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Picard Groups of Algebraic Groups and an Affineness Criterion

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Abstract. We prove that an algebraic group over a field k is affine precisely when its Picard group is torsion, and show that in this case the Picard group is finite when k is perfect, and the product of a finite group of order prime to p and a p -primary group of finite exponent when k is imperfect of characteristic p .

Résumé. Nous prouvons qu'un groupe algébrique sur un corps k est affine si et seulement si son groupe de Picard est de torsion, et que dans ce cas, le groupe de Picard est fini si k est parfait, et le produit d'un groupe fini d'ordre premier à p par un p -groupe d'exposant fini lorsque k est imparfait de caractéristique p .

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The main purpose of this short paper is to prove the following two results on algebraic groups over arbitrary fields.

Theorem 1. *Let G be a finite type group scheme over a field k . Then G is affine if and only if $\text{Pic}(G)$ is torsion.*

Theorem 2. *Let G be an affine group scheme of finite type over a field k .*

- (i) *If k is perfect, then $\text{Pic}(G)$ is finite.*
- (ii) *If $\text{char}(k) = p > 0$, then $\text{Pic}(G)$ is the product of a finite group and a p -primary group of finite exponent.*

In general, affine algebraic groups over imperfect fields need not have finite Picard groups. Indeed, [7, Prop. 5.9] shows that, when k is either separably closed (and imperfect) or a local function field, then this fails for almost every nontrivial k -form of \mathbf{G}_a . (For examples of nontrivial forms of \mathbf{G}_a over arbitrary imperfect fields, see, for example, [7, Ex. 5.8].)

We now turn to the proofs of the theorems above.

Lemma 3. *Let X be a scheme over a field k , and let $k' | k$ be a field extension such that $(k')^{p^n} \subset k$. Then for all i , the kernel of the pullback map*

$$H_{\text{ét}}^i(X, \mathbf{G}_m) \rightarrow H_{\text{ét}}^i(X_{k'}, \mathbf{G}_m)$$

is p^n -torsion.

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Proof. Let $f: \text{Spec}(k') \rightarrow \text{Spec}(k)$ denote the obvious morphism. The composition

$$(\mathbf{G}_m)_k \rightarrow f_*((\mathbf{G}_m)_{k'}) \xrightarrow{[p^n]} (\mathbf{G}_m)_k \tag{1}$$

is the p^n -th power map, hence induces multiplication by p^n on $H^i(X, \mathbf{G}_m)$. But pushforward through a radicial map is an exact functor on the category of étale sheaves, hence we have a canonical identification

$$H^i(X, f_*(\mathbf{G}_m)) = H^i(X_{k'}, \mathbf{G}_m)$$

which identifies the map on cohomology induced by the first map in (1) with the pullback map of the Lemma 3. It follows that anything killed by pullback is killed by p^n . \square

Proof of Theorem 2. We are free to pass to a finite field extension. Indeed, it suffices to show that we may pass to a finite separable extension or (when k is imperfect) a finite purely inseparable extension. The separable case follows from [5, Th. 1.13], while the purely inseparable case, which is only relevant when k is imperfect, follows from Lemma 3. We may in particular assume that $G_{\text{red}} \subset G$ is a smooth k -subgroup scheme. By [8, Lem. 2.2.9], the map $\text{Pic}(G) \rightarrow \text{Pic}(G_{\text{red}})$ is an isomorphism, so we may assume that G is smooth. Choose a finite extension k'/k such that $G_{k'} = \coprod G_i$, where the G_i are the connected components of $G_{k'}$, and such that $G_i(k') \neq \emptyset$ for each i . Renaming k' as k , we have that $G_i \simeq G^0$ as k -schemes, hence $\text{Pic}(G) \simeq \prod_i \text{Pic}(G^0)$. We may therefore replace G by G^0 , so we may assume that G is smooth and connected. Finally, passing to a finite purely inseparable extension (in the imperfect case) and applying [1, Ch. V, Th. 18.2 (ii)], we may assume that G is unirational, so $\text{Pic}(G)$ is finite by [7, Th. 1.3]. \square

Let G be a k -group scheme, and let $m, \pi_i: G \times G \rightarrow G$ ($i = 1, 2$) denote the multiplication and projection maps, respectively. We say that an element $\mathcal{L} \in \text{Pic}(G)$ is *primitive* if $m^*(\mathcal{L}) = \pi_1^*(\mathcal{L}) + \pi_2^*(\mathcal{L})$.

