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Number theory / *Théorie des nombres*

The number of nonunimodular roots of a reciprocal polynomial

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Abstract. We introduce a sequence P_d of monic reciprocal polynomials with integer coefficients having the central coefficients fixed as well as the peripheral coefficients. We prove that the ratio of the number of nonunimodular roots of P_d to its degree d has a limit L when d tends to infinity. We show that if the coefficients of a polynomial can be arbitrarily large in modulus then L can be arbitrarily close to 0. It seems reasonable to believe that if the coefficients are bounded then the analogue of Lehmer's Conjecture is true: either $L = 0$ or there exists a gap so that L could not be arbitrarily close to 0. We present an algorithm for calculating the limit ratio and a numerical method for its approximation. We estimated the limit ratio for a family of polynomials deduced from the powers of a given Salem number. We calculated the limit ratio of polynomials correlated to many bivariate polynomials having small Mahler measure introduced by Boyd and Mossinghoff.

Résumé. Nous introduisons une suite P_d de polynômes réciproques unitaires à coefficients entiers ayant les coefficients centraux fixes ainsi que les coefficients périphériques. Nous prouvons que le rapport du nombre de racines non unimodulaires de P_d sur son degré d a une limite L lorsque d tend vers l'infini. Nous montrons que si les coefficients d'un polynôme peuvent être arbitrairement grands en module alors L peut être arbitrairement proche de 0. Il semble raisonnable de croire que si les coefficients sont bornés, alors l'analogue de la conjecture de Lehmer est vrai : soit $L = 0$, soit il existe un écart tel que L ne puisse pas être arbitrairement proche de 0. Nous présentons un algorithme pour le calcul du rapport limite et une méthode numérique pour son approximation. Nous avons estimé le rapport limite pour une famille de polynômes déduits des puissances d'un nombre de Salem donné. Nous avons calculé le rapport limite des polynômes corrélés à de nombreux polynômes bivariés ayant une petite mesure de Mahler introduits par Boyd et Mossinghoff.

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1. Introduction

The Mahler measure $M(P)$ of a polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0 \in \mathbb{Z}[x]$ having $a_d \neq 0$ and zeros $\alpha_1, \alpha_2, \ldots, \alpha_d$ is defined as

$$M(P(x)) := |a_d| \prod_{j=1}^{d} \max\left(1, \left|\alpha_j\right|\right).$$

Let $I(P)$ denote the number of complex zeros of $P(x)$ which are < 1 in modulus, counted with multiplicities. Let $U(P)$ denote the number of zeros of $P(x)$ which are $= 1$ in modulus, (again, counting with multiplicities). Such zeros are called unimodular. Let $E(P)$ denote the number of complex zeros of $P(x)$ which are > 1 in modulus, counted with multiplicities. Then it is obviously that $I(P)+U(P)+E(P) = d$. A Pisot number can be defined as a real algebraic integer greater than 1 having its minimal polynomial $P(x)$ of degree d such that $I(P) = d-1$. The minimal polynomial of a Pisot number is called Pisot polynomial. A Salem number is a real algebraic integer > 1 having the minimal polynomial $P(x)$ of degree d such that $U(P) = d - 2 \geq 1$, $I(P) = 1$. We say that a polynomial of degree d is reciprocal if $P(x) = x^d P(1/x)$.

Definition. *A polynomial $P(x) \in \mathbb{Z}[x]$ is a Salem polynomial if it is reciprocal and can be written*

$$P(x) = A(x) \cdot R(x)$$

where $A(x)$ is the product of (irreducible) cyclotomic polynomials and $R(x)$ is the minimal polynomial of a Salem number.

If the moduli of the coefficients are small then a reciprocal polynomial has many unimodular roots. A Littlewood polynomial is a polynomial all of whose coefficients are 1 or −1. Mukunda [10] showed that every self-reciprocal Littlewood polynomial of odd degree at least 3 has at least 3 zeros on the unit circle. Drungilas [4] proved that every self-reciprocal Littlewood polynomial of odd degree $n \geq 7$ has at least 5 zeros on the unit circle and every self-reciprocal Littlewood polynomial of even degree $n \geq 14$ has at least 4 unimodular zeros. In [1] two types of very special Littlewood polynomials are considered: Littlewood polynomials with one sign change in the sequence of coefficients and Littlewood polynomials with one negative coefficient. The numbers $U(P)$ and $I(P)$ of such Littlewood polynomials P are investigated. In [2] Borwein, Erdélyi, Ferguson and Lockhart showed that there exists a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ with the n_m integral and all different so that the number of its real zeros in $[0, 2\pi)$ is $O(N^{9/10}(\log N)^{1/5})$ (here the frequencies $n_m = n_m(N)$ may vary with N). However, there are reasons to believe that a cosine polynomial $\sum_{m=1}^{N} \cos(n_m\theta)$ always has many zeros in the period.

Clearly, if α_j, is a root of a reciprocal $P(x)$ then $1/\alpha_j$ is also a root of $P(x)$ so that $I(P) = E(P)$. Let $C(P) = \frac{I(P)+E(P)}{2n}$ be the ratio of the number of nonunimodular zeros of P to its degree. Actually, it is the probability that a randomly chosen zero is not unimodular, and $C(P) = \frac{E(P)}{n}$.

Here we will investigate a special sequence of polynomials. Let n, k, l $a_0, a_1, \ldots, a_k$, $b_0, b_1, \ldots, b_l$ be integers such that $2n > k \geq 0$, $l \geq 0$, and let $P_{2n+2l}(x)$ be the monic, reciprocal polynomial with integer coefficients

$$P_{2n+2l}(x) = x^{n+l}\left(\sum_{j=0}^{l} b_j \left(x^{n+j} + \frac{1}{x^{n+j}}\right) + a_0 + \sum_{j=1}^{k} a_j \left(x^j + \frac{1}{x^j}\right)\right). \tag{1}$$

We should remark that we have already studied in [13] the special case of (1) for $l = 0$, $b_0 = 1$. We are looking for sequences (P_{2n+2l}) such that the ratio $C(P_{2n+2l})$ has a limit when n tends to ∞ and $0 < \lim_{n\to\infty} C(P_{2n+2l}) < 1$. If (P_{2n+2l}) is a sequence of Salem polynomials then this limit is trivially 0. Salem (see [12, Theorem IV, p. 30]) has found such a sequence. He discovered a simple way to construct two infinite sequences of Salem polynomials from a Pisot polynomial. The following example gives one of them.

