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An inductive approach to generalized abundance using nef reduction

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Abstract. We use the canonical bundle formula for parabolic fibrations to give an inductive approach to the generalized abundance conjecture using nef reduction. In particular, we observe that generalized abundance holds for a klt pair (X, B) if the nef dimension $n(K_X + B + L) = 2$ and $K_X + B \geq 0$ or $n(K_X + B + L) = 3$ and $\kappa(K_X + B) > 0$.

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1. Introduction

In [10], Lazić and Peternell propose the following

Conjecture 1 (Generalized Abundance). *Let (X, B) be a klt pair with $K_X + B$ pseudoeffective. If L is a nef divisor on X such that $K_X + B + L$ is also nef, then $K_X + B + L \equiv M$ for some semiample $\mathbb{Q}$-divisor M.*

They show that this conjecture holds in case $\dim X = 2$ and for $\dim X = 3$ if $\kappa(K_X + B) > 0$ (see [10, Corollary C, D]). The main purpose of this article is to prove the following result.

Theorem 2. *Let (X, B) be an n-dimensional klt pair with $K_X + B \geq 0$ and let $L \in \operatorname{Pic} X$ be nef such that $K_X + B + L$ is nef with nef dimension $n(K_X + B + L) = d$.*
Assume termination of klt flips [7, Conjecture 5-1-13] in dimension d, abundance conjecture [8, Conjecture 3.12] in dimension $\leq d$, semiampleness conjecture [10, page 2] in dimension $\leq d$ and generalized non-vanishing conjecture [10, page 2] in dimension $\leq d - 1$.
Then $K_X + B + L \equiv M$ for some semiample divisor M.

Note that the phrase "Conjecture $(*)$ holds in dimension d (resp. $\leq d$)" above means that $(*)$ holds for all klt pairs (X, B) such that $\dim X = d$ (resp. $\leq d$).

In case $L = 0$, such a theorem has been proved by [1]. The main ingredients of our proof are the canonical bundle formula for parabolic fibrations [5] and Nakayama-Zariski decomposition [12]. We have the following corollary

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Corollary 3. *Generalized abundance holds (in any dimension) in the following two cases:*

(1) $n(K_X + B + L) = 2$ *and* $K_X + B \geq 0$*, or*
(2) $n(K_X + B + L) = 3$ *and* $\kappa(K_X + B) > 0$.

Note that any divisor of nef dimension 1 is already numerically equivalent to a semiample divisor by [4, 2.4.4]. Thus generalized abundance also holds in this case.

2. Preliminaries

Definition 4 (Singularities of pairs). *Let* (X, B) *be a sub-pair consisting of a normal variety* X *and a* $\mathbb{Q}$*-divisor* B *on* X *such that* $K_X + B$ *is* $\mathbb{Q}$*-Cartier. It is called* sub-klt *if there exists a log resolution* $Y \xrightarrow{\mu} X$ *of* (X, B) *such that letting* B_Y *be defined by* $K_Y + B_Y = \mu^*(K_X + B)$*, all coefficients of* B_Y *are* < 1 *and* sub-lc *if all coefficients of* B_Y *are* ≤ 1*. If* $B \geq 0$*, we drop the prefix sub.*

Definition 5 (Nef reduction and nef dimension [4]). *Let* X *be a normal projective variety and let* $L \in \operatorname{Pic} X$ *be nef. Then there exists a dominant rational map* $\phi : X \dashrightarrow Y$ *with connected fibers which is proper and regular over an open subset of* Y *(i.e. there exists* $V \subset Y$ *nonempty open such* ϕ *restricts to a proper morphism* $\phi|_{\phi^{-1}(V)} : \phi^{-1}(V) \to V$*) where* Y *is also normal projective such that*

(1) *If* $F \subset X$ *is a general compact fiber of* ϕ *with* $\dim F = \dim X - \dim Y$*, then* $L|_F \equiv 0$.
(2) *If* $x \in X$ *is a very general point and* $C \subset X$ *a curve passing* x *such that* $\dim(\phi(C)) > 0$*, then* $(L \cdot C) > 0$.

