura's theorem [6, Thorem 1.4], it is known that if $R = k[x, y]/(x^2)$ with $\deg x = \deg y = 1$, then there exists a triangle equivalence $F : \underline{\mathsf{CM}}_0^{\mathbb{Z}}(R) \xrightarrow{\sim} \mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)$, where $\underline{\mathsf{CM}}_0^{\mathbb{Z}}(R)$ is the stable category of graded maximal Cohen-Macaulay R-modules M that satisfy $M_{\mathfrak{p}} \in \operatorname{proj} R_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R_{\mathfrak{p}} < \dim R$. Furthermore, in [13], it is shown that the equivalence $F : \underline{\mathsf{CM}}_0^{\mathbb{Z}}(R) \xrightarrow{\sim} \mathsf{K}^{\mathsf{b}}(\operatorname{proj}\Lambda)$ leads to an equivalence $\underline{\mathsf{CM}}^{\mathbb{Z}}(R) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)$. Thus, Corollary 9 (1) can also be obtained from this and graded Knörrer's periodicity theorem.

As a consequence of Theorem 8, we obtain the following corollary, which generalizes Theorem 2.

Corollary 11. *Let $A_\varepsilon = k_\varepsilon[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_{n-1}^2)$ be a skew graded (A_∞) hypersurface singularity with $n \geq 2$. Then A_ε has countably infinite Cohen–Macaulay representation type.*

In addition, Corollary 11 implies the following conclusion.

Corollary 12. *Let $A_\varepsilon = k_\varepsilon[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_{n-1}^2)$ be a skew graded (A_∞) hypersurface singularity with $n \geq 2$. Then A_ε is not a noncommutative graded isolated singularity; that is, the category $\operatorname{qgr} A_\varepsilon$ has infinite global dimension.*

This paper is organized as follows. In Section 2, some basic definitions and fundamental results are stated. In Section 3, the stable categories of graded maximal Cohen–Macaulay modules over skew graded (A_∞) hypersurface singularities are studied combinatorially. Proofs of Theorem 8 and Corollaries 11 and 12 are given in Section 4.

2. Preliminaries

Throughout this paper, k is an algebraically closed field of characteristic different from 2, and all algebras are over k.

2.1. *Stable categories of graded Maximal Cohen–Macaulay modules*

For an $\mathbb{N}$-graded algebra A, we write $\operatorname{GrMod} A$ for the category of graded right A-modules with A-module homomorphisms of degree zero, and $\operatorname{grmod} A$ for the full subcategory consisting of finitely generated graded modules. For a graded module $M \in \operatorname{GrMod} A$ and an integer $s \in \mathbb{Z}$, we define the *shift* $M(s) \in \operatorname{GrMod} A$ to be the graded module with i^{th} degree component $M(s)_i = M_{s+i}$. For $M, N \in \operatorname{GrMod} A$, we write $\operatorname{Ext}^i_{\operatorname{GrMod} A}(M, N)$ for the extension group in $\operatorname{GrMod} A$, and define $\operatorname{Ext}^i_A(M, N) := \bigoplus_{s \in \mathbb{Z}} \operatorname{Ext}^i_{\operatorname{GrMod} A}(M, N(s))$.

Let A a noetherian $\mathbb{N}$-graded algebra. Let $\operatorname{qgr} A = \operatorname{grmod} A / \operatorname{fdim} A$ denote the quotient category of $\operatorname{grmod} A$ by the Serre subcategory $\operatorname{fdim} A$ of finite-dimensional modules. The category $\operatorname{qgr} A$ plays the role of (the category of coherent sheaves on) the noncommutative projective scheme associated to A; see [1]. A noetherian $\mathbb{N}$-graded algebra A is called a *noncommutative graded isolated singularity* if $\operatorname{qgr} A$ has finite global dimension; see [19].

Recall that an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \in \mathbb{N}} A_i$ is said to be *connected graded* if $A_0 = k$. Let A be a noetherian connected graded algebra. Then A is called an *AS-regular* (respectively, *AS-Gorenstein*) algebra of dimension d if

- $\operatorname{gldim} A = d < \infty$ (respectively, $\operatorname{injdim}_A A = \operatorname{injdim}_{A^{\mathrm{op}}} A = d < \infty$), and
- $\operatorname{Ext}^i_A(k, A) \cong \operatorname{Ext}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d \\ 0 & \text{if } i \neq d \end{cases}$ for some $\ell \in \mathbb{Z}$.

Let A be a noetherian AS-Gorenstein algebra. We call $M \in \operatorname{grmod} A$ *graded maximal Cohen–Macaulay* if $\operatorname{Ext}^i_A(M, A) = 0$ for all $i \neq 0$. We write $\mathsf{CM}^{\mathbb{Z}}(A)$ for the full subcategory of $\operatorname{grmod} A$ consisting of graded maximal Cohen–Macaulay modules. Then $\mathsf{CM}^{\mathbb{Z}}(A)$ is a Frobenius category. The *stable category* of graded maximal Cohen–Macaulay modules, denoted by $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$, has the same objects as $\mathsf{CM}^{\mathbb{Z}}(A)$, and the morphism space is given by

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathbb{Z}}(A)}(M, N) = \operatorname{Hom}_{\mathsf{CM}^{\mathbb{Z}}(A)}(M, N)/P(M, N),$$

where $P(M, N)$ consists of degree zero A-module homomorphisms factoring through a graded projective module. By [9], $\underline{\mathsf{CM}}^{\mathbb{Z}}(A)$ canonically has a structure of triangulated category.

2.2. *The algebra $C(A)$*

The main algebraic framework used in this paper is due to Smith and Van den Bergh [17], which was originally developed by Buchweitz, Eisenbud, and Herzog [4].