Example 1. If $x^l+b_{l-1}x^{l-1}+\cdots+b_0$ is a Pisot polynomial, $b_l=1$, $k=a_0=0$, then (1) is a sequence of Salem polynomials.

Let the Pisot polynomial in Example 1 be the minimal polynomial of a Pisot number θ. We should remark that Salem proved in [11] that the family of Salem numbers associated to the family of Salem polynomials in the Example 1 tends to θ. Such family of Salem polynomials is not unique; the classification of all the families of Salem numbers converging to θ is unknown, and it constitutes a very difficult problem [8]. There is much we do not know about the distribution of Salem numbers on $(1,\infty)$ but it is believed that only a finite number of Salem numbers are less than 1.3 and the Lehmer conjecture (see [9, p. 23]) suggests that none of them is less than 1.176.

Theorem (Main Theorem). *If $k,l\geq 0$ are integers then for all fixed integers a_j, $j=0,1,\ldots,k$ and for all fixed integers b_j in* (1) *such that $b_j=b_{l-j}$, $j=0,1,\ldots,l$ the limit* $\lim_{n\to\infty}C(P_{2n+2l})$ *exists.*

Main Theorem enables us to introduce the following

Definition. *Let the limit of $C(P_{2n+2l})$ when n tends to infinity be called limit ratio and denoted $LC(P_{2n+2l})$.*

In the second section we present the proof of Main Theorem. In Example 1 $LC(P_{2n+2l})$ can be arbitrarily close to 0 as $n\to\infty$ but the condition $b_j=b_{l-j}$ is false. In Example 3 the condition $b_l=b_{l-j}$ is true, $LC(P_{2n+2l})$ can be arbitrarily close to 0 but the b_js are unbounded. We did not find an example of sequence $LC(P_{2n+2l})$ which satisfies the condition $b_j=b_{l-j}$, with b_j, $j=0,\ldots,l$ uniformly bounded in modulus and $LC(P_{2n+2l})$ arbitrarily close to 0. So we conjecture that such a sequence does not exist, in Conjecture 12. In the third section we establish the connection between the Mahler measure and the limit ratio and we calculate the limit ratio for many families of polynomials with small Mahler measure introduced by Boyd and Mossinghoff.

2. The Limit Ratio

The Main Theorem is a generalisation of Theorem 2.1 which we proved in [13], more precisely Theorem 2.1 can be obtained from the Main Theorem if we take $l=0$.

Proof of Main Theorem. The theorem will be proved if we show that $1-C(P_{2n+2l})$ has a limit when n tends to ∞. Since $1-C(P_{2n+2l})=\frac{U(P_{2n+2l})}{2n+2l}$ we have to count the unimodular roots of $P_{2n+2l}(x)$. The equation $P_{2n+2l}(x)=0$ is equivalent to

$$x^{n+l}\sum_{j=0}^{l}b_j\left(x^{n+j}+\frac{1}{x^{n+j}}\right)=x^{n+l}\left(-a_0-\sum_{j=1}^{k}a_j\left(x^j+\frac{1}{x^j}\right)\right). \tag{2}$$

Let $B(x)$ be the polynomial on the left side and let $A(x)$ be the polynomial on the right side of the previous equation.

Since $b_j=b_{l-j}$, $j=0,1,\ldots,l$ we have

$$\begin{aligned}B(x)&=x^{n+l}\sum_{j=0}^{l}b_j\left(x^{n+j}+\frac{1}{x^{n+j}}\right)\\&=\sum_{j=0}^{l}b_j\left(x^{2n+l+j}+x^{l-j}\right)\\&=\sum_{j=0}^{l}b_jx^{2n+l+j}+\sum_{j=0}^{l}b_{l-j}x^{l-j}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{l} b_j \left(x^{2n+l+j} + x^j\right) \\
&= \sum_{j=0}^{l} b_j x^j \left(x^{2n+l} + 1\right) \\
&= \left(x^{2n+l} + 1\right) \sum_{j=0}^{l} b_j x^j \\
&= \left(x^{2n+l} + 1\right) x^{l/2} \sum_{j=0}^{l} b_j x^{j-l/2}.
\end{aligned}
$$

Finally it follows that

$$B(x) = x^{n+l} \left(x^{n+l/2} + \frac{1}{x^{n+l/2}}\right) \sum_{j=0}^{l} b_j x^{j-l/2}. \tag{3}$$

Since we have to find unimodular roots we use the substitution $x = e^{it}$ in the equation (2). If l is even then we have

$$B(e^{it}) = e^{i(n+l)t} 2\cos[(n+l/2)t] \left(\sum_{j=0}^{l/2-1} 2b_j \cos[(l/2-j)t] + b_{l/2}\right). \tag{4}$$

If l is odd then it follows from (3)

$$B(e^{it}) = e^{i(n+l)t} 2\cos[(n+l/2)t] \left(\sum_{j=0}^{(l-1)/2} 2b_j \cos[(l/2-j)t]\right). \tag{5}$$

From the substitution $x = e^{it}$ it follows that x is unimodular if and only if t is real so that we have to count the real roots of $B(e^{it}) = A(e^{it})$, $(t \in [0, 2\pi))$. We denote with $E(t)$ the function defined by terms enclosed within brackets of (4) or of (5) i.e.

$$E(t) := \begin{cases} \sum_{j=0}^{l/2-1} 2b_j \cos(l/2-j)t + b_{l/2} & \text{if } l \text{ is even,} \\ \sum_{j=0}^{(l-1)/2} 2b_j \cos(l/2-j)t & \text{if } l \text{ is odd.} \end{cases} \tag{6}$$

