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A note on singular oscillatory integrals with certain rational phases

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Abstract. Let Ω be homogeneous of degree zero with mean value zero, P and Q real polynomials on $\mathbb{R}^n$ with $Q(0)=0$ and $\Omega \in B_q^{0,0}(S^{n-1})$ for some $q>1$. This note extends and improves a classical result of Stein and Wainger (*Ann. Math. Stud.* **112**, pp. 307-355, (1986)) to the following general form

$$\left|\text{p. v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}\frac{\Omega(x/|x|)}{|x|^n}\,dx\right| \le B,$$

where B depend only on $\|\Omega\|_{B_q^{0,0}(S^{n-1})}$, n and the degrees of P and Q, but not on their coefficients.

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1. Introduction

Let P be a polynomial on $\mathbb{R}^n$ of degree at most d with real coefficients and K be a homogeneous function of degree $-n$ on $\mathbb{R}^n$, that is,

$$K(x)=\frac{\Omega(x/|x|)}{|x|^n},$$

where Ω is an integrable function on the unit sphere S^{n-1} and satisfies $\int_{S^{n-1}}\Omega\,d\sigma=0$.

In [8] Stein showed that if $\Omega\in L^\infty(S^{n-1})$, then

$$\left|\text{p. v.}\int_{\mathbb{R}^n} e^{iP(x)}K(x)\,dx\right| \le C_{d,n,K}, \tag{1}$$

where $C_{d,n,K}$ is independent of the coefficients of P. The corresponding one-dimensional estimation was obtained by Stein and Wainger in [9], see also [7] for the sharp bound. Subsequently, Papadimitrakis and Parissis [6], Al-Qassem et al. [1] successively extended the estimate (1) to the cases of that $\Omega\in L\log L(S^{n-1})$ and $H^1(S^{n-1})$, the Hardy space defined on S^{n-1}. It is natural to ask the following question.

Question. Can one extend the estimate (1) to phases which are general rational functions?

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In 2003, Folch-Gabayet and Wright [3] showed that for general rational phases, the estimate (1) is not true. Meanwhile, they considered the rational phases of the form $P(x)+1/Q(x)$, where P and Q are real polynomials with $Q(0)=0$, and for $\Omega \in L\log L(S^{n-1})$, obtained the following estimate:

$$\left|\text{p. v.}\int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))}K(x)\,dx\right| \le A, \tag{2}$$

where A depends on $\|\Omega\|_{L\log L(S^{n-1})}$, n and the degrees of P and Q, but not otherwise on the coefficients of P and Q. It is well known that

$$L^{\infty}\left(S^{n-1}\right) \subsetneqq \bigcup_{r>1} L^r\left(S^{n-1}\right) \subsetneqq L\log L\left(S^{n-1}\right).$$

Therefore, the estimate (2) essentially improved and generalized the corresponding results in [6,8,9].

On the other hand, to study the mapping properties of singular integrals with rough kernels on $L^p(\mathbb{R}^n)$, Jiang and Lu introduced the following block spaces $B_q^{0,\nu}(S^{n-1})$ for $\nu > -1$ and $q > 1$ (see [5] for the details of block spaces).

Definition 1 ([5]). *A q-block on S^{n-1} is an L^q-function b ($1 < q \le \infty$) that satisfies*

$$\operatorname{supp}(b) \subseteq Q, \tag{i}$$

$$\|b\|_{L^q(S^{n-1})} \le |Q|^{1/q-1}, \tag{ii}$$

where $Q = S^{n-1} \cap \{y \in \mathbb{R}^n : |y-\zeta| < \rho$ for some $\zeta \in S^{n-1}$ and $\rho \in (0,1]\}$.

