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Stability of a weighted L2 projection in a weighted Sobolev norm

Stabilité d'une projection L2 à poids dans une norme de Sobolev à poids

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Abstract. We prove the stability of a weighted L^2 projection operator onto piecewise linear finite elements spaces in a weighted Sobolev norm. Namely, we consider the orthogonal projections $\pi_{N,\omega}$ from $L^2(\mathbb{D}, 1/\omega(x)\mathrm{d}x)$ to $\mathscr{X}_N$, where $\mathbb{D} \subset \mathbb{R}^2$ is the unit disk, $\omega(x) = \sqrt{1-|x|^2}$ and the spaces $(\mathscr{X}_N)_{N\in\mathbb{N}}$ consist of piecewise linear functions on a family of shape-regular and quasi-uniform triangulations of $\mathbb{D}$. We show that $\pi_{N,\omega}$ is stable in a weighted Sobolev norm, and prove an upper bound on the stability constant that does not depend on N. The result also holds when the disk $\mathbb{D}$ is replaced by a more general surface $\Gamma \subset \mathbb{R}^3$, replacing the weight ω by $\omega_\Gamma(x) := \sqrt{\mathrm{d}(x, \partial\Gamma)}$, the square root of the distance from x to the manifold boundary of Γ.

Résumé. On démontre la stabilité dans une norme de Sobolev à poids, de la projection orthogonale par rapport au produit scalaire d'un espace L^2 à poids, sur une famille d'éléments finis linéaires par morceaux. Plus précisément, soit $\pi_{N,\omega}$, de $L^2(\mathbb{D}, 1/\omega(x)\mathrm{d}x)$ dans $\mathscr{X}_N$, où $\mathbb{D} \subset \mathbb{R}^2$ est le disque unité, $\omega(x) = \sqrt{1-|x|^2}$ et les espaces $(\mathscr{X}_N)_{N\in\mathbb{N}}$ sont des espaces de fonctions continues et linéaires par morceaux sur une famille de triangulations régulière de $\mathbb{D}$. On montre que $\pi_{N,\omega}$ est stable dans une norme de Sobolev à poids, avec une borne supérieure sur la constante de stabilité qui ne dépend pas de N. Le résultat s'étend au cas de surfaces plus générales $\Gamma \subset \mathbb{R}^3$, en remplaçant le poids ω par $\omega_\Gamma(x) := \sqrt{\mathrm{d}(x, \partial\Gamma)}$, la racine carrée de la distance de x à $\partial\Gamma$, le bord de Γ.

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1. Introduction and motivation

Let $\mathbb{D} = \left\{x \in \mathbb{R}^2 \,\middle|\, |x|^2 < 1\right\}$ be the unit disk of $\mathbb{R}^2$, where $|x| = \sqrt{x_1^2 + x_2^2}$ stands for the Euclidean norm of x. Let $H^{1/2}(\mathbb{D})$ be the classical Sobolev space on $\mathbb{D}$, when $\mathbb{D}$ is regarded as a 2-dimensional manifold in $\mathbb{R}^3$. For example $H^{1/2}(\mathbb{D})$ can be defined, among other equivalent formulations (see e.g. [10, Chap. 3]), as the closure of $C^\infty(\overline{\mathbb{D}})$ for the norm

$$\|u\|_{1/2,\mathbb{D}}^2 := \inf\left\{\int_{\mathbb{R}^3} |F(x)|^2 + |\nabla F(x)|^2 \,\mathrm{d}x \,\middle|\, F \in C_c^\infty(\mathbb{R}^3) \text{ such that } F_{|\mathbb{D}} = u\right\}. \tag{1}$$

For $x \in \mathbb{D}$, let

$$\omega(x) := \sqrt{1-|x|^2}\,. \tag{2}$$

and let $A_{\mathbb{D}}$ be the *weighted Laplacian* defined by

$$A_{\mathbb{D}}\, u(x) := -\omega(x)\,\mathrm{div}(\omega \nabla u)(x)\,, \quad u \in C^\infty(\overline{\mathbb{D}})\,.$$

It is shown in [1] that $A_{\mathbb{D}}$ admits a self-adjoint extension on the *weighted L^2 space*

$$L^2_{1/\omega} := \left\{ u \in L^1_{\mathrm{loc}}(\mathbb{D}) \;\middle|\; \|u\|^2_{1/\omega} := \int_{\mathbb{D}} \frac{|u(x)|^2}{\omega(x)}\mathrm{d}x < +\infty \right\}, \tag{3}$$

with the domain of $\sqrt{Id + A_{\mathbb{D}}}$ (where Id stands for the identity operator) given by the *weighted Sobolev space*

$$T^1 := \left\{ u \in L^2_{1/\omega} \;\middle|\; \|u\|^2_{T^1} := \|u\|^2_{1/\omega} + \int_{\mathbb{D}} \omega(x)\,|\nabla u(x)|^2\,\mathrm{d}x < +\infty \right\}. \tag{4}$$