Lemma 4. *Given an exact sequence*

$$1 \longrightarrow G' \xrightarrow{j} G \longrightarrow G'' \longrightarrow 1 \tag{2}$$

of smooth connected group schemes over a field k , and a k -character $\chi: G' \rightarrow \mathbf{G}_m$, consider the pushout E of the sequence along χ , which is a \mathbf{G}_m -torsor over G'' . Then E is a primitive torsor.

Proof. We first claim that the conjugation action of G on $\widehat{G'}(k) := \text{Hom}_{k\text{-gps}}(G', \mathbf{G}_m)$ is trivial. Indeed, $\mathcal{D}G' \subset G'$ is a universally characteristic k -subgroup through which the character group factors, so in order to prove the claim we may replace the sequence (2) by the sequence obtained by forming the quotient by this subgroup, and we may assume that G' is commutative. Let $T \subset G'$ be the maximal k -torus. Then T is a characteristic k -subgroup of G' , hence normal in G . Further, G'/T has no nontrivial characters, hence has no nontrivial k -characters, so $\widehat{G'}(k) \subset \widehat{T}(k)$, and it suffices to check that the G -conjugation action on $\widehat{T}(k)$ is trivial. But in fact we claim that T is central in G , which is sufficient. Indeed, the automorphism functor $\text{Aut}_{T/k}$ of T is represented by an étale k -scheme. (This is well-known, but is easily seen by using Galois descent to pass to the case when $T \simeq \mathbf{G}_m^n$, in which case the automorphism functor is $\text{GL}_n(\mathbf{Z})$.) Conjugation induces a k -homomorphism $G \rightarrow \text{Aut}_{T/k}$ from a connected to an étale k -group scheme, which is therefore constant. This proves that the G -conjugation action on $\widehat{G'}(k)$ is trivial.

Now suppose we are in the situation of the lemma. We must show that there is an isomorphism $m^*(E) \simeq \pi_1^*(E) + \pi_2^*(E)$ of \mathbf{G}_m -torsors over $G'' \times G''$. This is equivalent to constructing a map $f: E \times E \rightarrow E$ such that $f(t_1 \cdot e_1, t_2 \cdot e_2) = (t_1 t_2) \cdot f(e_1, e_2)$, and such that the following diagram commutes:

$$\begin{array}{ccc} E \times E & \xrightarrow{f} & E \\ \downarrow & & \downarrow \\ G'' \times G'' & \xrightarrow{m} & G'' \end{array}$$

The torsor E may be described explicitly as $E = (\mathbf{G}_m \times G)/i(G')$, where i is the map $g' \mapsto (\chi(g'), j(g')^{-1})$ with the obvious \mathbf{G}_m -action and map to G' . Then one may define a map $g: (\mathbf{G}_m \times G) \times (\mathbf{G}_m \times G) \rightarrow \mathbf{G}_m \times G$ by the formula

$$(t_1, g_1) \times (t_2, g_2) \mapsto (t_1 t_2, g_1 g_2).$$

We claim that this descends to a map $f: E \times E \rightarrow E$. Once this is verified, the map clearly satisfies the required conditions above. A straightforward check reveals that what one needs to verify is that $\chi(g^{-1}g'g) = \chi(g')$ for all $g \in G$ and $g' \in G'$, and this is exactly the triviality of the G -conjugation action on $\widehat{G'}(k)$ proved above. \square

Lemma 5. *Given an exact sequence*

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{q} G'' \longrightarrow 1 \tag{3}$$

of smooth connected group schemes over a field k , then every element of

$$\ker(q^*: \text{Pic}(G'') \rightarrow \text{Pic}(G))$$

is primitive.