Definition 2 ([5]). *For $\nu > -1$ and $q > 1$, the block spaces $B_q^{0,\nu}$ on S^{n-1} are defined by*

$$B_q^{0,\nu}\left(S^{n-1}\right) = \left\{\Omega \in L^1\left(S^{n-1}\right) : \Omega(y') = \sum_s \lambda_s b_s(y'),\ M_q^{0,\nu}(\{\lambda_s\}) < \infty\right\}.$$

where each λ_s is a complex number, each b_s is a q-block supported in Q_s,

$$M_q^{0,\nu}(\{\lambda_s\}) = \sum_s |\lambda_s| \left\{1 + \left(\log^+ \frac{1}{|Q_s|}\right)^{1+\nu}\right\},$$

and

$$\|\Omega\|_{B_q^{0,\nu}(S^{n-1})} = \inf\left\{M_q^{0,\nu}(\{\lambda_s\}) : \Omega(x') = \sum_s \lambda_s b_s(x')\right\}.$$

It is easy to check that

$$B_q^{0,\nu_1}\left(S^{n-1}\right) \subsetneqq B_q^{0,\nu_2}\left(S^{n-1}\right), \quad \forall\ \nu_1 > \nu_2 > -1.$$

Moreover, it follows from [4,10] that for any $q > 1$,

$$\bigcup_{r>1} L^r\left(S^{n-1}\right) \subsetneqq B_q^{0,\nu}\left(S^{n-1}\right) \subset H^1\left(S^{n-1}\right) + L\left(\log^+ L\right)^{1+\nu}\left(S^{n-1}\right), \quad \forall\ \nu > -1,$$

and $B_q^{0,\nu}(S^{n-1}) \nsubseteq L\log^+ L(S^{n-1})$ for any $\nu \in (-1,0)$, in particular,

$$\bigcup_{r>1} L^r\left(S^{n-1}\right) \subsetneqq B_q^{0,0}\left(S^{n-1}\right),\ L\log^+ L\left(S^{n-1}\right) \subsetneqq H^1\left(S^{n-1}\right),$$

but the relationship between $B_q^{0,0}(S^{n-1})$ and $L\log^+ L(S^{n-1})$ remains open. Therefore, it is interesting to establish the estimate (2) under the assumption of that $\Omega \in B_q^{0,0}(S^{n-1})$, and more generally $\Omega \in H^1(S^{n-1})$.

In this paper, we will establish the estimate (2) provided that $\Omega \in B_q^{0,0}(S^{n-1})$. As for the case of that $\Omega \in H^1(S^{n-1})$, which is more interesting, it is still open. Our results can be formulated as follows.

Theorem 3. *Suppose that $K(x) = \Omega(x)/|x|^n$, where $\Omega \in B_q^{0,0}(S^{n-1})$ is homogeneous of degree zero with mean value zero and $q > 1$, P and Q are real polynomials with $Q(0) = 0$. Then*

$$\left| p.\ v. \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} K(x)\,dx \right| \le B, \tag{3}$$

where B depends on $\|\Omega\|_{B_q^{0,0}(S^{n-1})}$, n and the degrees of P and Q, but not otherwise on the coefficients of P and Q.

Remark 4. Employing the arguments in [3, Proposition 1.4], the following result shows that the requirement $Q(0) = 0$ in Theorem 3 is most likely not necessary.

Theorem 5. *With K and P as in Theorem 3, but now $Q(x) = a + v \cdot x$, where $a \in \mathbb{R}$ and $v \in \mathbb{R}^n$*

$$\left| p.v. \int_{\mathbb{R}^n} e^{i(P(x)+1/Q(x))} K(x)\,dx \right| \le B, \tag{4}$$

where B depends on $\|\Omega\|_{B_q^{0,0}(S^{n-1})}$, n and the degree of P but not otherwise on the coefficients of P.

As an immediate consequence of Theorems 3 and 5, we have the following result.

Corollary 6. *With K and Q as in Theorem 3 or 5, but now $P : \mathbb{R}^n \to \mathbb{R}^m$ being a polynomial mapping, convolution with the distribution*

$$L(\phi) = p.v. \int_{\mathbb{R}^n} \phi(P(x)) e^{i/Q(x)} K(x)\,dx, \qquad \phi \in \mathscr{S}(\mathbb{R}^m),$$

is bounded on $L^2(\mathbb{R}^m)$.