Let S be a d-dimensional noetherian AS-regular algebra with Hilbert series $(1-t)^{-d}$. Then S is Koszul by [16, Theorem 5.11]. Let $f \in S$ be a homogeneous regular central element of degree 2, and let $A = S/(f)$. Then A is a $(d-1)$-dimensional noetherian AS-Gorenstein algebra. Moreover, A is Koszul by [17, Lemma 5.1 (1)], and there exists a homogeneous central regular element $w \in A^!_2$ such that $A^!/(w) \cong S^!$ by [17, Lemma 5.1 (2)]. We can define the algebra

$$C(A) := A^![w^{-1}]_0.$$

By [17, Lemma 5.1 (3)], we have $\dim_k C(A) = \dim_k (S^!)^{(2)} = 2^{d-1}$.

Theorem 13 ([17, Proposition 5.2]). *With notation as above, we have an equivalence $\underline{\mathsf{CM}}^{\mathbb{Z}}(A) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A))$, where $\mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A))$ denotes the bounded derived category of finite-dimensional modules over $C(A)$.*

Theorem 14 ([15, Theorem 5.5]; see also [17, Proposition 5.2], [10, Theorem 6.3]). *With notation as above, the following are equivalent.*

(1) *A has finite Cohen–Macaulay representation type.*
(2) *A is a noncommutative graded isolated singularity.*
(3) *$C(A)$ is semisimple.*

2.3. *Graphs*

A *graph G* consists of a set of vertices $V(G)$ and a set of edges $E(G)$ between two vertices. In this paper, we always assume that $V(G)$ is a finite set and G has neither loops nor multiple edges. An edge between two vertices $v, w \in V(G)$ is written by $vw \in E(G)$. For a vertex $v \in V(G)$, let $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$. A graph G' is called the *induced subgraph* of G induced by $V' \subset V(G)$ if $vw \in E(G')$ whenever $v, w \in V'$ and $vw \in E(G)$. For a subset $W \subset V(G)$, we denote by $G \setminus W$ the induced subgraph of G induced by $V(G) \setminus W$.

Definition 15.

(1) *We say that v is a* isolated vertex *of a graph G if v is a vertex of G such that $N_G(v) = \emptyset$.*
(2) *We say that vw is an* isolated edge *of a graph G if vw is an edge of G such that $N_G(v) = \{w\}$ and $N_G(w) = \{v\}$.*

Now let us focus on the notions of switching and relative switching of graphs.

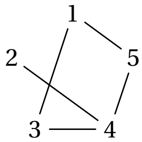
Definition 16 ([8, Section 11.5]). *Let G be a graph and $v \in V(G)$. The* switching $\mu_v(G)$ *of G at v is defined to be the graph $\mu_v(G)$ with $V(\mu_v(G)) = V(G)$ and*

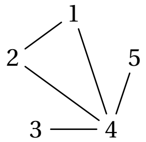
$$E(\mu_v(G)) = \left\{vw \,\middle|\, w \in V(G) \setminus N_G(v)\right\} \cup E(G \setminus \{v\}).$$

For $v, w \in V(G)$, we define $\mu_w\mu_v(G) := \mu_w(\mu_v(G))$.

Remark 17.

(1) The same notion is called mutation in [15] and [11].
(2) For $v \in V(G)$, we have $\mu_v\mu_v(G) = G$. For $v, w \in V(G)$, we have $\mu_w\mu_v(G) = \mu_v\mu_w(G)$.

Example 18. If $G =$ 

, then $\mu_1(G) =$ 

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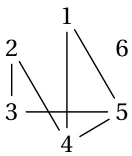
Definition 19 ([15, Definition 6.6]). *Let $v, w \in V(G)$ be distinct vertices. Then the* relative switching $\mu_{v\leftarrow w}(G)$ *of G at v with respect to w is defined to be the graph $\mu_{v\leftarrow w}(G)$ with $V(\mu_{v\leftarrow w}(G)) = V(G)$ and*

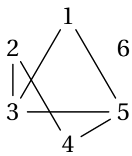
$$E\left(\mu_{v\leftarrow w}(G)\right) = \left\{vu \,\middle|\, u \in N_G(w) \setminus N_G(v)\right\} \cup \left\{vu \,\middle|\, u \in N_G(v) \setminus N_G(w)\right\} \cup E(G \setminus \{v\}).$$

For v, v' $w, w' \in V(G)$ with $v \neq w$ and $v' \neq w'$, we define $\mu_{v'\leftarrow w'}\mu_{v\leftarrow w}(G) := \mu_{v'\leftarrow w'}\left(\mu_{v\leftarrow w}(G)\right)$.

Remark 20.

(1) In the original paper [15], relative switching is called relative mutation.
(2) For distinct $v, w \in V(G)$, we have $\mu_{v\leftarrow w}\mu_{v\leftarrow w}(G) = G$.

Example 21. If $G =$ 

, then $\mu_{1\leftarrow 2}(G) =$ 

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For a graph G, let $M(G)$ denote the adjacency matrix of G with entries in $\mathbb{F}_2$. Define

$$\Delta(G) = \begin{pmatrix} & & & 1 \\ & M(G) & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} \in M_{n+1}(\mathbb{F}_2).$$

Definition 22. *For a matrix $M \in M_{n+1}(\mathbb{F}_2)$, we say that M* satisfies the condition *(L) if the n^{th} column $\boldsymbol{v}_n \in \mathbb{F}_2^{n+1}$ of M can be expressed as a linear combination of the other columns $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{n-1}, \boldsymbol{v}_{n+1} \in \mathbb{F}_2^{n+1}$ of M.*

Lemma 23. *Let G be a graph with $V(G) = \{1, \ldots, n\}$. If $G' = \mu_v(G)$ for some $v \in V(G)$, then*

(1) $\mathrm{null}_{\mathbb{F}_2}(\Delta(G)) = \mathrm{null}_{\mathbb{F}_2}(\Delta(G'))$, *and*
(2) $\Delta(G)$ *satisfies the condition (L) if and only if so does* $\Delta(G')$.