ϕ *is called the* ***nef reduction map*** *of* L *and* $\dim Y$ *the* ***nef dimension*** $n(L)$ *of* L.

Remark 6. Note that if $\phi : X \dashrightarrow Y$ is proper and regular over an open subset of Y, then there exists a resolution $\widehat{\phi} : \widehat{X} \to Y$ of ϕ such that the exceptional divisor $Ex(\widehat{X}/X)$ is $\widehat{\phi}$-vertical. For example, $\widehat{X}$ can be chosen to be the closure of the graph of ϕ.

Notation 7. For a pseudoeffective divisor D on a smooth projective variety X, $P_\sigma(D)$ and $N_\sigma(D)$ will denote the positive and negative parts of the Nakayama-Zariski decomposition of D (see [12, Chapter 3] for details). $\kappa(D)$ will denote the Iitaka dimension and $\nu(D)$ the numerical dimension (see [6, Definition 2.10]) of D.

3. Inductive approach to generalized abundance

Proof of Theorem 2. We will follow some ideas of [1, Theorem 5.1] and [6, Lemma 4.4]. Let Φ be the nef reduction map of $K_X + B + L$. Let $\widehat{X}$ be the normalization of the closure of the graph of Φ. We have an induced commutative diagram:

$$\begin{array}{ccc} \widehat{X} & & \\ \downarrow{\scriptstyle\beta} & \searrow{\scriptstyle\phi} & \\ X & \overset{\Phi}{\dashrightarrow} & Y \end{array}$$

Note that since Φ is proper and regular over an open subset of Y, $Ex(\beta)$ is ϕ-vertical. We will make base changes that preserve this property. Let $F \subset \widehat{X}$ be a general fiber of ϕ. Then

$$\left(K_{\hat{X}} + B_{\hat{X}} + L_{\hat{X}}\right)\big|_F \equiv 0. \tag{1}$$

Here $L_{\hat{X}} := \beta^* L$ and $B_{\hat{X}}$ is defined by $K_{\hat{X}} + B_{\hat{X}} = \beta^*(K_X + B)$. Now since $K_{\hat{X}} + B_{\hat{X}} \geq 0$, we have $K_F + B_F \geq 0$ which implies $K_F + B_F \sim_{\mathbb{Q}} 0$. Indeed, suppose that $K_F + B_F > 0$. Then there exists a curve $C \subset F$ such that $((K_F + B_F) \cdot C) > 0$. Then $((K_F + B_F + L_F) \cdot C) > 0$. This contradicts (1). We also conclude that $L_{\hat{X}}|_F \equiv 0$.

By [10, Lemma 3.1], there exists $\sigma : Y' \to Y$ birational from a smooth projective variety Y' such that letting X' denote the normalization of the main component of $\widehat{X} \times_Y Y' \to Y'$ and $\phi' : X' \to Y'$ the induced morphism, there exists a nef $\mathbb{Q}$-Cartier divisor $L_{Y'}$ on Y' such that $L_{X'} \equiv \phi'^*(L_{Y'})$. For a divisor D, let D^+ and D^- denote its positive and negative parts respectively. Letting F' denote a general fiber of ϕ', note that $(B^-_{X'})|_{F'} \sim 0$. Thus

$$\left(K_{X'} + B_{X'}\right)|_{F'} \sim \left(K_{X'} + B^+_{X'}\right)\Big|_{F'} \sim 0.$$

Then by [5, Section 4], there exists a birational morphism $\widetilde{Y} \to Y'$, $\widetilde{X}$ birational to the main component of $X' \times_{Y'} \widetilde{Y}$ where $\widetilde{X}$ and $\widetilde{Y}$ are both smooth projective such that letting $\widetilde{\phi} : \widetilde{X} \to \widetilde{Y}$ denote the induced morphism, there exists a $\mathbb{Q}$-divisor Δ on $\widetilde{X}$ such that $\widetilde{\phi} : (\widetilde{X}, B^+_{\widetilde{X}} + \Delta) \to \widetilde{Y}$ is a klt trivial fibration ([9, Definition 2.1]) where Δ^+ is exceptional over $\widetilde{Y}$ and X' and $\widetilde{\phi}_* \mathcal{O}_{\widetilde{X}}(\lfloor l\Delta^- \rfloor) \cong \mathcal{O}_{\widetilde{Y}}$ for all $l \in \mathbb{N}$.