Moreover, there exists a constant $C > 0$ such that

$$\frac{1}{C}\left(\sqrt{\mathrm{Id}+A_{\mathbb{D}}}\,u, u\right)_{1/\omega} \le \|u\|^2_{1/2,\mathbb{D}} \le C\left(\sqrt{\mathrm{Id}+A_{\mathbb{D}}}\,u, u\right)_{1/\omega}, \quad \forall\, u \in T^1\,, \tag{5}$$

where $(\cdot,\cdot)_{1/\omega}$ is the inner product of $L^2_{1/\omega}$ (cf. [1, Thm. 3]). The singular weight ω plays an essential role in the above formula. Indeed, if we instead take $\omega \equiv 1$ in what precedes, i.e. if we replace $A_{\mathbb{D}}$ by the standard negative Laplace operator $-\Delta$, consider the $L^2(\mathbb{D})$ inner product $(\cdot,\cdot)$, and use the non-weighted Sobolev space $H^1_0(\mathbb{D})$, then one has

$$\|u\|^2_{H^{1/2}_{00}} \sim \left(\sqrt{\mathrm{Id}-\Delta}\,u, u\right), \quad \forall\, u \in H^1_0(\mathbb{D})\,,$$

where the left hand side now involves the $H^{1/2}_{00}(\mathbb{D})$ norm (see e.g. [3, p. 9]), which is strictly stronger than the $H^{1/2}(\mathbb{D})$ norm. Let us also mention the connection between the weight ω and the singularity of the geometry considered (i.e. the presence of the sharp edge $\partial\mathbb{D}$, when $\mathbb{D}$ is considered as a manifold in $\mathbb{R}^3$). Namely, given the restriction u of a smooth function in $\mathbb{R}^3$, the unique bounded extension of u which is harmonic in $\mathbb{R}^3 \setminus \overline{\mathbb{D}}$ has normal derivatives behaving like $O(1/\omega)$ towards the edge of $\mathbb{D}$ (see e.g. [7]).

Provided that we can approximate $A_{\mathbb{D}}$, Eq. (5) may offer an efficient method to estimate the $H^{1/2}$ norm of u. To this aim, consider a finite-dimensional subspace $\mathscr{X}_N$ of $H^{1/2}(\mathbb{D}) \cap T^1$ and let $A_{N,\mathbb{D}} : \mathscr{X}_N \to \mathscr{X}_N$ be the *discrete* weighted Laplacian defined by

$$\left(A_{\mathbb{D},N} u_N, v_N\right)_{1/\omega} := (A_{\mathbb{D}} u_N, v_N)_{1/\omega}\,, \quad u_N, v_N \in \mathscr{X}_N\,.$$

The point is that $A_{\mathbb{D},N}$ can be computed numerically and, in turn, $\sqrt{I_N + A_{\mathbb{D},N}}$ can be obtained from $A_{\mathbb{D},N}$ via matrix functional calculus, e.g. using contour integration techniques. As a result, the discrete norm

$$\|u_N\|^2_{A_{\mathbb{D}},N} := \left(\sqrt{I_N + A_{\mathbb{D},N}}\,u_N, u_N\right) \quad u_N \in \mathscr{X}_N\,, \tag{6}$$

can be evaluated efficiently and accurately. Whether this discrete norm is a suitable replacement for the continuous norm Eq. (8) is related to the stability property of weighted projection operator. More precisely, we have the following result:

Theorem 1 (cf. [1, Thm. 6]). *Let $\pi_{N,\omega} : L^2_{\frac{1}{\omega}} \to \mathscr{X}_N$ be the $L^2_{1/\omega}$-orthogonal projection onto $\mathscr{X}_N$, and assume that there exists a constant $C_\pi > 0$ such that*

$$\forall\, u \in T^1\,, \quad \left\|\pi_{N,\omega} u\right\|_{T^1} \le C_\pi \|u\|_{T^1}\,. \tag{7}$$

Then there holds

$$\forall\, u_N \in \mathscr{X}_N\,, \quad \frac{1}{C}\|u_N\|^2_{A_{\mathbb{D}},N} \le \|u_N\|^2_{1/2,\mathbb{D}} \le C C_\pi \|u_N\|^2_{A_{\mathbb{D}},N}\,. \tag{8}$$

The purpose of this paper is to prove that, when $\mathscr{X}_N$ is chosen as a space of piecewise linear continuous functions on a regular mesh of $\mathbb{D}$, then the stability estimate (7) does hold, with a constant C_π independent of the mesh size (see Theorem 2).