Proof. By definition of switching, we have

$$\left(E + E_{v,n+1}\right)\Delta(G)\left(E + E_{n+1,v}\right) = \begin{pmatrix} & & & 1 \\ & M(\mu_v(G)) & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} = \Delta(G'),$$

where E is the identity matrix and $E_{i,j}$ is the matrix such that the (i,j)-entry is 1 and the other entries are all 0. This yields the assertions of the Lemma 23. $\square$

Lemma 24. *Let G be a graph with $V(G) = \{1, \dots, n\}$ with $n \geq 2$. Assume that 1 is an isolated vertex of G. Let $v, w \in V(G_\varepsilon)$ be distinct vertices with $v \neq 1, w \neq n$. If $G' = \mu_{v \leftarrow w}(G)$, then*

(1) $\mathrm{null}_{\mathbb{F}_2}(\Delta(G)) = \mathrm{null}_{\mathbb{F}_2}(\Delta(G'))$, *and*
(2) $\Delta(G)$ *satisfies the condition (L) if and only if so does* $\Delta(G')$.

Proof. Since 1 is an isolated vertex of G, the first row (respectively, the first column) of $\Delta(G)$ is $(0 \cdots 0\ 1)$ (respectively,

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}).$$

By definition of relative switching, we have

$$\left(E + E_{v,w} + E_{v,1}\right) \Delta(G) \left(E + E_{w,v} + E_{1,v}\right) = \begin{pmatrix} & & & 1 \\ & M(\mu_{v\leftarrow w}(G)) & & \vdots \\ & & & 1 \\ 1 & \cdots & 1 & 0 \end{pmatrix} = \Delta(G'),$$

where E is the identity matrix and $E_{i,j}$ is the matrix such that the (i, j)-entry is 1 and the other entries are all 0. This yields the assertions of the Lemma 24. □

Switching and relative switching will be used to compute the stable categories of graded maximal Cohen–Macaulay modules over skew graded (A_∞) hypersurface singularities in the following sections.

3. Graphical Methods for Computing Stable Categories

Throughout this section, let $A_\varepsilon = k_\varepsilon[x_1, \dots, x_n]/(x_1^2 + \cdots + x_{n-1}^2)$ be a skew graded (A_∞) hypersurface singularity with $n \geq 2$ and $\deg x_i = 1$, and let G_ε be the graph associated to $k_\varepsilon[x_1, \dots, x_n]$. The purpose of this section is to study $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)$ using G_ε. Note that $k_\varepsilon[x_1, \dots, x_n]$ is an n-dimensional noetherian Koszul AS-regular domain with Hilbert series $(1-t)^{-n}$, so A_ε is an $(n-1)$-dimensional noetherian Koszul AS-Gorenstein algebra.

Lemma 25.

(1) $A_\varepsilon^!$ *is isomorphic to*

$$k\langle x_1, \dots, x_n\rangle / \left(\varepsilon_{ij} x_i x_j + x_j x_i,\ x_1^2 - x_\ell^2,\ x_n^2 \,\middle|\, 1 \leq i, j \leq n, i \neq j, 1 \leq \ell \leq n-1\right).$$

(2) $w := x_1^2 \in A_\varepsilon^!$ *is a central regular element such that* $A_\varepsilon^!/(w) \cong k_\varepsilon[x_1, \dots, x_n]^!$.
(3) $C(A_\varepsilon) = A_\varepsilon^![w^{-1}]_0$ *is isomorphic to*

$$k\langle t_2, \dots, t_n\rangle / \left(t_i t_j + \varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1} t_j t_i,\ t_\ell^2 - 1,\ t_n^2 \,\middle|\, 2 \leq i, j \leq n, i \neq j, 2 \leq \ell \leq n-1\right).$$

Proof. (1) and (2) follow from direct calculation.

(3) Write $t_i := x_1 x_i w^{-1} \in C(A_\varepsilon)$ for $2 \leq i \leq n$. Then it is easy to see that $\{t_2, \dots, t_n\}$ is a set of generators of $C(A_\varepsilon)$. Since we have

$$\begin{aligned} t_i t_j &= \left(x_1 x_i w^{-1}\right)\left(x_1 x_j w^{-1}\right) = -\varepsilon_{1i} x_1^2 x_i x_j w^{-2} = -\varepsilon_{1i} x_i x_j w^{-1} = \varepsilon_{1i}\varepsilon_{ji} x_j x_i w^{-1} \\ &= -\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1}\left(-\varepsilon_{1j} x_1^2 x_j x_i w^{-2}\right) = -\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1}\left(x_1 x_j w^{-1}\right)\left(x_1 x_i w^{-1}\right) = -\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1} t_j t_i, \end{aligned}$$

for $2 \leq i, j \leq n, i \neq j$,

$$t_\ell^2 = \left(x_1 x_\ell w^{-1}\right)\left(x_1 x_\ell w^{-1}\right) = -\varepsilon_{1\ell} x_\ell^2 w^{-1} = -\varepsilon_{1\ell} x_1^2 w^{-1} = -\varepsilon_{1\ell}$$

for $2 \leq \ell \leq n-1$, and

$$t_n^2 = \left(x_1 x_n w^{-1}\right)\left(x_1 x_n w^{-1}\right) = -\varepsilon_{1n} x_n^2 w^{-1} = 0,$$

there exists a surjection $k\langle t_2,\dots,t_n\rangle/(t_it_j+\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1}t_jt_i,\ t_\ell^2+\varepsilon_{1\ell},\ t_n^2)\to C(A_\varepsilon)$. This is an isomorphism because the algebras have the same dimension. Since $\varepsilon_{1\ell}\neq 0$, the assignment $t_\ell\mapsto\sqrt{-\varepsilon_{1\ell}}\,t_\ell$ for $2\le\ell\le n-1$ and $t_n\mapsto t_n$ induces the isomorphism

$$k\langle t_2,\dots,t_n\rangle\big/\big(t_it_j+\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1}t_jt_i,\ t_\ell^2+\varepsilon_{1\ell},\ t_n^2\big)\xrightarrow{\sim}k\langle t_2,\dots,t_n\rangle\big/\big(t_it_j+\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1}t_jt_i,\ t_\ell^2-1,\ t_n^2\big).$$