$$\begin{array}{ccccccc}
\widetilde{X} & \xrightarrow{\pi} & X' & \longrightarrow & \widehat{X} & \xrightarrow{\beta} & X \\
\downarrow{\scriptstyle \widetilde{\phi}} & & \downarrow{\scriptstyle \phi'} & & \downarrow{\scriptstyle \phi} & \swarrow & \\
\widetilde{Y} & \longrightarrow & Y' & \longrightarrow & Y & &
\end{array}$$

By [6, Lemma 2.15], the last condition implies that the vertical part $(\Delta^-)^v$ is $\widetilde{\phi}$-degenerate (see [6, Definition 2.14]). Letting $B_{\widetilde{Y}}$ and $M_{\widetilde{Y}}$ denote the induced discriminant and moduli divisors on $\widetilde{Y}$, we have

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + \Delta^+ - \Delta^- \sim_{\mathbb{Q}} \widetilde{\phi}^*\left(K_{\widetilde{Y}} + B_{\widetilde{Y}} + M_{\widetilde{Y}}\right).$$

Since $(\Delta^-)^v$ is $\widetilde{\phi}$-degenerate, it follows from the definition of $B_{\widetilde{Y}}$ that $B_{\widetilde{Y}} \geq 0$. Since $K_{\widetilde{X}} + B^+_{\widetilde{X}} = \pi^*(K_{X'} + B^+_{X'}) + E$, where $E \geq 0$ is π-exceptional, if $\widetilde{F} \subset \widetilde{X}$ is a general fiber of $\widetilde{\phi}$,

$$\left(K_{\widetilde{X}} + B^+_{\widetilde{X}}\right)\Big|_{\widetilde{F}} = E|_{\widetilde{F}}.$$

Note that $E|_{\widetilde{F}}$ is an exceptional divisor for the induced birational morphism $\pi|_{\widetilde{F}} : \widetilde{F} \to F'$. Indeed, assume otherwise. Then there exists $U' \subset Y'$ open such that letting $V' := \phi'^{-1}(U')$, codim $(\pi(E) \cap V') \leq 1$. But then E can't be π-exceptional which is a contradiction. We conclude that $\kappa((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}) = \nu((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}) = 0$. Thus we can run a relative $(K_{\widetilde{X}} + B^+_{\widetilde{X}})$-MMP with ample scaling over $\widetilde{Y}$ to get

$$\begin{array}{ccc}
\widetilde{X} & \overset{\psi}{\dashrightarrow} & \widetilde{X}_m \\
\downarrow{\scriptstyle \widetilde{\phi}} & \swarrow{\scriptstyle \widetilde{\phi}_m} & \\
\widetilde{Y} & &
\end{array}$$

such that letting $K_{\widetilde{X}_m} + B^+_{\widetilde{X}_m} = \psi_*(K_{\widetilde{X}} + B^+_{\widetilde{X}})$ and $\Delta_{\widetilde{X}_m} := \psi_*\Delta$, we have $(K_{\widetilde{X}_m} + B^+_{\widetilde{X}_m})|_{\widetilde{F}_m} \sim_{\mathbb{Q}} 0$, $\widetilde{F}_m$ being the general fiber of $\widetilde{\phi}_m$ (see the first paragraph of the proof of [6, Lemma 4.4] for details). Consider the induced klt-trivial fibration

$$\widetilde{\phi}_m : \left(\widetilde{X}_m, B_{\widetilde{X}_m} + \Delta^+_{\widetilde{X}_m} - \Delta^-_{\widetilde{X}_m}\right) \to \widetilde{Y}.$$