Employing the arguments in proving [3, Theorem 1.1], the main ingredient of the proof of Theorem 3 or 5 in the current paper is to establish the following integral estimate, which has an independent interest.

Proposition 7. *Let $\Omega \in B_q^{0,0}(S^{n-1})$ for some $q > 1$, $P(x) = \sum_{|\alpha|=d} c_\alpha x^\alpha$ be a homogeneous polynomial of degree d on $\mathbb{R}^n$. Write $m_P = \sum_{|\alpha|=d} |c_\alpha|$. Then*

$$\int_{S^{n-1}} |\Omega(\omega)| \cdot \left| \log\left(\frac{|P(\omega)|}{m_P} \right) \right| d\sigma(\omega) \lesssim C_d \|\Omega\|_{B_q^{0,0}(S^{n-1})}. \tag{5}$$

The rest of this paper is organized as follows. In Section 2 we will give some auxiliary lemmas. The proofs of our main results will be given in Section 3. We remark that some ideas in our arguments are taken from [3].

Finally, we make some conventions on notation. Throughout this paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Let A, B be complex-valued quantities. We use $A \lesssim B$ or $A = O(B)$ to denote the estimate $|A| \le C|B|$. We use $A \sim B$ to denote the estimate $A \lesssim B \lesssim A$. For $1 \le \gamma \le \infty$, γ' is the conjugate index of γ, and $1/\gamma + 1/\gamma' = 1$.

2. Preliminaries

In this section, we recall and establish some auxiliary lemmas, which will be used in our arguments later.

Lemma 8 ([3]). *For any polynomial $Q(r) = \sum_{j=1}^{d} b_k r^k$ on $\mathbb{R}^+$, there is a finite collection $\{G_j\}_{j=1}^M$ of disjoint intervals, called "gaps", of $\mathbb{R}^+$ with $M = O(1)$ such that*

(1) *The complement $\mathbb{R}^+ \setminus \bigcup_{j=1}^M \{G_j\}$ is the union of $M-1$ disjoint "dyadic" intervals, that is, the ratio of the endpoints of such intervals is ~ 1.*

(2) *For each G_j, there is a $k = k_j, 1 \le k_j \le d$, such that for $r \in G_j$,*

$$|Q(r)| \sim \left|b_{k_j}\right| r^{k_j} \quad \text{and} \quad |Q^{'}(r)| \sim \left|b_{k_j}\right| r^{k_j - 1}$$

Also if L_j and R_j denote the left and right endpoints of G_j respectively, then

(i) *if $R_j < \infty$, then $R_j = C_d[|b_l|/|b_m|]^{1/m-l}$ for some $1 \le l < m \le d$ and*
(ii) *if $L_j > 0$, then $L_j = C_d[|b_r|/|b_s|]^{1/s-r}$ for some $1 \le r < s \le d$.*

Lemma 9 ([8]). *Suppose ϕ is real-valued and smooth on (a,b), and that $|\phi^{(k)}| \ge \lambda > 0$ for all $t \in (a,b)$. Then*

$$\left|\int_a^b e^{i\phi(t)}dt\right| \le C_k\lambda^{-1/k}$$

when either $k \ge 2$, or $k = 1$ and $\phi'(t)$ is monotonic.