Hence we have $C(A_\varepsilon)\cong k\langle t_2,\dots,t_n\rangle/(t_it_j+\varepsilon_{1i}\varepsilon_{ij}\varepsilon_{j1}t_jt_i,\ t_\ell^2-1,\ t_n^2)$. □

Example 26.

(1) If $A_\varepsilon=k[x_1,x_2]/(x_1^2)$ or $A_\varepsilon=k\langle x_1,x_2\rangle/(x_1x_2+x_2x_1,\ x_1^2)$, then $C(A_\varepsilon)\cong k[t]/(t^2)$ by Lemma 25 (3), so $C(A_\varepsilon)$ is isomorphic to the algebra Λ given by the quiver with relations (1).

(2) If $A_\varepsilon=k[x_1,x_2,x_3]/(x_1^2+x_2^2)$, then $C(A_\varepsilon)\cong k\langle s,t\rangle/(st+ts,\ s^2-1,\ t^2)$ by Lemma 25 (3). Let

$$e_1=\frac{1}{2}(1+s+t+st),\quad e_2=\frac{1}{2}(1-s-t-st),\quad a=\frac{1}{2}(t+st),\quad b=\frac{1}{2}(t-st).$$

Then $\{e_1,e_2,a,b\}$ is a k-basis of $k\langle s,t\rangle/(st+ts,\ s^2-1,\ t^2)$. Since e_1,e_2 are orthogonal idempotents with $e_1+e_2=1$, and

$$e_1ae_2=a,\quad e_2be_1=b,\quad ab=0,\quad ba=0,$$

it follows that $C(A_\varepsilon)\cong k\langle s,t\rangle/(st+ts,\ s^2-1,\ t^2)\cong\Gamma$, where Γ is the algebra given by the quiver with relations (2).

(3) If $A_\varepsilon=k\langle x_1,x_2,x_3\rangle/(x_1x_2+x_2x_1,\ x_1x_3+x_3x_1,\ x_2x_3+x_3x_2,\ x_1^2+x_2^2)$, then $C(A_\varepsilon)\cong k[s,t]/(s^2-1,\ t^2)$ by Lemma 25 (3). Since $k[s,t]/(s^2-1,\ t^2)\cong k[s]/(s^2-1)\otimes k[t]/(t^2)\cong k^2\otimes k[t]/(t^2)\cong k[t]/(t^2)\times k[t]/(t^2)$, we have $C(A_\varepsilon)\cong\Lambda\times\Lambda$, where Λ is the algebra given by the quiver with relations (1).

We first give two theorems concerning switching and relative switching, which are analogs of [15, Lemmas 6.5 and 6.7].

Theorem 27 (Switching theorem). *Let $A_\varepsilon, A_{\varepsilon'}$ be skew graded (A_∞) hypersurface singularities. If $G_{\varepsilon'}=\mu_v(G_\varepsilon)$ for some $v\in V(G_\varepsilon)$, then $C(A_\varepsilon)\cong C(A_{\varepsilon'})$ and $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)\cong\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Proof. Since $G_{\varepsilon'}=\mu_v(G_\varepsilon)$, we see that $\varepsilon'_{ij}\varepsilon'_{jh}\varepsilon'_{hi}=\varepsilon_{ij}\varepsilon_{jh}\varepsilon_{hi}$ for every $1\le i<j<h\le n$, so $C(A_\varepsilon)\cong C(A_{\varepsilon'})$ by Lemma 25 (3), and $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)\cong\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$ by Theorem 13. □

Theorem 28 (Relative switching theorem). *Let $A_\varepsilon, A_{\varepsilon'}$ be skew graded (A_∞) hypersurface singularities. Assume that 1 is an isolated vertex of G_ε. Let $v,w\in V(G_\varepsilon)$ be distinct vertices with $v\neq 1, w\neq n$. If $G_{\varepsilon'}=\mu_{v\leftarrow w}(G_\varepsilon)$, then $C(A_\varepsilon)\cong C(A_{\varepsilon'})$ and $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)\cong\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Proof. If $w=1$, then $\mu_{v\leftarrow w}(G_\varepsilon)=G_\varepsilon$ since w is an isolated vertex, so the result is obvious. Assume now that $w\neq 1$. Since $\varepsilon_{1i}=-1$ for all $1\le i\le n$,

$$C(A_\varepsilon)\cong k\langle t_2,\dots,t_n\rangle\big/\big(t_it_j+\varepsilon_{ij}t_jt_i,\ t_\ell^2-1,\ t_n^2\,\big|\,2\le i,j\le n,\ i\neq j,\ 2\le\ell\le n-1\big)$$

by Lemma 25 (3). Let D be the algebra generated by $s_2,\dots,s_n$ with defining relations

$$\begin{aligned}
&s_is_j+\varepsilon_{ij}s_js_i\quad\big(2\le i,j\le n,\ i\neq j,\ i\neq v,\ j\neq v\big),\\
&s_vs_j-\varepsilon_{vj}\varepsilon_{wj}s_js_v\quad\big(2\le j\le n,\ j\neq v,\ j\neq w\big),\\
&s_vs_w+\varepsilon_{vw}s_ws_v,\\
&s_\ell^2-1\quad(2\le\ell\le n-1,\ \ell\neq v),\\
&s_v^2+\varepsilon_{vw}\quad(\text{when } v\neq n),\\
&s_n^2.
\end{aligned}$$