Note: the fact that $\widetilde{\phi}_m$ is klt-trivial can be shown by choosing a smooth resolution of indeterminacy $\widetilde{X} \xleftarrow{p} W \xrightarrow{q} \widetilde{X}_m$ of ψ and using the fact that $p^*(K_{\widetilde{X}} + B^+_{\widetilde{X}}) = q^*(K_{\widetilde{X}_m} + B^+_{\widetilde{X}_m}) + E$ where $E \geq 0$ is q-exceptional. By [2, Lemma 2.6], $\widetilde{\phi}_m$ and $\widetilde{\phi}$ induce the same discriminant and moduli divisors on $\widetilde{Y}$. In the case of $\widetilde{\phi}_m$, $(\Delta^-_{\widetilde{X}_m})|_{\widetilde{F}_m} \sim_{\mathbb{Q}} 0$, thus by [3, Theorem 3.3], $M_{\widetilde{Y}}$ is b-nef and b-good. Hence, by the arguments in the proof of [3, Theorem 4.1], we can write

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + \Delta^+ - \Delta^- \sim_{\mathbb{Q}} \widetilde{\phi}^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}\right).$$

where $(\widetilde{Y}, \Delta_{\widetilde{Y}})$ is klt. We write this as

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + \Delta^+ - (\Delta^-)^h \sim_{\mathbb{Q}} \widetilde{\phi}^* \left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}\right) + (\Delta^-)^v \tag{2}$$

where ^{h} and ^{v} denote the horizontal and vertical parts with respect to $\widetilde{\phi}$ respectively. As observed above, $(K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}}$ is $\pi|_{\widetilde{F}}$-exceptional and $(K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}} \sim_{\mathbb{Q}} (\Delta^-)^h|_{\widetilde{F}}$. Thus $(\Delta^-)^h|_{\widetilde{F}} = N_\sigma((K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}})(= (K_{\widetilde{X}} + B^+_{\widetilde{X}})|_{\widetilde{F}})$. Now it follows from the definition of N_σ that

$$N_\sigma\left(\left(K_{\widetilde{X}} + B^+_{\widetilde{X}}\right)\Big|_{\widetilde{F}}\right) \le N_\sigma\left(K_{\widetilde{X}} + B^+_{\widetilde{X}}\right)\Big|_{\widetilde{F}}. \tag{3}$$

Since $\widetilde{F}$ is a general fiber and $(\Delta^-)^h$ has no vertical components, it follows that

$$(\Delta^-)^h \le N_\sigma\left(K_{\widetilde{X}} + B^+_{\widetilde{X}}\right) \tag{4}$$

Note that $P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}})$ is effective. Indeed, let $\mu : \widetilde{X} \to X$ be the induced birational morphism. Then we have $K_{\widetilde{X}} + B^+_{\widetilde{X}} = \mu^*(K_X + B) + B^-_{\widetilde{X}}$ where $B^-_{\widetilde{X}}$ is effective and μ-exceptional. Thus

$$\kappa\left(K_{\widetilde{X}} + B^+_{\widetilde{X}}\right) = \kappa\left(\mu^*(K_X + B)\right) = \kappa(K_X + B) \ge 0.$$

Then $\kappa(P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}})) = \kappa(K_{\widetilde{X}} + B^+_{\widetilde{X}}) \ge 0$ by [6, Lemma 2.9]. It follows that $K_{\widetilde{X}} + B^+_{\widetilde{X}} - (\Delta^-)^h \ge 0$. By [12, Lemma 1.8], (4) implies that $N_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}} - (\Delta^-)^h) = N_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}}) - (\Delta^-)^h$ and thus $P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}} - (\Delta^-)^h) = P_\sigma(K_{\widetilde{X}} + B^+_{\widetilde{X}})$. Now we take P_σ in (2). Since $\kappa(D) = \kappa(P_\sigma(D))$ for any pseudoeffective divisor D ([6, Lemma 2.9]), the fact that Δ^+ and $B^-_{\widetilde{X}}$ are exceptional over X, the $\widetilde{\phi}$-degeneracy of $(\Delta^-)^v$ and [6, Lemma 2.16] imply that

$$\kappa(K_X + B) = \kappa\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}\right) \tag{5}$$