Proof. By descent theory, $\ker(q^*)$ may be identified with the Čech cohomology group $\check{H}^1(q, \mathbf{G}_m)$ associated to the fppf cover q . This group is the middle cohomology of the sequence

$$\Gamma(G, \mathbf{G}_m) \xrightarrow{p_1^* - p_2^*} \Gamma(G \times_{G''} G, \mathbf{G}_m) \xrightarrow{p_{23}^* - p_{13}^* + p_{12}^*} \Gamma(G \times_{G''} G \times_{G''} G, \mathbf{G}_m), \tag{4}$$

where the p_i and p_{ij} denote the obvious projections. In particular, we obtain a surjection from the kernel of the second map above to $\ker(q^*)$. We have isomorphisms

$$\begin{aligned} G \times_k G' &\xrightarrow{\sim} G \times_{G''} G, & (g, g') &\mapsto (g, g g') \\ G \times_k G' \times_k G' &\xrightarrow{\sim} G \times_{G''} G \times_{G''} G, & (g, g'_1, g'_2) &\mapsto (g, g g'_1, g g'_1 g'_2). \end{aligned}$$

Under these isomorphisms, the second map in the sequence (4) is transported to the map

$$\Gamma(G \times_k G', \mathbf{G}_m) \xrightarrow{f_{23}^* - f_{13}^* + f_{12}^*} \Gamma(G \times_k G' \times_k G', \mathbf{G}_m), \tag{5}$$

where the f_{ij} are defined by the following formulas:

$$(g, g'_1, g'_2) \mapsto \begin{cases} (g g'_1, g'_2), & f_{23} \\ (g, g'_1 g'_2), & f_{13} \\ (g, g'_1), & f_{12}. \end{cases}$$

Now suppose given $h \in \Gamma(G \times_k G', \mathbf{G}_m)$ which lies in the kernel of the map (5). Then restricting the equality $(h \circ f_{23})(h \circ f_{13})^{-1}(h \circ f_{12}) = 1$ to the point $(1, 1, 1)$ of $G \times_k G' \times_k G'$ shows that $h(1, 1) = 1$. It follows from the Rosenlicht Unit Theorem (more precisely, a corollary of that theorem [2, Cor. 1.2]) that h is a character of $G \times_k G'$: say $h = \chi \chi'$, where χ is a character of G and χ' a character of G' . A simple calculation shows that the fact that h lies in the kernel of the map (5) is equivalent to the identity $\chi(gg') = 1$ for all $g \in G, g' \in G'$. Thus $\chi = 1$ – that is, $h = \chi' \in \widehat{G'}(k)$ is a character of G' . We thus obtain a surjection

$$\widehat{G'}(k) \twoheadrightarrow \ker(q^*).$$

This surjection is functorial in the exact sequence, hence, via the identification of $\ker(q^*)$ with $\check{H}^1(q, \mathbf{G}_m)$, it is none other than the pushout map which sends a character $\chi: G' \rightarrow \mathbf{G}_m$ to the \mathbf{G}_m -torsor over G'' obtained by pushing out the sequence (3) along χ . (Actually, it could also be the inverse of this map, since the identification of $\ker(q^*)$ with $\check{H}^1(q, \mathbf{G}_m)$ is only well-defined up to a universal choice of sign.) It follows that any element of $\ker(q^*)$ is obtained by such a pushout, and Lemma 4 implies that such a pushout is primitive. \square

Lemma 6. *Suppose given an exact sequence*

$$1 \longrightarrow H \longrightarrow G \xrightarrow{q} A \longrightarrow 1$$

of smooth connected group schemes over a field k , with A a nonzero abelian variety. Then for any ample $\mathcal{L} \in \text{Pic}(A)$, the pullback $q^(\mathcal{L}) \in \text{Pic}(G)$ is not torsion.*