Define a map $\phi : k\langle s_2, \dots, s_n\rangle \to k\langle t_2, \dots, t_n\rangle/(t_i t_j + \varepsilon_{ij} t_j t_i,\ t_\ell^2 - 1,\ t_n^2)$ by $s_i \mapsto t_i$ $(2 \le i \le n, i \ne v)$ and $s_v \mapsto t_v t_w$. Then one can verify that ϕ sends all defining relations of D to zero, so we get an induced map $\overline{\phi} : D \to k\langle t_2, \dots, t_n\rangle/(t_i t_j + \varepsilon_{ij} t_j t_i,\ t_\ell^2 - 1,\ t_n^2)$. It is easily seen that $\overline{\phi}$ is an isomorphism.

Moreover, since $G_{\varepsilon'} = \mu_{v\leftarrow w}(G_\varepsilon)$, it follows that

$$\begin{aligned} &\varepsilon'_{ij} = \varepsilon_{ij} \quad \bigl(1 \le i, j \le n,\ i \ne j,\ i \ne v,\ j \ne v\bigr), \\ &\varepsilon'_{vj} = -\varepsilon_{vj}\varepsilon_{wj} \quad \bigl(1 \le j \le n,\ j \ne v,\ j \ne w\bigr), \\ &\varepsilon'_{vw} = \varepsilon_{vw}. \end{aligned}$$

and 1 is an isolated vertex of $G_{\varepsilon'}$, so we see that D is isomorphic to $C(A_{\varepsilon'})$ by Lemma 25 (3). Therefore, we obtain $C(A_\varepsilon) \cong C(A_{\varepsilon'})$. The last equivalence follows from Theorem 13. □

We then give two ways to reduce the number of variables in the computation of $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)$, which are analogs of [15, Lemmas 6.17 and 6.18]. We will see that the first one is coming from the noncommutative Knörrer's periodicity theorem [15, Theorem 3.9] and the second one is coming from [15, Theorem 4.12 (3)].

Theorem 29 (Special case of noncommutative graded Knörrer's periodicity theorem). *Let S be a noetherian AS-regular algebra and f a homogeneous regular central element of positive even degree $2m$. Then there exists an equivalence*

$$\underline{\mathsf{CM}}^{\mathbb{Z}}\bigl(S/(f)\bigr) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}\bigl(S[y,z]\big/\bigl(f + y^2 + z^2\bigr)\bigr),$$

where $\deg y = \deg z = m$.

Proof. This is a special case of [15, Theorem 3.9]. □

Theorem 30 (Knörrer reduction). *Suppose that vw is an isolated edge of G_ε, where $v \ne n, w \ne n$, and $V(G_\varepsilon) \setminus \{v, w, n\} \ne \emptyset$. If $G_{\varepsilon'} = G_\varepsilon \setminus \{v, w\}$, then $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Proof. Let $G_{\varepsilon''} = \mu_w\mu_v(G_\varepsilon)$. Then x_v, x_w are central elements in $k_{\varepsilon''}[x_1, \dots, x_n]$. Since $G_{\varepsilon'} = G_\varepsilon \setminus \{v, w\} = G_{\varepsilon''} \setminus \{v, w\}$, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon''}) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$ by Theorems 27 and 29. □

Let S be a connected algebra and σ a graded algebra automorphism of S. A *graded Ore extension* of S by σ is a graded algebra $S[y;\sigma]$ which is defined as follows: $\deg y \ge 1$, $S[y;\sigma] = S[y]$ as a graded left free S-module, and the multiplication of $S[y;\sigma]$ is given by $ay = y\sigma(a)$ for $a \in S$. Let ξ denote the graded algebra automorphism of S defined by $a \mapsto (-1)^{\deg a} a$. For example, $k[x_1, x_2][y;\xi]$ is isomorphic to $k\langle x_1, x_2, y\rangle/(x_1x_2 - x_2x_1, x_1y + yx_1, x_2y + yx_2)$.

Let S be a d-dimensional noetherian AS-regular algebra with Hilbert series $(1-t)^{-d}$. For a homogeneous regular central element $f \in S$ of degree 2 and $A = S/(f)$, we define $S^\dagger := S[y;\xi]$ and $A^\dagger := S^\dagger/(f + y^2)$, where $\deg y = 1$. Since $f + y^2 \in S_2^\dagger$ is a regular central element, we further define $S^{\dagger\dagger} := (S^\dagger)^\dagger = (S[y;\xi])[z;\xi]$ and $A^{\dagger\dagger} := (A^\dagger)^\dagger = S^{\dagger\dagger}/(f + y^2 + z^2)$, where $\deg z = 1$.