Lemma 10. *Let $\gamma > 1, \Omega \in L^\gamma(S^{n-1})$ and $P(x) = \sum_{|\alpha|=d} c_\alpha x^\alpha$ be a homogeneous polynomial of degree d on $\mathbb{R}^n$. Write $m_P = \sum_{|\alpha|=d}|c_\alpha|$. Then*

$$\int_{S^{n-1}} |\Omega(\omega)|\cdot\left|\log\left(\frac{|P(\omega)|}{m_P}\right)\right| d\sigma(\omega) \lesssim C_d\gamma'\|\Omega\|_{L^\gamma(S^{n-1})}. \tag{6}$$

Proof. We may assume $m_P = 1$. By the Hölder inequality,

$$\int_{S^{n-1}} |\Omega(\omega)|\cdot\left|\log\left(|P(\omega)|\right)\right| d\sigma(\omega) \le \|\Omega\|_{L^\gamma(S^{n-1})}\left(\int_{S^{n-1}}\left|\log\left(|P(\omega)|\right)\right|^{\gamma'} d\sigma(\omega)\right)^{1/\gamma'}.$$

And

$$\begin{aligned}
\left(\int_{S^{n-1}}\left|\log\left(|P(\omega)|\right)\right|^{\gamma'} d\sigma(\omega)\right)^{1/\gamma'} &\lesssim \left(\int_{1/2}^1 r^{n-1}\int_{S^{n-1}}\left|\log\left(|P(\omega)|\right)\right|^{\gamma'} d\sigma(\omega)dr\right)^{1/\gamma'}\\
&\lesssim \left(\sum_{k\geqslant 0}\left(k\gamma'\right)^{\gamma'}\int_{1/2}^1 r^{n-1}dr\int_{\left\{\omega\in S^{n-1}:\, r^d|P(\omega)|\leqslant 2^{-k\gamma'}\right\}} d\sigma(\omega)\right)^{1/\gamma'}\\
&\leqslant \gamma'\left(\sum_{k\geqslant 0} k^{\gamma'}\int_{\left\{\frac12\leqslant|x|\leqslant 1:\,2^{-k\gamma'-1}\leqslant|P(x)|\leqslant 2^{-k\gamma'}\right\}} dx\right)^{1/\gamma'}\\
&\leqslant \gamma'\left(\sum_{k\geqslant 0} k^{\gamma'}\left|\left\{\frac12\leqslant|x|\leqslant 1:\,|P(x)|\leqslant 2^{-k\gamma'}\right\}\right|\right)^{1/\gamma'},
\end{aligned}$$

which reduces matters to obtaining uniform sublevel set estimates for P under the normalisation $m_P = 1$. Using the fact that all norms are equivalent on the space of polynomials of degree at most d, we can find a derivative ∂^α, $0 \le |\alpha| \le d$, such that $1 \lesssim |\partial^\alpha P(x)|$ on $|x| \leqslant 1$. If $\alpha = 0$, then the above sublevel sets are empty for large k and so we may assume $\alpha > 0$. In this case, using the mean value theorem, one can show that $|\{|x| \leqslant 1 : |P(x)| \leqslant 2^{-k\gamma'}\}| \lesssim 2^{-k\gamma'/|\alpha|}$ (see e.g. [2]). Thus

$$\begin{aligned}
\left(\sum_{k\geqslant 1} k^{\gamma'}\left|\left\{\frac12\leqslant|x|\leqslant 1:\,|P(x)|\leqslant 2^{-k\gamma'}\right\}\right|\right)^{1/\gamma'} &\leqslant \left(\sum_{k\geqslant 1} k^{\gamma'}2^{-k\gamma'/|\alpha|}\right)^{1/\gamma'}\\
&\le \sum_{k\geqslant 1} k2^{-k/d} \lesssim C_d.
\end{aligned}$$

This implies the desired conclusion and completes the proof of Lemma 10. □

3. Proofs of Main Results

In this section, we present the proofs of Proposition 7 and Theorem 3.