If $t \in \mathbb{R}$ we can divide the equation $B(e^{it}) = A(e^{it})$ with $2e^{i(n+l)t} \neq 0$ and obtain

$$\cos[(n+l/2)t] E(t) = -a_0/2 - \sum_{j=1}^{k} a_j \cos jt$$

Let Γ be the graph of $E(t)$, let Γ_1 be the graph of $f_1(t) = \cos(n+l/2)t\, E(t)$. Obviously for all n Γ_1 is settled between graphs of $E(t)$ and $-E(t)$ and in certain points Γ touches Γ_1. For that reason we call $E(t)$ the envelope of $f_1(t)$. Let Γ_2 be the graph of

$$f_2(t) := -a_0/2 - \sum_{j=1}^{k} a_j \cos jt. \tag{7}$$

Then $U(P)$ is equal to the number of intersection points of Γ_1 and Γ_2. These intersection points are obviously settled between curves $y = -|E(t)|$ and $y = |E(t)|$. Graph Γ_2 of the continuous function f_2 and graph Γ are fixed i.e. they do not depend on n, therefore there are r subintervals I_j, such that r is a finite integer, $I_j = [\alpha_j, \beta_j]$, $0 < \beta_{j-1} < \alpha_j < \beta_j < \alpha_{j+1} < 2\pi$, such that if $t \in I_j$ then $|f_2(t)| \le |E(t)|$, where α_j, β_j are solutions of

$$|E(t)| = |f_2(t)|. \tag{8}$$

We need the following theorem of Erdős and Turán [5] to finish the proof of Main Theorem.

Theorem 2. *(Erdős, Turán) Let* $F(x) = \sum_{k=0}^{d} a_k x^k \in \mathbb{C}[x]$ *with* $a_d a_0 \neq 0$*, and let*

$$N(F;\alpha,\beta) = \#\{roots\ r \in \mathbb{C}\ of F\ with\ \alpha \le \arg(r) \le \beta\}.$$

Then for all $0 \le \alpha < \beta \le 2\pi$,

$$\left|\frac{N(F;\alpha,\beta)}{d} - \frac{\beta-\alpha}{2\pi}\right| \le \frac{16}{\sqrt{d}}\left[\log\left(\frac{|a_0|+\cdots+|a_d|}{\sqrt{|a_0 a_d|}}\right)\right]^{1/2}.$$

Using Theorem 2 of Erdős and Turán we obtain

$$\left|\frac{N\left(P_{2n+2l};\alpha_j,\beta_j\right)}{2n+2l} - \frac{\beta_j-\alpha_j}{2\pi}\right| \le \frac{16}{\sqrt{2n+2l}}\left[\log\left(\frac{2\sum_{j=0}^{l}|b_j| + |a_0| + 2\sum_{j=1}^{k}|a_j|}{\sqrt{|b_0 b_l|}}\right)\right]^{1/2}.$$

If we introduce a constant

$$D := \left[\log\left(\frac{2\sum_{j=0}^{l}|b_j| + |a_0| + 2\sum_{j=1}^{k}|a_j|}{\sqrt{|b_0 b_l|}}\right)\right]^{1/2}$$

then it follows that

$$\frac{\beta_j-\alpha_j}{2\pi} - \frac{16}{\sqrt{2n+2l}}D \le \frac{N\left(P_{2n+2l};\alpha_j,\beta_j\right)}{2n+2l} \le \frac{\beta_j-\alpha_j}{2\pi} + \frac{16}{\sqrt{2n+2l}}D.$$

If we summarize the previous inequalities for $j = 1,2,\ldots,r$ then we get

$$\sum_{j=1}^{r}\frac{\beta_j-\alpha_j}{2\pi} - r\frac{16}{\sqrt{2n+2l}}D \le \sum_{j=1}^{r}\frac{N\left(P_{2n+2l};\alpha_j,\beta_j\right)}{2n+2l} \le \sum_{j=1}^{r}\frac{\beta_j-\alpha_j}{2\pi} + r\frac{16}{\sqrt{2n+2l}}D.$$

Finally we have to notice that $\sum_{j=1}^{r} N(P_{2n+2l};\alpha_j,\beta_j) = U(P_{2n+2l})$ and find the limit when n tends to infinity. Using the Theorem 2 it follows that

$$\lim_{n\to\infty}\frac{U(P_{2n+2l})}{2n+2l} = \sum_{j=0}^{r}\frac{\beta_j-\alpha_j}{2\pi}$$

because $\lim_{n\to\infty} r\frac{16}{\sqrt{2n+2l}}D = 0$. □

It is well known that $S_1(x) = x^4 - x^3 - x^2 - x + 1$ is a Salem polynomial having two real roots: a Salem number $\gamma > 1$, $1/\gamma$ and two complex unimodular roots $\theta,\bar{\theta}$. Let $S_m(x) = x^4 + b_{1,m}x^3 + b_{2,m}x^2 + b_{3,m}x + 1$ be the Salem polynomial of the Salem number γ^m so that its coefficients should be $b_{0,m} = b_{4,m} = 1$,

$$b_{1,m} = b_{3,m} = -\left(\gamma^m + 1/\gamma^m + \theta^m + \bar{\theta}^m\right), \tag{9}$$

$$b_{2,m} = 2 + \theta^m\gamma^m + \theta^m/\gamma^m + \bar{\theta}^m\gamma^m + \bar{\theta}^m/\gamma^m. \tag{10}$$

Example 3. Let $T_{2n+8,m}$ denote

$$T_{2n+8,m}(x) = x^{n+4}\left(\sum_{j=0}^{4} b_{j,m}\left(x^{n+j} + \frac{1}{x^{n+j}}\right) + 2\right).$$

Theorem 4. *With the notation introduced in Example 3 the following is true*

$$\lim_{m\to\infty} LC\left(T_{2n+8,m}(x)\right) = 0.$$

Proof. In this example $l = 4$ is even, $k = 0$, $a_0 = 2$. We have to use (6) to calculate the envelope: $E_m(t) = 2\cos(2t) + 2b_{1,m}\cos t + b_{2,m}$. We have to solve (8) that is equivalent with $E_m(t) = 1$ or $E_m(t) = -1$. Since $\cos 2t = 2\cos^2 t - 1$ the equations are quadratic in $\cos(t)$, so that, solving $E_m(t) = \pm 1$, we take the solutions in $[-1,1]$. From $E_m(t) = 1$ we get

$$\cos\alpha_m = \frac{1}{4}\left(-b_{1,m} - \sqrt{b_{1,m}^2 - 4b_{2,m} + 12}\right).$$