Theorem 31 ([15, Theorem 4.12 (3)]). *Let S be a d-dimensional noetherian AS-regular algebra with Hilbert series $(1-t)^{-d}$, $f \in S$ a homogeneous regular central element of degree 2, and $A = S/(f)$. Then $C(A^{\dagger\dagger}) \cong C(A^\dagger) \times C(A^\dagger)$.*

Theorem 32 (Two point reduction). *Let $A_\varepsilon, A_{\varepsilon'}$ be skew graded (A_∞) hypersurface singularities. Suppose that $v, w \in V(G_\varepsilon)$ are two distinct isolated vertices different from n. If $G_{\varepsilon'} = G_\varepsilon \setminus \{v\}$, then $C(A_\varepsilon) \cong C(A_{\varepsilon'}) \times C(A_{\varepsilon'})$ and $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'}) \times \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})$.*

Proof. If $|V(G_\varepsilon)| = 3$, then this follows from Example 26 (1) and (3). Assume that $|V(G_\varepsilon)| \geq 4$. Let $A = A_\varepsilon/(x_v, x_w)$. Then A is a skew graded (A_∞) hypersurface singularity such that $A_{\varepsilon'} \cong A^\dagger$ and $A_\varepsilon \cong A^{\dagger\dagger}$, so $C(A_\varepsilon) \cong C(A_{\varepsilon'}) \times C(A_{\varepsilon'})$ by Lemma 31. Furthermore, we obtain

$$\begin{aligned}\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) &\cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A_\varepsilon)) \cong \mathsf{D}^{\mathsf{b}}\big(\operatorname{mod}(C(A_{\varepsilon'}) \times C(A_{\varepsilon'}))\big)\\ &\cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A_{\varepsilon'})) \times \mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A_{\varepsilon'})) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'}) \times \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})\end{aligned}$$

by Theorem 13. □

Here, we prove a combinatorial lemma, which is a modification of [11, Lemma 3.1]. For non-negative integers a and b, let $G(a,b)$ denote the graph with set of vertices $\{u_i, u'_i \mid i = 1, \dots, a\} \cup \{u''_j \mid j = 1, \dots, b\}$ and set of edges $\{u_i u'_i \mid i = 1, \dots, a\}$. Namely, $G(a,b)$ consists of a isolated edges and b isolated vertices.

Lemma 33. *Let G be a graph with $V(G) = \{1, \dots, n\}$ and $n \geq 2$. Assume that 1 is an isolated vertex of G. Then there exists a sequence of relative switchings $\mu_{v_1 \leftarrow w_1}, \dots, \mu_{v_m \leftarrow w_m}$ such that*

$$\mu_{v_m \leftarrow w_m} \mu_{v_{m-1} \leftarrow w_{m-1}} \cdots \mu_{v_1 \leftarrow w_1}(G) \cong G(\alpha, \beta),$$

where $v_i \in \{2, \dots, n\}$, $w_j \in \{2, \dots, n-1\}$ for all $1 \leq i, j \leq m$, $2\alpha + \beta = n$, and $\beta \geq 1$.

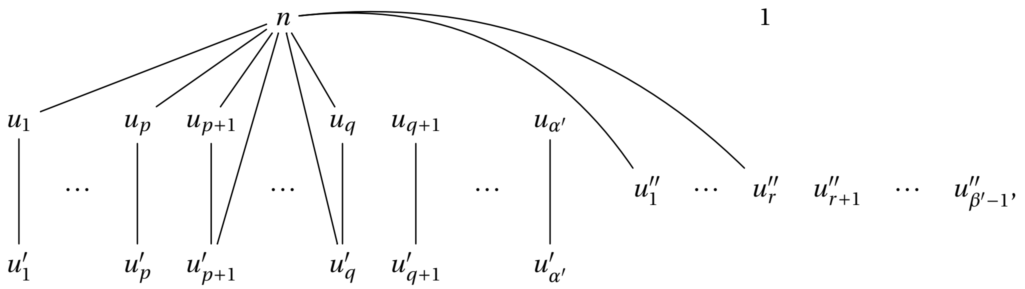
Proof. We prove the claim by induction on n. If $n = 2$, then the claim is trivial because G is already equal to $G(0,2)$. Suppose that $n \geq 3$. Let $G' = G \setminus \{n\}$. By the induction hypothesis, there exists a sequence of relative switchings $\mu_{v_1 \leftarrow w_1}, \dots, \mu_{v_h \leftarrow w_h}$ such that

$$\mu_{v_h \leftarrow w_h} \mu_{v_{h-1} \leftarrow w_{h-1}} \cdots \mu_{v_1 \leftarrow w_1}(G') \cong G(\alpha', \beta'),$$

where $v_i \in \{2, \dots, n-1\}$, $w_j \in \{2, \dots, n-2\}$ for all $1 \leq i, j \leq h$, $2\alpha' + \beta' = n-1$, and $\beta' \geq 1$. Let

$$G_1 = \mu_{v_h \leftarrow w_h} \mu_{v_{h-1} \leftarrow w_{h-1}} \cdots \mu_{v_1 \leftarrow w_1}(G).$$

Then $G_1 \setminus \{n\} \cong G(\alpha', \beta')$ and 1 is still an isolated vertex of G_1. Thus G_1 has the form



where $\{u_1, u'_1, \dots, u_{\alpha'}, u'_{\alpha'}\} \cup \{u''_1, \dots, u''_{\beta'-1}\} = \{2, \dots, n-1\}$ and the set of edges of G_1 is

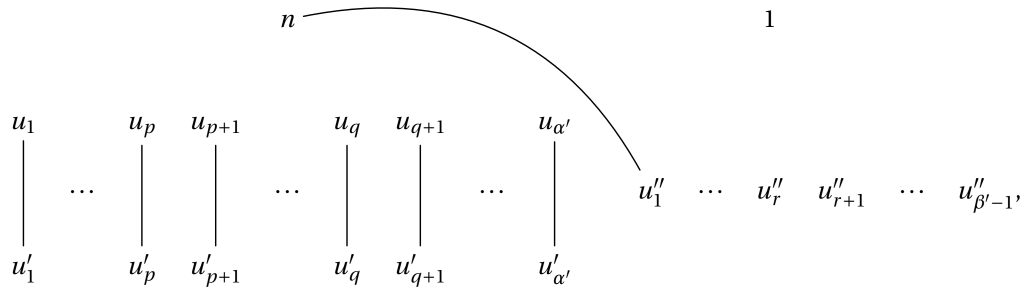
$$\{u_1 u'_1, \dots, u_{\alpha'} u'_{\alpha'}\} \cup \{nu_1, \dots, nu_p\} \cup \left\{nu_{p+1}, nu'_{p+1}, \dots, nu_q, nu'_q\right\} \cup \{nu''_1, \dots, nu''_r\}$$

for some $0 \leq p \leq q \leq \alpha'$ and $0 \leq r \leq \beta' - 1$.