Proof of Proposition 7. Since $\Omega \in B_q^{0,0}(S^{n-1})$, we know by Definition 2 that there is a decomposition: $\Omega(x') = \sum_s \lambda_s b_s(x')$, where each b_s is a q-block, supported in Q_s and

$$\sum_s |\lambda_s| \left(1 + \log^+ \frac{1}{|Q_s|}\right) < \infty.$$

Therefore,

$$\begin{aligned}\int_{S^{n-1}} |\Omega(\omega)| \cdot \left|\log\left(\frac{|P(\omega)|}{m_P}\right)\right| d\sigma(\omega) &\le \sum_s |\lambda_s| \int_{S^{n-1}} |b_s(\omega)| \cdot \left|\log\left(\frac{|P(\omega)|}{m_P}\right)\right| d\sigma(\omega) \\ &\le \left(\sum_{|Q_s| \ge e^{-q'}} + \sum_{|Q_s| < e^{-q'}}\right) |\lambda_s| \int_{S^{n-1}} |b_s(\omega)| \cdot \left|\log\left(\frac{|P(\omega)|}{m_P}\right)\right| d\sigma(\omega) \\ &=: I + II.\end{aligned}$$

Recall that for each b_s, $\mathrm{supp}(b_s) \subset Q_s$ and $\|b_s\|_{L^q(S^{n-1})} \le |Q_s|^{1/q-1} = |Q_s|^{-1/q'}$. If $|Q_s| \ge e^{-q'}$, we take $\gamma = q$ and obtain

$$\gamma' \|b_s\|_{L^\gamma(S^{n-1})} \le q' \|b_s\|_{L^q(S^{n-1})} \le q' |Q_s|^{-1/q'} \lesssim 1;$$

if $|Q_s| < e^{-q'}$, we take $\gamma = \log|Q_s|/(1+\log|Q_s|)$, then $1 < \gamma < q$, $\gamma' = \log(1/|Q_s|)$ and

$$\gamma' \|b_s\|_{L^\gamma(S^{n-1})} \le \log \frac{1}{|Q_s|} \|b_s\|_{L^q(S^{n-1})} |Q_s|^{1/\gamma - 1/q} \le \log \frac{1}{|Q_s|} |Q_s|^{-1/\gamma'} \lesssim \log \frac{1}{|Q_s|}.$$

These, combining with Lemma 10, lead to

$$I \le C_d \sum_{|Q_s| \ge e^{-q'}} |\lambda_s| q' \|b_s\|_{L^q(S^{n-1})} \lesssim C_d \sum_{|Q_s| \ge e^{-q'}} |\lambda_s|,$$

and for $\gamma = \log|Q_s|/(1+\log|Q_s|)$,

$$II \le C_d \sum_{|Q_s| < e^{-q'}} |\lambda_s| \gamma' \|b_s\|_{L^\gamma(S^{n-1})} \lesssim C_d \sum_{|Q_s| > e^{-q'}} |\lambda_s| \log \frac{1}{|Q_s|}.$$

Consequently,

$$\int_{S^{n-1}} |\Omega(\omega)| \cdot \left|\log\left(\frac{|P(\omega)|}{m_P}\right)\right| d\sigma(\omega) \lesssim C_d \sum_s |\lambda_s| \left(1 + \log \frac{1}{|Q_s|}\right),$$

which completes the proof of Proposition 7. □

Proof of Theorem 3. The arguments are completely similar to those in proving [3, Theorem 1.1]. The only difference is replacing [3, Lemma 2.2] by Proposition 7 in the current setting. For completeness, we present the details as follows.

We may assume $P(0) = 0$. Using polar coordinates write the integral in (3) as

$$I = \int_{S^{n-1}} \Omega(\omega) \int_0^\infty e^{i[P_\omega(r) + 1/Q_\omega(r)]} \frac{1}{r} dr d\sigma(\omega),$$

where $Q(x) = Q_\omega(r) = \sum_{j=1}^{d'} q_j(\omega) r^j$, $P(x) = P_\omega(r) = \sum_{k=1}^{d} p_j(\omega) r^k$ and p_j, q_k are homogeneous polynomials of degree j and k. Using Lemma 8, we may write $I = \sum I_{j,k} + O(1)$, where

$$I_{j,k} = \int_{S^{n-1}} \Omega(\omega) \int_{G_j \cap F_k} e^{i[P_\omega(r) + 1/Q_\omega(r)]} \frac{1}{r} dr d\sigma(\omega).$$

Here $\{G_j\}$ and $\{F_k\}$ are the "gaps" of $Q_\omega(r)$ and $P_\omega(r)$, respectively. Note that although the inner integral of $I_{j,k}$ depends on ω in a complicated way, we know the form of the endpoints of G_j and F_k as given by Lemma 8 and so it is at least measurable as a function of ω. It suffices to bound each $I_{j,k}$ separately.