From $E_m(t) = -1$ we get

$$\cos\beta_m = \frac{1}{4}t\left(-b_{1,m} - \sqrt{b_{1,m}^2 - 4b_{2,m} + 4}\right).$$

It remains to calculate

$$\lim_{m\to\infty}\left(\cos\beta_m - \cos\alpha_m\right) = \lim_{m\to\infty}\frac{2}{\sqrt{b_{1,m}^2 - 4b_{2,m} + 12} + \sqrt{b_{1,m}^2 - 4b_{2,m} + 4}}.$$

To show that the last limit is 0 it is sufficient to show that $b_{1,m}^2 - 4b_{2,m}$ tends to $+\infty$ when $m \to \infty$. Using (9) and (10)

$$b_{1,m}^2 - 4b_{2,m} = \left(\gamma^{2m} - 2\gamma^m\theta^m - 2\gamma^m\bar\theta^m\right) + \left(1/\gamma^{2m} - 2\bar\theta^m/\gamma^m - 2\theta^m/\gamma^m + \theta^{2m} + \bar\theta^{2m} + 2\theta^m\bar\theta^m - 6\right).$$

The terms inside the first pair of parentheses are equal to

$$\gamma^m\left(\gamma^m - 2\theta^m - 2\bar\theta^m\right) \ge \gamma^m\left(\gamma^m - 4\right)$$

so that they tend to $+\infty$ when $m \to \infty$. Since all terms inside the second pair of parentheses are bounded or tend to zero it follows that $b_{1,m}^2 - 4b_{2,m}$ tends to $+\infty$ when $m \to \infty$. □

Let us now consider the case $b_0 = b_1 = \cdots = b_l = 1$. To determine the envelope in Theorem 7 we need the following lemmas which can be easily proved.

Lemma 5.

$$\sin\frac{t}{2}\left(\sum_{j=1}^{m} 2\cos jt + 1\right) = \sin\frac{(2m+1)t}{2}$$

Proof.

$$\begin{aligned}\sin\frac{t}{2}\left(\sum_{j=1}^{m} 2\cos jt + 1\right) &= \sin\frac{t}{2}\left(\sum_{j=0}^{m} 2\cos jt - 1\right)\\ &= \sin\frac{t}{2}\left(2\sum_{j=0}^{m}\cos jt\right) - \sin\frac{t}{2}\\ &= 2\cos\frac{mt}{2}\sin\frac{(m+1)t}{2} - \sin\frac{t}{2}\\ &= \sin\frac{(2m+1)t}{2} + \sin\frac{t}{2} - \sin\frac{t}{2}\\ &= \sin\frac{(2m+1)t}{2}\end{aligned}$$

□

Lemma 6.

$$\sin\frac{t}{2}\left(\sum_{j=1}^{m} 2\cos\frac{\left(2j-1\right)t}{2}\right) = \sin mt$$

Proof. The formula is obviously true for $m = 1$ because $2\sin\frac{t}{2}\cos\frac{t}{2} = \sin t$. We suppose that the formula is true for $m = k$ i.e.

$$\sin\frac{t}{2}\left(\sum_{j=1}^{k} 2\cos\frac{\left(2j-1`\right)t}{2}\right) = \sin kt.$$

Using the product-to-sum formula it follows that the formula is true for $m = k+1$:

$$\sin\frac{t}{2}\left(\sum_{j=1}^{k+1} 2\cos\frac{(2j-1)t}{2}\right) = \sin kt + 2\sin\frac{t}{2}\cos\frac{(2k+1)t}{2} = \sin kt + \sin(k+1)t - \sin kt = \sin(k+1)t.$$

We conclude recursively that the formula holds for every natural number m. □

Theorem 7. *If $b_0 = b_1 = \cdots = b_l = 1$ in (1) then*

$$E(t) = \frac{\sin\frac{(l+1)t}{2}}{\sin\frac{t}{2}}. \tag{11}$$

Proof. If l is even then (6) gives

$$E(t) = \sum_{j=0}^{l/2-1} 2\cos\left[\left(l/2 - j\right)t\right] + 1.$$

If we change the index of summation $J := l/2 - j$ and then reverse the order of summation we get

$$E(t) = \sum_{J=1}^{l/2} 2\cos Jt + 1. \tag{12}$$

Finally using Lemma 5 it follows that

$$\frac{\sin\frac{(l+1)t}{2}}{\sin\frac{t}{2}}.$$

If l is odd then (6) gives

$$E(t) = \sum_{j=0}^{(l-1)/2} 2\cos\left[\left(l/2 - j\right)t\right].$$

If we change the index of summation $J := 1/2 + l/2 - j$ and then reverse the order of summation we get

$$E(t) = \sum_{J=1}^{(l+1)/2} 2\cos\left[(J - 1/2)\,t\right]. \tag{13}$$

Finally using Lemma 6 we get

$$E(t) = \frac{\sin\frac{(l+1)t}{2}}{\sin\frac{t}{2}}.$$

□

In [3] Boyd and Mossinghoff introduced the following

Definition. *Let $\varphi_A(x)$ denote the polynomial $(x^A - 1)/(x-1)$, and write*

$$P_{A,B}(x,y) = x^{\max(A-B,0)}\left(\varphi_A(x) + \varphi_B(x)y + x^{B-A}\varphi_A(x)y^2\right).$$

Example 8. Let $H_{2n+2l}(x)$ denote

$$H_{2n+2l}(x) = x^{n+l}\left(\sum_{j=0}^{l}\left(x^{n+j} + \frac{1}{x^{n+j}}\right) + 1\right).$$

We can show that

$$H_{2n+2l}(x) = P_{l+1,1}\left(x, x^{n+l}\right)/x^l.$$

It is convenient to substitute $l = m-1$ in the previous example.