It is hard to overstate the importance of the $H^{1/2}$ in the context of second-order elliptic boundary value problems. Having at hand a cheap way to evaluate this norm on a manifold with boundary may have important applications. As an example, this is the basis of the preconditioning method for the weakly singular integral equation on $\mathbb{D}$ proposed in [1].

Notice that Theorem 1 is completely analogous to the result mentioned in [3, p. 9] according to which the $H^{1/2}_{00}(\mathbb{D})$ is uniformly equivalent to the norm generated by the square root of the discrete (non-weighted) Laplacian on $\mathscr{X}_N$. In this case, the stability assumption concerns the (non-weighted) $L^2(\mathbb{D})$ projection in the standard $H^1(\mathbb{D})$ norm, which is a very well-studied topic (cf. e.g. [4, 5, 8]).

In contrast, the T^1 stability of $\pi_{N,\omega}$, is not studied in the literature, to the best of our knowledge, and cannot be obtained by routine application of standard arguments. In fact, the singularity of the weight ω poses a significant challenge for the analysis for several reasons. Firstly, pullback to a reference element, and scaling arguments, which are at the heart of the proofs of the H^1-stability of the L^2 projection, are completely ineffective to analyze $\pi_{N,\omega}$. Indeed, the spaces T^1 and $L^2_{1/\omega}$ lack simple scaling properties. Secondly, the analysis involves some weighted Poincaré-type inequalities, which are not similar to the ones encountered in the literature (e.g. [9, 11]). One of the major steps in our proof is a careful estimate of the domain-dependent constant in such inequalities (cf. Theorem 7).

Given the importance of the H^1-stability of the L^2 projection in various fields of numerical analysis — e.g. to analyze multigrid methods and domain decomposition methods [12], or to prove the quasi-optimality of the Galerkin approximation for parabolic problems [8] — we believe that the corresponding result in weighted spaces might also find additional applications to the one presented above, and thus, the proof might be of interest in itself.

2. Notation and statement of the main result

Let $(P_N)_{N\in\mathbb{N}}$ be a sequence of polygonal approximations of the disk. That is, $P_N \subset \overline{\mathbb{D}}$ and the vertices of P_N all lie in $\partial\mathbb{D}$. Furthermore, the maximal distance between two consecutive vertices of P_N is denoted by h_N, and we assume that

$$\lim_{N\to\infty} h_N = 0\,. \tag{9}$$

For each N, we consider a regular triangulation $\mathscr{T}_N$ of P_N, i.e. a set of pairwise disjoint open triangles, with the usual conformity assumptions (two triangles of $\mathscr{T}_N$ can only intersect along a common vertex or edge, or not at all) an such that

$$\bigcup_{\tau\in\mathscr{T}_N} \bar{\tau} = \overline{P_N}\,. \tag{10}$$

For each $\tau \in \mathscr{T}_N$, let h_τ and Δ_τ be the diameter and the area of τ, respectively. We assume that there exist constants c_1, C_1 and c_2, independent of N and τ, such that

$$c_1 h_N \le h_\tau \le C_1 h_N\,, \quad \text{(global quasi-uniformity)}, \tag{11}$$

$$\frac{\Delta_\tau}{h_\tau^2} \ge c_2\,, \quad \text{(uniform shape-regularity)}. \tag{12}$$

To construct piecewise linear functions that are continuous on $\mathbb{D}$ (which then makes them elements of T^1), special attention must be paid to the triangles on the boundary of the triangulation. If τ has two vertices A and B in $\partial\mathbb{D}$, let U_τ be the open region of $\mathbb{D}$ enclosed, on the one hand, by the smallest arc of $\partial\mathbb{D}$ linking A to B, and on the other hand, by the straight line segment $[A,B]$. Let K_τ be the open convex region resulting from the union of τ and U_τ, i.e.

$$\overline{K_\tau} := \bar{\tau} \cup \overline{U_\tau}\,. \tag{13}$$

When τ has one or zero vertex in $\partial\mathbb{D}$, we simply put $K_\tau = \tau$. Then, the set $\{K_\tau\}_{\tau\in\mathscr{T}_N}$ partitions $\mathbb{D}$ in the sense that

$$\bigcup_{\tau\in\mathscr{T}_N} \overline{K_\tau} = \overline{\mathbb{D}}\,. \tag{14}$$