We have $|Q_\omega(r)| \sim |q_{j_l}(\omega)| r^{j_l}$ and $|Q'_\omega(r)| \sim |q_{j_l}(\omega)| r^{j_l - 1}$ on G_j, for some $1 \le j_l \le d'$, and $|P_\omega(r)| \sim |p_{k_m}(\omega)| r^{k_m}$ and $|P'_\omega(r)| \sim |p_{k_m}(\omega)| r^{k_m - 1}$ on F_k, for some $1 \le k_m \le d$.

Therefore away from where $r^{j_l+k_m} \sim (|p_{k_m}(\omega)| \cdot |q_{j_l}(\omega)|)^{-1}$ the size of the phase $\phi_\omega(r) = P_\omega(r) + 1/Q_\omega(r)$ and its derivative is understood. In fact, on $R_{j,k} = G_j \cap F_k \cap [C(|p_{k_m}(\omega)| \cdot |q_{j_l}(\omega)|)^{-1/(k_m+j_l)}, \infty)$ (for C large enough), we have

$$\left|\phi_\omega(r)\right| \sim \left|p_{k_m}(\omega)\right| r^{k_m} \quad \text{and} \quad \left|\phi'_\omega(r)\right| \sim \left|p_{k_m}(\omega)\right| r^{k_m-1}.$$

An application of van der Corput's Lemma 9 shows

$$\left|\int_{\{r\in R_{j,k}:r\ge\Theta\}} e^{i\phi_\omega(r)}\frac{1}{r}dr\right| = O(1),$$

where $\Theta = |p_{k_m}(\omega)|^{1/k_m}$. Since we are applying Lemma 9 with $k=1$, we need to first split the integration of the above integral into $O(1)$ intervals, where $\phi'_\omega(r)$ is monotone. In the complementary interval, $r<\Theta$, due to the size of $\phi_\omega(r)$ on $R_{j,k}$, we see that

$$\left|\int_{\{r\in R_{j,k}:r<\Theta\}} \left[e^{i\phi_\omega(r)}\frac{1}{r}-1\right]dr\right| = O(1).$$

Therefore for the part of $I_{j,k}$ over $R_{j,k}$,

$$\int_{S^{n-1}}\Omega(\omega)\int_{R_{j,k}} e^{i\phi_\omega(r)}\frac{1}{r}dr\,d\sigma(\omega) = \int_{S^{n-1}}\Omega(\omega)\int_{\{r\in R_{j,k}:r<\Theta\}}\frac{dr}{r}d\sigma(\omega)+O(1).$$

Similarly for $L_{j,k} = G_j\cap F_k\cap(-\infty, \delta(|p_{k_m}(\omega)|\cdot|q_{j_l}(\omega)|)^{-1/(k_m+j_l)}]$ (for δ small enough), we have

$$\int_{S^{n-1}}\Omega(\omega)\int_{L_{j,k}} e^{i\phi_\omega(r)}\frac{1}{r}dr\,d\sigma(\omega) = \int_{S^{n-1}}\Omega(\omega)\int_{\{r\in L_{j,k}:r\ge\Lambda\}}\frac{dr}{r}d\sigma(\omega)+O(1),$$