Proof. Replacing \mathcal{L} with a power of itself, it suffices to show that $q^*(\mathcal{L}) \neq 0$. But if it were 0, then Lemma 5 would imply that \mathcal{L} is primitive. In particular, the homomorphism $\phi_{\mathcal{L}}: A \rightarrow A$ defined by the formula $a \mapsto t_a^*(\mathcal{L}) \otimes \mathcal{L}^{-1}$ – where t_a denotes translation by a – is the 0 map. But by [6, Ch. II, §6, p. 60, Application I], the ampleness of \mathcal{L} is equivalent to the map $\phi_{\mathcal{L}}$ being an isogeny, and this yields a contradiction because A is nonzero. \square

Lemma 7. *Let k be a field, and suppose given an isogeny $\phi: G_1 \rightarrow G_2$ of finite type k -group schemes (that is, a faithfully flat k -homomorphism with finite kernel). If, in the diagram below, the rows are exact sequences of finite type k -group schemes with L_i affine and A_i abelian varieties ($i = 1, 2$), and L_1 smooth and connected, then the diagram of solid arrows extends uniquely to a commutative diagram with dotted arrows, and the map f is an isogeny.*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L_1 & \longrightarrow & G_1 & \xrightarrow{\pi} & A_1 & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi & & \downarrow f & & \\ 1 & \longrightarrow & L_2 & \longrightarrow & G_2 & \longrightarrow & A_2 & \longrightarrow & 1. \end{array}$$

Proof. That the diagram extends uniquely, if it extends at all, is clear. That it extends is equivalent to the claim that the composition $L_1 \hookrightarrow G_1 \xrightarrow{\phi} G_2 \rightarrow A_2$ is trivial, and this follows from the fact that any k -homomorphism from a smooth connected affine k -group to an abelian variety is trivial. Finally, it remains to check that f is an isogeny. It is surjective because ϕ is, so if it is not an isogeny then its kernel contains a nonzero abelian variety $B \subset \ker(f)$. Then $\pi^{-1}(B) \rightarrow L_2$ has finite kernel F , hence letting $G' := \pi^{-1}(B)/F$ and L' the image of $\pi^{-1}(B)$ in L_2 , we obtain a k -isomorphism $\psi: G' \xrightarrow{\sim} L'$ with the former an extension of a nonzero abelian variety C by a finite type affine k -group M , and the latter an affine k -group of finite type. Then $G'/M \simeq L'/\psi(M)$, with the former a nonzero abelian variety and the latter affine. This is a contradiction, so f is an isogeny. \square

Proof of Theorem 1. Theorem 2 implies that affine algebraic groups have torsion Picard groups, so it only remains to show that finite type k -groups which are not affine have non-torsion elements in their Picard groups. Because G is affine if and only if G^0 is, we may assume that G is connected.

First we treat the case in which k is perfect. Then $G_{\text{red}} \subset G$ is a smooth connected k -subgroup scheme, and G is affine precisely when G_{red} is, so we may assume that G is smooth and connected. Chevalley’s Theorem (which again uses the perfection of k) then furnishes an exact sequence

$$1 \longrightarrow L \longrightarrow G \longrightarrow A \longrightarrow 1$$

with L a smooth connected affine k -group scheme, and A an abelian variety. Because G is not affine by assumption, A is nonzero. Lemma 6 implies that any ample line bundle on A pulls back to a non-torsion element of $\text{Pic}(G)$.

Next we treat the case in which $\text{char}(k) = p > 0$. By [3, VII_A, Prop. 8.3], there is a normal infinitesimal k -subgroup scheme $I \trianglelefteq G$ such that $H := G/I$ is smooth (and necessarily connected, because G is). We claim that H is not affine. Indeed, $\pi: G \twoheadrightarrow H$ has affine kernel I . The I -torsor over H defined by π becomes trivial under pullback along the faithfully flat quasi-compact map $\pi: G \rightarrow H$. Therefore, after a faithfully flat quasi-compact base change, π becomes an affine morphism. It follows that π is itself affine [4, § 2.7.1(xiii)]. Therefore, if H were affine, then G would also be, in violation of our assumption.