Let

$$\begin{aligned}G_2 &= \mu_{n \leftarrow u'_p} \cdots \mu_{n \leftarrow u'_1}(G_1),\\ G_3 &= \mu_{n \leftarrow u_q} \mu_{n \leftarrow u'_q} \cdots \mu_{n \leftarrow u_{p+1}} \mu_{n \leftarrow u'_{p+1}}(G_2),\\ G_4 &= \mu_{u''_r \leftarrow u''_1} \cdots \mu_{u''_2 \leftarrow u''_1}(G_3).\end{aligned}$$

Note that $\mu_{n\leftarrow u'_p}\cdots\mu_{n\leftarrow u'_1}$ removes edges $nu_1,\dots,nu_p$ from G_1, $\mu_{n\leftarrow u_q}\mu_{n\leftarrow u'_q}\cdots\mu_{n\leftarrow u_{p+1}}\mu_{n\leftarrow u'_{p+1}}$ removes edges $nu_{p+1},nu'_{p+1},\dots,nu_q,nu'_q$ from G_2, and $\mu_{u''_r\leftarrow u''_1}\cdots\mu_{u''_2\leftarrow u''_1}$ removes edges $nu''_2,\dots,nu''_r$ from G_3. Hence G_4 has the form



where the set of edges of G_4 is $\{u_1u'_1,\dots,u_{\alpha'}u'_{\alpha'}\}\cup\{nu''_1\}$ if $r\geq 1$ and $\{u_1u'_1,\dots,u_{\alpha'}u'_{\alpha'}\}$ if $r=0$. That is, G_4 is isomorphic to $G(\alpha'+1,\beta'-1)$ if $r\geq 1$ and $G(\alpha',\beta'+1)$ if $r=0$. Therefore, we have proved the claim. □

4. Proofs of Theorem 8 and Corollaries 11 and 12

In this section, we provide proofs of Theorem 8 and Corollaries 11 and 12.

Proof of Theorem 8. Let $N_{G_\varepsilon}(1)=\{u_1,\dots,u_h\}$ and define

$$G_{\varepsilon'}:=\mu_{u_h}\cdots\mu_{u_1}(G_\varepsilon).$$

Then 1 is an isolated vertex of $G_{\varepsilon'}$. By Lemma 33, there exists a sequence of relative switchings $\mu_{v_1\leftarrow w_1},\dots,\mu_{v_m\leftarrow w_m}$ such that

$$G_{\varepsilon''}:=\mu_{v_m\leftarrow w_m}\cdots\mu_{v_1\leftarrow w_1}(G_{\varepsilon'})\cong G(\alpha,\beta),$$

where $v_i\in\{2,\dots,n\}$, $w_j\in\{2,\dots,n-1\}$ for all $1\leq i,j\leq m$, $2\alpha+\beta=n$, and $\beta\geq 1$.

By switching theorem (Theorem 27) and relative switching theorem (Theorem 28), we have

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)\cong\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'})\cong\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon''}).\tag{3}$$

By Lemmas 23 (1) and 24 (1),

$$r=\operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon)=\operatorname{null}_{\mathbb{F}_2}(\Delta_{\varepsilon'})=\operatorname{null}_{\mathbb{F}_2}(\Delta_{\varepsilon''})=\operatorname{null}_{\mathbb{F}_2}\big(\Delta(G(\alpha,\beta))\big)=\beta-1\tag{4}$$

(note that $\beta\geq 1$). Moreover, by Lemmas 23 (2) and 24 (2),

$$\Delta_\varepsilon\text{ satisfies (L)}\iff\Delta_{\varepsilon'}\text{ satisfies (L)}\iff\Delta_{\varepsilon''}\text{ satisfies (L)}.\tag{5}$$

(1) Since Δ_ε satisfies (L), so does $\Delta_{\varepsilon''}$ by (5), that is, the n^{th} column of $\Delta_{\varepsilon''}$ can be expressed as a linear combination of the other columns. Since $G_{\varepsilon''}\cong G(\alpha,\beta)$ and 1 is an isolated vertex of $G_{\varepsilon''}$, we see that the n^{th} column of $\Delta_{\varepsilon''}$ is equal to

$$\begin{pmatrix}0\\ \vdots\\ 0\\ 1\end{pmatrix},$$

so n is an isolated vertex of $G_{\varepsilon''}$.

Using Knörrer reduction (Theorem 30) α times and two point reduction (Theorem 32) $\beta-2$ times, we have

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon''})\cong\underbrace{\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''})\times\cdots\times\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''})}_{2^{\beta-2}},\tag{6}$$

where

$$G_{\varepsilon'''} := \quad 1 \qquad n \qquad \text{(two isolated vertices)}.$$

Since $A_{\varepsilon'''} \cong k\langle x_1, x_2\rangle/(x_1x_2 + x_2x_1, x_1^2)$, Example 26 (1) and Theorem 13 imply

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''}) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda). \tag{7}$$

Hence we conclude

$$\begin{aligned}
\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon}) &\cong \underbrace{\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''}) \times \cdots \times \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''})}_{2^{\beta-2}} && \text{(by (3) and (6))}\\
&\cong \underbrace{\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda) \times \cdots \times \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)}_{2^{\beta-2}} && \text{(by (7))}\\
&\cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod}\Lambda^{2^{\beta-2}}\right)\\
&\cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod}\Lambda^{2^{r-1}}\right) && \text{(by (4))},
\end{aligned}$$

as desired.