Theorem 9. *If m is an integer greater than 1 then*

$$\frac{2}{\pi(2m+1)}\frac{\sin\frac{(m-1)\pi}{2m}}{\sin\frac{\pi}{2m}} < LC\left(H_{2n+2m-2}(x)\right) < \frac{2}{6m-\pi}\frac{\sin\frac{(m-1)\pi}{2m}}{\sin\frac{\pi}{2m}}$$

Proof. Since $b_0 = b_1 = \cdots = b_{m-1}$ in the previous example, we can use Theorem 7 to determine the envelope: $E_m(T) = \frac{\sin\frac{mT}{2}}{\sin\frac{T}{2}}$. We have to solve (8) that is equivalent with $|E_m(T)| = 1/2$, $T \in [0, 2\pi]$ because $k = 0$, $a_0 = 1$. If we substitute $T = 2t$ it follows that we have to solve

$$2|\sin mt| = \sin t, \quad t \in [0, \pi]$$

because we have to determine the sum of length of all intervals where $2|\sin mt| < |\sin t|$ on $[0, \pi]$.

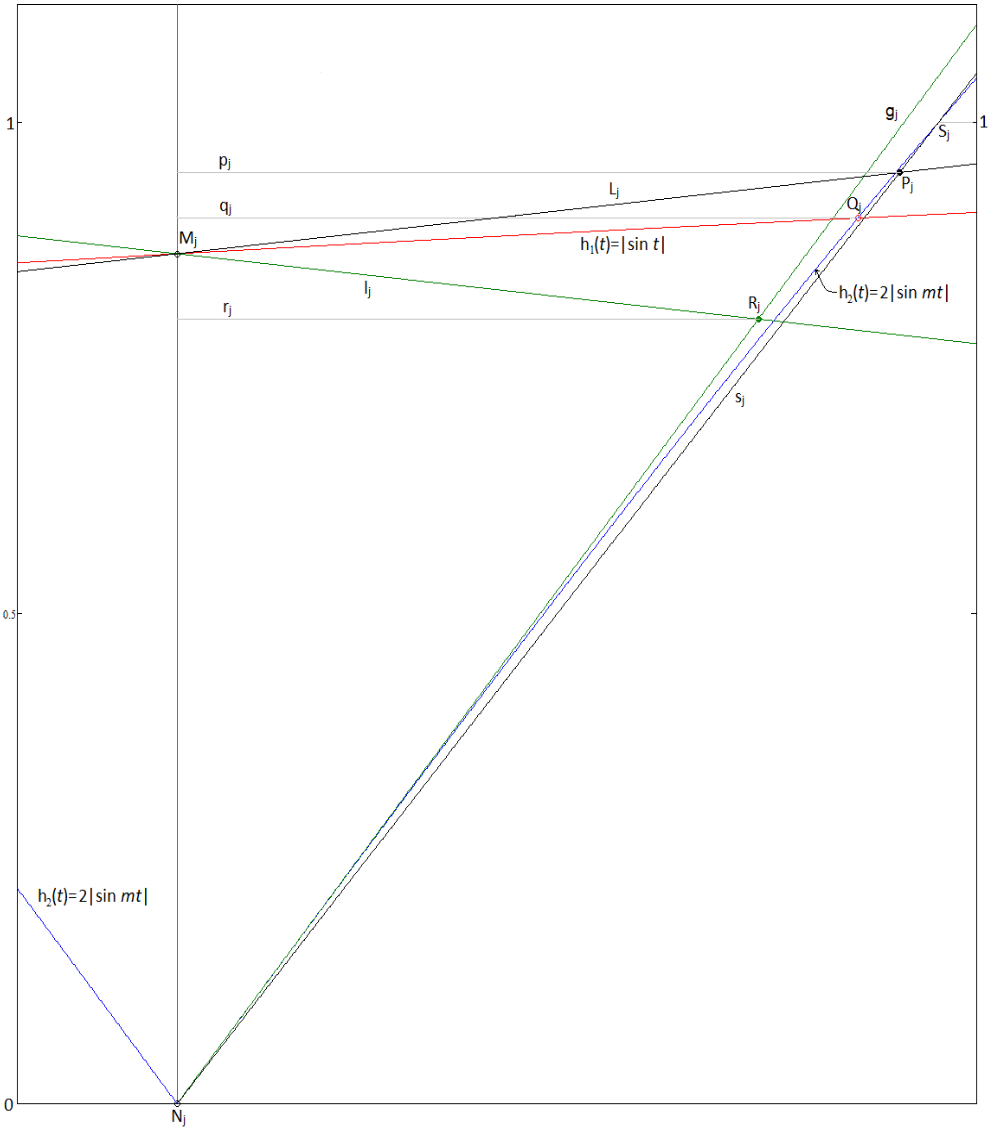

Figure 1. The estimation of the length of an interval where $2|\sin mt| < |\sin t|$.

Let G_1 be the graph of $h_1(t) = |\sin t|$ and let G_2 be the graph of $h_2(t) = 2|\sin mt|$. Let L_j be the line passing through $M_j(\frac{j\pi}{m}, \sin\frac{j\pi}{m})$ with the slope 1, and let l_j be the line passing through M_j with the slope -1 (see. Figure 1). Let g_j be the tangent line of $2|\sin mt|$ at $N_j(\frac{j\pi}{m}, 0)$ with the slope $2m$ and let s_j be the secant line of $2|\sin mt|$ passing through N_j and $S_j(\frac{j\pi}{m} + \frac{\pi}{6m}, 1)$. Let Q_j be the unique intersection point of G_1 and G_2 on the segment $I_j = [\frac{j\pi}{m}, \frac{j\pi}{m} + \frac{\pi}{2m}]$. Since $\frac{2}{\pi} < 1$ there is the unique intersection point P_j of s_j and L_j, and also the unique intersection point R_j of g_j and l_j. On I_j function h_2 increases and is concave down so that if p_j, q_j, r_j are distances from points P_j, Q_j, R_j, respectively, to the vertical line M_jN_j then $r_j < q_j < p_j$. To calculate p_j, r_j it is convenient to use horizontal translation of all these objects such that N_j moves to the origin O.