Then (2) gives

$$K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}} + \Delta^+ - (\Delta^-)^h \equiv \widetilde{\phi}^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right) + (\Delta^-)^v \tag{6}$$

where Δ^+ is exceptional over X' and $\widetilde{Y}$ and $(\Delta^-)^v$ is $\widetilde{\phi}$-degenerate. Since $L_{\widetilde{X}}|_{\widetilde{F}} \equiv 0$, as observed above, we have

$$N_\sigma\left(\left(K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}}\right)\Big|_{\widetilde{F}}\right) = N_\sigma\left(\left(K_{\widetilde{X}} + B^+_{\widetilde{X}}\right)\Big|_{\widetilde{F}}\right) \ge (\Delta^-)^h|_{\widetilde{F}} \text{ implying } (\Delta^-)^h \le N_\sigma\left(K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}}\right)$$

and exactly as before, we conclude that

$$P_\sigma\left(K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}} + \Delta^+ - (\Delta^-)^h\right) = P_\sigma\left(K_{\widetilde{X}} + B^+_{\widetilde{X}} + L_{\widetilde{X}} + \Delta^+\right).$$

But the latter equals

$$P_\sigma\left(\mu^*(K_X + B + L) + B^-_{\widetilde{X}} + \Delta^+\right) = P_\sigma\left(\mu^*(K_X + B + L)\right) = \mu^*(K_X + B + L)$$

since Δ^+ and $B^-_{\widetilde{X}}$ are exceptional over X and $K_X + B + L$ is nef. Now taking P_σ in (6) and using [6, Lemma 2.16], we get

$$\mu^*(K_X + B + L) = P_\sigma\left(\widetilde{\phi}^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right)\right).$$

By [10, Theorem 5.3], there exists a smooth projective $\overline{Y}$ and $w : \overline{Y} \to \widetilde{Y}$ birational such that $P_\sigma(w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}))$ is numerically equivalent to a semiample divisor. Letting $\overline{X}$ denote a desingularization of the main component of $\widetilde{X} \times_{\widetilde{Y}} \overline{Y}$ and $v : \overline{X} \to \widetilde{X}$, $\overline{\phi} : \overline{X} \to \overline{Y}$ the induced morphisms, we have

$$\begin{aligned}
&\overline{\phi}^* P_\sigma\left(w^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right)\right) = P_\sigma\left(\overline{\phi}^* w^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right)\right)\\
&\left(\text{since } P_\sigma\left(w^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\overline{Y}}\right)\right) \text{ is nef; see [10, Lemma 2.5]}\right)\\
&= P_\sigma\left(v^*\widetilde{\phi}^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right)\right) = v^* P_\sigma\left(\widetilde{\phi}^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right)\right)\\
&\left(\text{since } P_\sigma\left(\widetilde{\phi}^*\left(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}\right)\right) \text{ is nef}\right) = v^*\mu^*(K_X + B + L)
\end{aligned} \tag{$*$}$$

is numerically equivalent to a semiample divisor. Thus $K_X + B + L$ is numerically equivalent to a semiample divisor. □

Remark 8. The above result can also be proved by using the MMP instead of the canonical bundle formula. See for example the proof of [11, Theorem 3.5].

Proof of Corollary 3.

(1) If $n(K_X + B + L) = 2$, then $\dim \widetilde{Y} = 2$ and $K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} \geq 0$ by (5). Termination of flips for threefold pairs and abundance for surface and threefold pairs are classical. Semiampleness and generalized non-vanishing conjectures hold for surface pairs by [10, Corollary C, page 4]. Thus the case $n(K_X + B + L) = 2$ i.e. when $\dim \widetilde{Y} = 2$ is immediate.

(2) If $n(K_X + B + L) = 3$, then $\dim \widetilde{Y} = 3$ and by (5), $\kappa(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}}) > 0$. Then by [10, Remark 5.4], $P_\sigma(w^*(K_{\widetilde{Y}} + \Delta_{\widetilde{Y}} + L_{\widetilde{Y}}))$ is numerically equivalent to a semiample divisor for some birational morphism $w : \overline{Y} \to \widetilde{Y}$ from a smooth projective variety $\overline{Y}$. Then so is $K_X + B + L$ by $(*)$ above. □

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