(2) Since Δ_{ε} does not satisfy (L), it follows from (5) that $\Delta_{\varepsilon''}$ does not satisfy (L), that is, the n^{th} column of $\Delta_{\varepsilon''}$ cannot be expressed as a linear combination of the other columns. Since $G_{\varepsilon''} \cong G(\alpha, \beta)$ and 1 is an isolated vertex of $G_{\varepsilon''}$, we see that the n^{th} column of $\Delta_{\varepsilon''}$ is not equal to

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},$$

so n is not an isolated vertex of $G_{\varepsilon''}$. In other words, n constitutes an isolated edge of $G_{\varepsilon''}$.

Using Knörrer reduction (Theorem 30) $\alpha - 1$ times and two point reduction (Theorem 32) $\beta - 1$ times, we have

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon''}) \cong \underbrace{\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''}) \times \cdots \times \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''})}_{2^{\beta-1}} \tag{8}$$

where

$$G_{\varepsilon'''} := \quad 1 \qquad i \text{ —— } n \qquad \text{(one isoleted vertex and one isolated edge)}$$

for some i. Let $G_{\varepsilon''''} = \mu_1(G_{\varepsilon'''})$. Then

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''}) \cong \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon''''}) \tag{9}$$

by switching theorem (Theorem 27). Since $A_{\varepsilon''''} \cong k[x_1, x_2, x_3]/(x_1^2 + x_2^2)$, Example 26 (2) and Theorem 13 imply

$$\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon''''}) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma). \tag{10}$$

Hence we conclude

$$\begin{aligned}
\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon}) &\cong \underbrace{\underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''}) \times \cdots \times \underline{\mathsf{CM}}^{\mathbb{Z}}(A_{\varepsilon'''})}_{2^{\beta-1}} && \text{(by (3) and (8))}\\
&\cong \underbrace{\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma) \times \cdots \times \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma)}_{2^{\beta-1}} && \text{(by (9) and (10))}\\
&\cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod}\Gamma^{2^{\beta-1}}\right)\\
&\cong \mathsf{D}^{\mathsf{b}}\left(\operatorname{mod}\Gamma^{2^{r}}\right) && \text{(by (4))},
\end{aligned}$$

as desired. $\square$

Proof of Corollary 11. By [3, Theorem A], we see that Λ and Γ are derived-discrete algebras. Furthermore, it follows from [3, Theorem B] that $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda)$ and $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma)$ have countably many indecomposable objects. Thus $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda^{2^{r-1}})$ and $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma^{2^r})$ also have countably many indecomposable objects. By Theorem 8, we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda^{2^{r-1}})$ or $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma^{2^r})$, so in either case, $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon)$ has countably many indecomposable objects. This says that A_ε has countably many indecomposable non-projective graded maximal Cohen–Macaulay modules up to isomorphism, so A_ε is countable Cohen–Macaulay representation type. But A_ε is not finite Cohen-Macaulay representation type. Indeed, suppose that A_ε is finite Cohen–Macaulay representation type. Then $C(A_\varepsilon)$ is semisimple by Theorem 14, so $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod} C(A_\varepsilon))$ is a semisimple triangulated category in the sense of [17, Section 4.2]. However, since $\Lambda^{2^{r-1}}$ and Γ^{2^r} are not semisimple, $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Lambda^{2^{r-1}})$ and $\mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma^{2^r})$ are not semisimple triangulated categories. This is a contradiction. □

Proof of Corollary 12. By Corollary 11, A_ε is not finite Cohen–Macaulay representation type, so A_ε is not a noncommutative graded isolated singularity by Theorem 14. □

In closing, we give an example.

Example 34. Let us consider the case $n = 4$. Let $A_\varepsilon = k_\varepsilon[x_1, x_2, x_3, x_4]/(x_1^2 + x_2^2 + x_3^2)$ be the skew graded (A_∞) hypersurface singularity given by

$$\varepsilon = (\varepsilon_{ij}) = \begin{pmatrix} 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \end{pmatrix}.$$

Then

$$G_\varepsilon = \begin{tikzcd}[column sep=small] & 1 \arrow[dl, dash] \arrow[dr, dash] & \\ 2 \arrow[rr, dash] \arrow[dr, dash] & & 4 \\ & 3 & \end{tikzcd} \qquad \text{and} \qquad \Delta_\varepsilon = \begin{pmatrix} 0&1&0&1&1 \\ 1&0&1&1&1 \\ 0&1&0&0&1 \\ 1&1&0&0&1 \\ 1&1&1&1&0 \end{pmatrix}.$$

One can check that the 4^{th} column

$$\boldsymbol{v}_4 = \begin{pmatrix} 1\\1\\0\\0\\1 \end{pmatrix} \in \mathbb{F}_2^5$$

of Δ_ε cannot be written as a linear combination of the other columns

$$\boldsymbol{v}_1 = \begin{pmatrix} 0\\1\\0\\1\\1 \end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix} 1\\0\\1\\1\\1 \end{pmatrix}, \boldsymbol{v}_3 = \begin{pmatrix} 0\\1\\0\\0\\1 \end{pmatrix}, \boldsymbol{v}_5 = \begin{pmatrix} 1\\1\\1\\1\\0 \end{pmatrix} \in \mathbb{F}_2^5$$

of Δ_ε, and $\operatorname{null}_{\mathbb{F}_2}(\Delta_\varepsilon) = 1$. Hence we have $\underline{\mathsf{CM}}^{\mathbb{Z}}(A_\varepsilon) \cong \mathsf{D}^{\mathsf{b}}(\operatorname{mod}\Gamma^2)$ by Theorem 8; compare with Corollary 9 (1) in the commutative case.

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