Then L_j moves to $L'_j : y = t + \sin\frac{j\pi}{m}$ and s_j moves to $s'_j : y = \frac{6m}{\pi}t$. The solution t of the system of these two equations is equal to p_j so that

$$p_j = \frac{\sin\frac{j\pi}{m}}{\frac{6m}{\pi} - 1}.$$

Also l_j moves to $l'_j : y = -t + \sin\frac{j\pi}{m}$ and g_j moves to $g'_j : y = 2mt$. The solution t of the system of the last two equations is equal to r_j so that

$$r_j = \frac{\sin\frac{j\pi}{m}}{2m+1}.$$

Similarly, let $\overline{Q}_j$ be the unique intersection point of G_1 and G_2 on the segment $\overline{I}_j = [\frac{j\pi}{m} - \frac{\pi}{2m}, \frac{j\pi}{m}]$. Let $\overline{q}_j$ be the distances from point $\overline{Q}_j$ to the vertical line M_jN_j. Since the line $x = \frac{\pi}{2}$ is the axis of symmetry of G_1 as well as of G_2 it follows that $\overline{q}_j = q_{m-j}$ thus the sum of length of all intervals where $2|\sin mt| < |\sin t|$ on $[0,\pi]$ is equal to double $\sum_{j=1}^{m-1} q_j$. It follows from $r_j < q_j < p_j$ that $\frac{2}{\pi}\sum_{j=1}^{m-1} r_j < \frac{2}{\pi}\sum_{j=1}^{m-1} q_j < \frac{2}{\pi}\sum_{j=1}^{m-1} p_j$ so that

$$\frac{2}{\pi(2m+1)}\sum_{j=1}^{m-1}\sin\frac{j\pi}{m} < \frac{2}{\pi}\sum_{j=1}^{m-1} q_j < \frac{2}{6m-\pi}\sum_{j=1}^{m-1}\sin\frac{j\pi}{m}.$$

Finally if we use the formula for the sum of sines with arguments in arithmetic progression we obtain the claim of the theorem. □

Corollary 10. *If A is an adherent point of the sequence $(LC(P_{m,1}(x)))_{m\geq 1}$ then*

$$\frac{2}{\pi^2} \leq A \leq \frac{2}{3\pi}$$

Proof. We can easily show that the sequence of the lower bounds in the claim of previous theorem has the limit equal to $\frac{2}{\pi^2} \approx 0.2026$ and that the sequence of the upper bounds has the limit equal to $\frac{2}{3\pi} \approx 0.2122$ when $m \to \infty$. □

Conjecture 11. *The limit of the sequence in Corollary 10 exists with an approximate value of:*

$$\lim_{m\to\infty} LC(P_{m,1}(x)) \approx 0.209. \tag{14}$$

3. Approximating $\lim_{n\to\infty} C(P_{2n+2l})$

It is necessary to explain how we approximated the limit in (14). In the proof of Theorem 1 we actually declared the following steps of an algorithm for determining $\lim_{n\to\infty} C(P_{2n+2l})$:

(1) determine all real roots t_j of the equations $f_2(t) = E(t)$ and $f_2(t) = -E(t)$, where $E(t)$, $f_2(t)$ are defined in (6) and (7),
(2) arrange them as an increasing sequence $0 = t_0 < t_1 < \ldots < t_p = 2\pi$,
(3) determine r intervals $I_j = [\alpha_j, \beta_j]$ such that if $\alpha_j < t < \beta_j$ then $|f_2(t)| \leq |E(t)|$, $\alpha_j, \beta_j \in \{t_0, t_1, \ldots, t_p\}$,
(4) calculate $\lim_{n\to\infty} C(P_{2n+2l}) = 1 - \sum_{j=1}^{r}(\beta_j - \alpha_j)/(2\pi)$.

If we bring to mind (6) it follows that the equation $f_2(t) = \pm E(t)$ i.e. $-a_0/2 - \sum_{j=1}^{k} a_j \cos jt = \pm E(t)$ is algebraic in $\cos t$ so that t_j can be expressed by arccosine of an algebraic real number $\alpha \in [-1,1]$ thus only solutions of this kind should be taken into account.

If $f_0(t)$ is defined:

$$f_0(t) = \begin{cases} 1, & |f_2(t)| \geq |E(t)| \\ 0, & \text{otherwise} \end{cases}$$

then

$$\lim_{n\to\infty} C(P_{2n+2l}) = \frac{1}{2\pi}\int_0^{2\pi} f_0(t)dt. \tag{15}$$

We can approximate numerically the integral in (15) i.e. $\lim_{n\to\infty} C(P_{2n+2l})$. Suppose the interval $[0,2\pi]$ is divided into p equal subintervals of length $\Delta t = 2\pi/p$ so that we introduce a partition of $[0,2\pi]$ $0 = t_0 < t_1 < \ldots < t_p = 2\pi$ such that $t_j - t_{j-1} = \Delta t$. Then we chose numbers $\xi_j \in [t_j, t_{j-1}]$ and count all ξ_j such that $|f_2(\xi_j)| \geq |E(t)$, $j = 1,2,\ldots,p$. If there are s such ξ_j then $\lim_{n\to\infty} C(P_{2n+2l})$ is approximately equal to $\frac{s}{p}$.

$$\lim_{n\to\infty} C(P_{2n+2l}) \approx \frac{1}{p}\sum_{j=1}^{p} f_0\left(j\frac{2\pi}{p}\right)$$

where we chosed $\xi_j = 2j\pi/p$.

If we introduce the substitution $t = 2\pi u$ in (15) we get

$$\lim_{n\to\infty} C(P_{2n+2l}) = \int_0^1 f_0(2\pi u)du = \int_U du. \tag{16}$$

where $U = \{u \in [0,1] : |f_2(2\pi u)| \geq |E(2\pi u)|\}$.