where $\Lambda = |q_{j_l}(\omega)|^{1/j_l}$. Therefore

$$\begin{aligned} I_{j,k} &= \int_{S^{n-1}}\Omega(\omega)\int_{G_j\cap F_k} e^{i[P_\omega(r)+1/Q_\omega(r)]}\frac{1}{r}dr\,d\sigma(\omega)\\ &= \int_{S^{n-1}}\Omega(\omega)\int_{\{r\in R_{j,k}:r\le\Theta\}}\frac{dr}{r}d\sigma(\omega)\\ &\quad+\int_{S^{n-1}}\Omega(\omega)\int_{\{r\in L_{j,k}:r\ge\Lambda\}}\frac{dr}{r}d\sigma(\omega)+O(1)\end{aligned} \tag{7}$$

and these two last integrals can be shown to be $O(1)$ by repeatedly applying Proposition 7.

In fact, by the structures of $R_{j,k}$ and L_{jk}, the integrals in (7):

$$\int_{S^{n-1}}\Omega(\omega)\int_{\{r\in R_{j,k}:r\le\Theta\}}\frac{dr}{r}d\sigma(\omega), \quad\text{and}\quad \int_{S^{n-1}}\Omega(\omega)\int_{\{r\in L_{j,k}:r\ge\Lambda\}}\frac{dr}{r}d\sigma(\omega)$$

can be written in the form

$$\int_{S^{n-1}}\Omega(\omega)\int_{E(\omega)}\frac{dr}{r}d\sigma(\omega),$$

where $E(\omega)$ is the intersection of $O(1)$ intervals of the form $[a(\omega),\infty)$ or $(-\infty, a(\omega)]$ with

$$a(\omega)\in\Big\{\big(\left|p_{k_1}(\omega)\right|/\left|p_{k_2}(\omega)\right|\big)^{1/(k_2-k_1)}, \big(\left|q_{j_1}(\omega)\right|/\left|q_{j_2}(\omega)\right|\big)^{1/(j_2-j_1)},$$
$$\left|p_k(\omega)\right|^{-1/k}, \left|q_j(\omega)\right|^{-1/j}, \big(\left|p_k(\omega)\right|\cdot\left|q_j(\omega)\right|\big)^{-1/(j+k)}\Big\}.$$

Without loss of the generality, we may say that $E(\omega)$ is the intersection of M half infinite intervals such as $[a(\omega),\infty)$ or $(-\infty, a(\omega)]$. Let us write $E(\omega) = [a(\omega),\infty)\cap E'(\omega)$, where $a(\omega)$, say, is $(|p_{k_1}(\omega)|/|p_{k_2}(\omega)|)^{1/(k_2-k_1)}$ and $E'(\omega)$ is the intersection of $M-1$ half infinite intervals as in $E(\omega)$. We can then use Proposition 7 to write

$$\left|\int_{S^{n-1}}\Omega(\omega)\int_{E(\omega)}\frac{dr}{r}d\sigma(\omega)\right| \le \left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,\infty)\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right|+O(1),$$

where $A_1 = (m_{p_{k_1}}/m_{p_{k_2}})^{1/(k_2-k_1)}$, which is independent of ω. Indeed,

$$\left|\int_{S^{n-1}}\Omega(\omega)\int_{[a(\omega),\infty)\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right| \le \left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,\infty)\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right| + \left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,a(\omega)]\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right|,$$

and for the second integral, by Lemma 9 and Proposition 7, we have

$$\begin{aligned}\left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,a(\omega)]\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right| &\le \left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,a(\omega)]}\frac{dr}{r}d\sigma(\omega)\right| \\ &\le \frac{1}{|k_2-k_1|}\left[\int_{S^{n-1}}|\Omega(\omega)|\left|\log\left(\frac{|p_{k_1}(\omega)|}{m_{p_{k_1}}}\right)\right|d\sigma(\omega)+\int_{S^{n-1}}|\Omega(\omega)|\left|\log\left(\frac{|p_{k_2}(\omega)|}{m_{p_{k_2}}}\right)\right|d\sigma(\omega)\right] \\ &\lesssim O(1).\end{aligned}$$

For the other forms of $a(\omega)$, the argument is similar.