By Chevalley’s Theorem, $H_{k_{\text{perf}}}$ can be written as an extension of an abelian variety by a smooth connected affine k_{perf} -group scheme. It follows that there is such an extension over k^{1/p^n} for some integer $n \geq 0$. Extending scalars via the p^n -th-power isomorphism $k^{1/p^n} \xrightarrow{\sim} k$, we obtain an exact sequence

$$1 \longrightarrow L \longrightarrow H^{(p^n)} \longrightarrow A \longrightarrow 1$$

with L a smooth connected affine k -group and A/k a nonzero abelian variety (because H , hence also $H^{(p^n)}$, is not affine). Denote by q the composite map

$$G \rightarrow H \xrightarrow{F_{H/k}^{(p^n)}} H^{(p^n)} \rightarrow A,$$

where the map in middle is the n -fold relative Frobenius map. Let \mathcal{L} be an ample line bundle on A . We claim that the pullback $q^*(\mathcal{L}) \in \text{Pic}(G)$ is not torsion.

It suffices to check this after extending scalars to \bar{k} , hence we are now free to assume that $k = \bar{k}$. Then $G_{\text{red}} \subset G$ is a smooth connected k -subgroup scheme, hence by Chevalley there is an exact sequence

$$1 \longrightarrow M \longrightarrow G_{\text{red}} \xrightarrow{g} B \longrightarrow 1$$

with M a smooth connected affine k -group and B/k an abelian variety. The composition

$$G_{\text{red}} \subset G \rightarrow H \xrightarrow{F_{H/k}^{(p^n)}} H^{(p^n)}$$

is an isogeny: each map in this sequence is set-theoretically surjective, and G_{red} and $H^{(p^n)}$ are smooth, so the map is fppf, and its kernel is finite because that is true of each map in the above composition. Lemma 7 therefore implies that we have a unique commutative diagram with exact rows

$$\begin{array}{ccccccc} 1 & \longrightarrow & M & \longrightarrow & G_{\text{red}} & \xrightarrow{g} & B \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \downarrow f \\ 1 & \longrightarrow & L & \longrightarrow & H^{(p^n)} & \longrightarrow & A \longrightarrow 1, \end{array}$$

and that f is an isogeny. In particular, $f^*(\mathcal{L})$ is still ample, hence by Lemma 6, $g^*f^*(\mathcal{L}) \in \text{Pic}(G_{\text{red}})$ is not torsion. The commutative diagram

$$\begin{array}{ccc} G_{\text{red}} & \longrightarrow & G \\ \downarrow g & & \downarrow q \\ B & \xrightarrow{f} & A \end{array}$$

then shows that $q^*(\mathcal{L})$ is not torsion either. □

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References

- [1] A. Borel, *Linear Algebraic Groups*, 2nd ed., Graduate Texts in Mathematics, vol. 126, Springer, 1991.
- [2] B. Conrad, “Units On Product Varieties”, 2006, available at <http://math.stanford.edu/~conrad/papers/unitthm.pdf>.
- [3] M. Demazure, A. Grothendieck, *Schémas en Groupes*, Lecture Notes in Mathematics, vol. I, II, III, Springer, 1970.
- [4] A. Grothendieck, “Éléments de géométrie algébrique. IV : Étude locale des schémas et des morphismes de schémas, Seconde partie”, *Publ. Math., Inst. Hautes Étud. Sci.* **24** (1965), p. 5-231.
- [5] R. Guralnick, D. B. Jaffe, W. Raskind, R. Wiegand, “On the Picard Group: Torsion and the Kernel Induced by a Faithfully Flat Map”, *J. Algebra* **183** (1996), no. 2, p. 420-455.

- [6] D. B. Mumford, *Abelian Varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Published for the Tata Institute of Fundamental Research, Bombay by Oxford University Press, 1970.
- [7] Z. Rosengarten, "Translation-Invariant Line Bundles On Linear Algebraic Groups", *J. Algebr. Geom.* (2020).
- [8] ———, "Tate Duality In Positive Dimension Over Function Fields", <https://arxiv.org/abs/1805.00522>, to appear in *Memoirs of the American Mathematical Society*, 2021.