The definition of the Mahler measure could be extended to polynomials in several variables. We recall Jensen's formula which states that

$$\int_0^1 \log\left|P\left(e^{2\pi i\theta}\right)\right| \mathrm{d}\theta = \log|a_0| + \sum_{j=1}^{d} \log\max\left(\left|\alpha_j\right|, 1\right).$$

Thus

$$M(P) = \exp\left\{\int_0^1 \log\left|P\left(e^{2\pi i\theta}\right)\right| \mathrm{d}\theta\right\},$$

so M(P) is just the geometric mean of $|P(z)|$ on the torus T. Hence a natural candidate for $M(F)$ is

$$M(F) = \exp\left\{\int_0^1 \mathrm{d}\theta_1 \cdots \int_0^1 \log\left|F\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_r}\right)\right| \mathrm{d}\theta_r\right\}.$$

Boyd and Mossinghoff in [3] listed in a table 48 bivariate polynomials having small Mahler measure. Here we calculated the limit ratio of polynomials correlated to bivariate polynomials quadratic in y and added them to the table. Flammang studied some other measures, defined for agebraic integers, in [7] [6].

If we bring to mind the calculation of the Mahler measure in Exercise 2.24 and especially in Exercise 2.25 in the new book of McKee and Smyth [9]:

$$M(P) = \exp\left(\int_U \log\frac{\left|f_2(2\pi u)\right| + \sqrt{f_2^2(2\pi u) - E^2(2\pi u)}}{|E(2\pi u)|} du\right)$$

where $U = \{u \in [0,1] : |f_2(2\pi u)| \geq |E(2\pi u)|\}$ then we can determine the correlation between Mahler measure and the limit ratio

$$LC(P) = \int_U du.$$

Table 1 presents $f_2(2\pi u)$ and $E(2\pi u)$ for certain families of polynomials, quadratic in y.

In Table 2 we present limit points calculated in [3] of Mahler measure of bivariate polynomials $P(x, y)$, quadratic in y, in ascending order. We complemented the table of Boyd and Mossinghoff by the limit points of the ratio of number of nonunimodular roots of the polynomial $P(x, x^n)$ to its degree when $n \to \infty$. As in [3] polynomials $P_{a,b}(x, y)$, $Q_{a,b}(x, y)$, $R_{a,b}(x, y)$, $S_{a,b,\epsilon}(x, y)$, defined in Table 1, are labeled as $P(a, b)$, $Q(a, b)$, $R(a, b)$, $S(a, b, \mathrm{sgn}(\epsilon))$ respectively, in Table 2. Some polynomials are identified by the sequences, for example the third smallest known limit point $(1+x)+(1-x^2+x^4)y+(x^3+x^4)y^2$, is identified by [++000, +0−0+, 000++], as in [3]. Polynomials in

Table 1. $f_2(2\pi u)$ and $E(2\pi u)$ for certain families of polynomials.

Family	Definition	$f_2(2\pi u)$	$E(2\pi u)$
$P_{a,b}(x,y)$	$x^{\max(a-b,0)}\left(\sum_{j=0}^{a-1} x^j + \sum_{j=0}^{b-1} x^j y + x^{b-a}\sum_{j=0}^{a-1} x^j y^2\right)$	$\sin\left(\frac{b}{2}\pi u\right)$	$2\sin\left(\frac{a}{2}\pi u\right)$
$Q_{a,b}(x,y)$	$x^{\max(a-b,0)}(1+x^a+(1+x^b)y+x^{b-a}(1+x^a)y^2$	$\cos\left(\frac{b}{2}\pi u\right)$	$2\cos\left(\frac{a}{2}\pi u\right)$
$R_{a,b}(x,y)$	$x^{\max(a-b,0)}(1+x^a+(1-x^b)y-x^{b-a}(1+x^a)y^2$	$\sin\left(\frac{b}{2}\pi u\right)$	$2\cos\left(\frac{a}{2}\pi u\right)$
$S_{a,b,\epsilon}(x,y)$	$1+(x^a+\epsilon)(x^b+\epsilon)y+x^{a+b}y^2,\ \epsilon=\pm 1$	$\cos\left(\frac{a+b}{2}\pi u\right)+\epsilon\cos\left(\frac{b-a}{2}\pi u\right)$	1

Table 2 are written explicitly in Table D.2 of [9]. We excluded the polynomials not quadratic in y. It is interesting to compare the Mahler measure and the limit ratio of polynomials in two variables.

(1) The Mahler measure is ≥ 1 while the limit ratio is in $[0,1]$.
(2) Mahler measures of two polynomials can be equal though their limit ratios are different (see examples (2) and (2') in Table 2.
(3) Mahler Measures of two polynomials increase while the corresponding limit ratios decrease.
(4) The polynomial $P_{2,3}$ has the smallest Mahler measure and the smallest limit ratio.
(5) The second smallest Mahler measure comes from $P_{2,1}$ and $P_{1,3}$ while the second smallest limit ratio corresponds to $R_{1,5}$.

We showed in Example 3 and Theorem 4 that the limit ratio can be arbitrary close to zero. It is clear that in this example the coefficients of the polynomials are unbounded. Our calculations show that if the coefficients are bounded then the limit ratio can not be arbitrary close to zero. Also, Theorem 9 supports our opinion that the analogue of Lehmer's conjecture is true:

Conjecture 12. *If N is a natural number ≥ 1 there is some $c(N) > 0$ such that any sequence P_{2n+2l} of integer polynomials defined in* (1)*, satisfying the condition $b_j = b_{l-j}$, having the coefficients $\leq N$ in modulus, that has the limit ratio strictly below $c(N)$ has the limit ratio equal to 0.*

Table 2: Limit points of Mahler measure and limit points of the ratio of number of nonunimodular roots of a polynomial to its degree.