Similarly, we have

$$\left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,\infty)\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right| \le \left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,\infty)\cap[A_2,\infty)\cap E''(\omega)}\frac{dr}{r}d\sigma(\omega)\right| + O(1),$$

or

$$\left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,\infty)\cap E'(\omega)}\frac{dr}{r}d\sigma(\omega)\right| \le \left|\int_{S^{n-1}}\Omega(\omega)\int_{[A_1,\infty)\cap(-\infty,A_2]\cap E''(\omega)}\frac{dr}{r}d\sigma(\omega)\right| + O(1),$$

where A_2 is independent of ω, and $E''(\omega)$ is the intersection of $M-2$ half infinite intervals as in $E(\omega)$. Continuing this process, after M iterations, we obtain that

$$\left|\int_{S^{n-1}}\Omega(\omega)\int_{E(\omega)}\frac{dr}{r}d\sigma(\omega)\right| \le \left|\int_{S^{n-1}}\Omega(\omega)\int_{E}\frac{dr}{r}d\sigma(\omega)\right| + O(1),$$

where E is the intersection of M intervals of the form $[A,\infty)$ or $(-\infty,A]$, where A is independent of ω. Therefore, by the mean value zero of Ω on S^{n-1}, we have

$$\int_{S^{n-1}}\Omega(\omega)\int_{E}\frac{dr}{r}d\sigma(\omega) = 0.$$

This implies that

$$\left|\int_{S^{n-1}}\Omega(\omega)\int_{E(\omega)}\frac{dr}{r}d\sigma(\omega)\right| \le \left|\int_{S^{n-1}}\Omega(\omega)\int_{E}\frac{dr}{r}d\sigma(\omega)\right| + O(1) = O(1),$$

and completes the proof of Theorem 3. ☐

Proof of Theorem 5. By replacing [3, Lemma 2.2] by Proposition 7, the proof is similar to the proof of [3, Proposition 1.4]. We omit the details here. ☐

References

[1] H. M. Al-Qassem, L. C. Cheng, A. Fukui, Y. Pan, "Weighted Estimates for Oscillatory Singular Integrals", *J. Funct. Spaces Appl.* **2013** (2013), article no. 543748 (3 pages).

[2] A. Carbery, M. Christ, J. Wright, "Multidimensional van der Corput and sublevel set estimates", *J. Am. Math. Soc.* **12** (1999), no. 4, p. 981-1015.

[3] M. Folch-Gabayet, J. Wright, "An estimation for a family of oscillatory integrals", *Stud. Math.* **154** (2003), no. 1, p. 89-97.

[4] M. Keitoku, E. Sato, "Block spaces on the unit sphere in $\mathbb{R}^n$", *Proc. Am. Math. Soc.* **119** (1993), no. 2, p. 453-455.

[5] S. Lu, M. H. Taibleson, G. Weiss, *Spaces Generated by Blocks*, Publishing House of Beijing Normal University, 1989.

[6] M. Papadimitrakis, I. R. Parissis, "Singular oscillatory integrals on $\mathbb{R}^n$", *Math. Z.* **266** (2010), no. 1, p. 169-179.

[7] I. R. Parissis, “A sharp bound for the Stein–Wainger oscillatory integral”, *Proc. Am. Math. Soc.* **136** (2008), no. 3, p. 963-972.
[8] E. M. Stein, “Oscillation integrals in Fourier analysis”, in *Beijing Lectures in Harmonic Analysis (Beijing, 1984)*, Annals of Mathematics Studies, vol. 112, Princeton University Press, 1986, p. 307-355.
[9] E. M. Stein, S. Wainger, “The estimation of an integral arising in multiplier transformations”, *Stud. Math.* **35** (1970), p. 101-104.
[10] X. Ye, X. Zhu, “A note on certain block spaces on the unit sphere”, *Acta Math. Sin., Engl. Ser.* **22** (2006), no. 6, p. 1843-1846.