	Mahler measure	Polynomial $\mathscr{P}$	$\lim_{n\to\infty} C(\mathscr{P})$	Exact value of $\lim_{n\to\infty} C(\mathscr{P})$, sequence
1.	1.2554338662666087457	P(2, 3)	0.1328095098966884	$1-2\arccos\left(\frac{\sqrt{2}}{2}-\frac{1}{2}\right)/\pi$
2.	1.2857348642919862749	P(2, 1)	0.1608612465103325	$1-2\arccos(1/4)/\pi$
2'.	1.2857348642919862749	P(1, 3)	0.3333333333333333	1/3
3.	1.3090983806523284595		0.2970136797597501	[++000, +0−0+, 000++]
4.	1.3156927029866410935	P(3, 5)	0.1646453474320021	$\frac{4}{\pi}\arctan\frac{1}{\sqrt{2\sqrt{94-26\sqrt{13}}+4\sqrt{13}-13}}+$ $+\frac{-4}{\pi}\arctan\frac{1}{\sqrt{\frac{1}{2}\sqrt{\frac{544\sqrt{5}}{121}+\frac{992}{121}}+\frac{4\sqrt{5}}{11}+\frac{17}{11}}}$
6.	1.3253724973075860349	P(3, 4)	0.1739784246485862	
7.	1.3320511054374193142	P(2, 5)	0.2634504964561481	
8.	1.3323961294587154121	S(1, 3,+)	0.3814904582918582	$\arccos\left(\sqrt{17}/4-1/4\right)/\pi+1/6$
9.	1.3381374319388410775	P(3, 2)	0.1871346248477649	$\frac{2}{\pi}\left[\pi-\arccos\frac{1+\sqrt{17}}{8}-\arccos\frac{1-\sqrt{17}}{8}\right]$
10.	1.3399999217381835332	P(4, 7)	0.1784746137157699	
11.	1.3405068829308471079	P(3, 1)	0.1895159205822178	$\frac{2}{\pi}\left[\arcsin(\sqrt{14}/4)-\arcsin\left(\sqrt{10}/4\right)\right]$
13.	1.3500148321630142650	P(3, 7)	0.2403097841316317	

15.	1.3511458956697046903	P(4, 5)	0.1902698620670582	
16.	1.3524680625188602961	P(5, 9)	0.1860703555283188	
17.	1.3536976494626355711	Q(1, 6)	0.1893226580984896	
18.	1.3567481051456008311	P(4, 3)	0.1964065801899085	
19.	1.3567859884526454967	P(5, 8)	0.1908351326172760	
20.	1.3581296324044179208		0.3755212901021780	$0.4-\alpha_1+0.8-2/3+1-\alpha_2$, α_1,α_2 roots of $32z^6-48z^4+16z^2+2z-0.5$ [++, +0-[-]-0+,++]
21.	1.3585455903960511404	P(4, 1)	0.1981783524823832	
22.	1.3592080686995589268	P(4, 9)	0.2295536290347317	
23.	1.3598117752819405021	P(6, 11)	0.1908185635976727	
24.	1.3598158989877492950	S(1, 6,+)	0.3638326121576760	
26.	1.3602208408592842371	P(5, 7)	0.1947758787175794	
27.	1.3627242816569882815	P(5, 6)	0.1976969967166677	
28.	1.3636514981864992177	S(3, 5,+)	0.3616163835316277	
31.	1.3645459857899151366	P(7, 13)	0.1940425569464528	
32.	1.3646557293930641449	P(5, 11)	0.2236027778291241	
33.	1.3650623157174417179	S(2, 7,−)	0.3360946113639976	
34.	1.3654687370557201592	P(5, 4)	0.2007692138817449	
36.	1.3661459663116649518	P(5, 3)	0.2014521139875612	
37.	1.3665709746056369455	P(5, 2)	0.2018615118309531	
38.	1.3668078899273126149	P(5, 1)	0.2020844014923849	
39.	1.3668830708592258921	R(1, 5)	0.1417550822341309	
40.	1.3669909125179202255	P(7, 12)	0.1970232013102869	
41.	1.3677988580117157740	P(8, 15)	0.1963614081210482	
43.	1.3681962517212729703	P(6, 13)	0.2199360577499605	
44.	1.3682140096679950123	P(1, 9)	0.2082012946810569	
45.	1.3683434385467330804		0.3045732337814742	[++00000, ++0−0++,00000++]
46.	1.3687474425069274154	P(6, 7)	0.2014928273535877	
47.	1.3689491694959833864	P(7, 11)	0.1994880038265199	
48.	1.3697823199880122791	S(1, 9,+)	0.3622499773114010	

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References

[1] P. Borwein, S. Choi, R. Ferguson, J. Jankauskas, "On Littlewood polynomials with prescribed number of zeros inside the unit disk", *Can. J. Math.* **67** (2015), no. 3, p. 507-526.

[2] P. Borwein, T. Erdélyi, R. Ferguson, R. Lockhart, "On the zeros of cosine polynomials: solution to a problem of Littlewood", *Ann. Math.* **167** (2008), no. 3, p. 1109-1117.

[3] D. W. Boyd, M. J. Mossinghoff, "Small limit points of Mahler's measure", *Exp. Math.* **14** (2005), no. 4, p. 403-414.

[4] P. Drungilas, "Unimodular roots of reciprocal Littlewood polynomials", *J. Korean Math. Soc.* **45** (2008), no. 3, p. 835-840.

[5] P. Erdős, P. Turán, "On the Distribution of Roots of Polynomials", *Ann. Math.* **51** (1950), no. 1, p. 105-119.

[6] V. Flammang, "The N-measure for algebraic integers having all their conjugates in a sector", *Rocky Mt. J. Math.* **50** (2020), no. 6, p. 2035-2045.

[7] ———, "The S-measure for algebraic integers having all their conjugates in a sector", *Rocky Mt. J. Math.* **50** (2020), no. 4, p. 1313-1321.

[8] C. Guichard, J.-L. Verger-Gaugry, "On Salem numbers, expansive polynomials and Stieltjes continued fractions", *J. Théor. Nombres Bordeaux* **27** (2015), no. 3, p. 769-804.

[9] J. McKee, C. Smyth, *Around the Unit Circle. Mahler measure, integer matrices and roots of unity.*, Universitext, Springer, 2021.

[10] K. Mukunda, "Littlewood Pisot numbers", *J. Number Theory* **117** (2006), no. 1, p. 106-121.

[11] R. Salem, "Power series with integral coefficients", *Duke Math. J.* **12** (1945), p. 153-172.

[12] ———, *Algebraic numbers and Fourier analysis*, D. C. Heath and Company, 1963.

[13] D. Stankov, "The number of unimodular roots of some reciprocal polynomials", *C. R. Math. Acad. Sci. Paris* **358** (2020), no. 2, p. 159-168.