With these definitions, let

$$\mathscr{X}_N := \left\{u \in C^0(\overline{\mathbb{D}}) \;\middle|\; u_{|K_\tau} \text{ is affine for all } \tau \in \mathscr{T}_N\right\}. \tag{15}$$

It is clear that $\mathscr{X}_N$ is a finite-dimensional subspace of T^1. We can now define

$$\pi_{N,\omega} : L^2_{1/\omega} \to \mathscr{X}_N\,, \tag{16}$$

the $L^2_{1/\omega}$-orthogonal projection onto $\mathscr{X}_N$. The main result of this paper is the following:

Theorem 2. *There exists a constant $C_\pi > 0$ independent of N such that,*

$$\forall\, u \in T^1\,, \quad \left\|\pi_{N,\omega} u\right\|_{T^1} \le C_\pi \left\|u\right\|_{T^1}\,. \tag{17}$$

Remark 3. A similar result also holds when $\mathbb{D}$ is replaced by a surface $\Gamma \subset \mathbb{R}^3$, with surface measure $d\sigma$, and with a shape regular and quasi-uniform family of triangulations $\mathscr{T}_N(\Gamma)$. Let

$$\omega_\Gamma(x) := \sqrt{d(x,\partial\Gamma)}$$

where $\partial\Gamma$ is the manifold boundary of Γ and set

$$L^2_{1/\omega_\Gamma} = \left\{u \in L^1_{\mathrm{loc}}(\Gamma) \;\middle|\; \|u\|^2_{1/\omega_\Gamma} := \int_\Gamma \frac{|u(x)|^2}{\omega_\Gamma(x)}\mathrm{d}\sigma(x) < \infty\right\},$$

$$T^1(\Gamma) := \left\{u \in L^2_{1/\omega_\Gamma} \;\middle|\; \|u\|^2_{T^1(\Gamma)} := \|u\|^2_{1/\omega_\Gamma} + \int_\Gamma \omega_\Gamma(x)\,|\nabla_\Gamma u(x)|^2\,\mathrm{d}\sigma(x) < \infty\right\},$$

where ∇_Γ is the tangential gradient on Γ. Let $\mathscr{X}_N(\Gamma)$ be the set of continuous piecewise linear functions on $\mathscr{T}_N(\Gamma)$ and $\pi_{N,\omega_\Gamma} : L^2_{1/\omega_\Gamma} \to \mathscr{X}_N(\Gamma)$ the L^2_{1/ω_Γ}-orthogonal projection onto $\mathscr{X}_N(\Gamma)$. Then there exists a constant $C_{\pi,\Gamma}$ independent of N such that

$$\forall\, u \in T^1(\Gamma)\,, \quad \left\|\pi_{N,\omega_\Gamma}\right\|_{T^1(\Gamma)} \le C_{\pi,\Gamma} \left\|u\right\|_{T^1(\Gamma)}\,.$$

The proof contains no essential additional difficulty compared to that Theorem 2, so in the rest of this paper, we restrict our attention to the case of the unit disk $\mathbb{D}$ for the sake of clarity and conciseness.

The remainder of this paper is organized as follows. In Section 3, we state a first lemma to reduce the proof of Theorem 2 to the proof of three key properties (A1)–(A3). In Sections 4 and 5, we derive weighted Poincaré and local inverse inequalities, respectively. Finally, in Section 6, we define a quasi-interpolant I_N and show that it meets the requirements.

In the proofs, we use the letter C to denote a generic positive constant that is independent of the index $N \in \mathbb{N}$ of the triangulation $\mathscr{T}_N$. The value of C is allowed to change from line to line. Nevertheless, we refrain from doing so in the result statements, to ensure the clarity of our discussion. We will also use the following notation: for an open set $U \subset \mathbb{D}$, let

$$\|u\|^2_{U,1/\omega} := \int_U \frac{|u(x)|^2}{\omega(x)}\mathrm{d}x \quad \text{and} \quad \|u\|^2_{U,T^1} := \|u\|^2_{U,1/\omega} + \int_U \omega(x)\,|u(x)|^2\,\mathrm{d}x\,.$$

3. Three key properties

Our analysis of $\pi_{N,\omega}$ relies on three main ingredients.

(A1) A *quasi-interpolant* $I_N : L^2_{1/\omega} \to \mathscr{X}_N$ that is uniformly T^1-continuous, i.e. there exists a constant $C_I > 0$ such that

$$\forall\, N \in \mathbb{N},\ \forall\, u \in T^1, \quad \|I_N u\|_{T^1} \le C_I \|u\|_{T^1}, \tag{18}$$

and such that, for each $N \in \mathbb{N}$ and $\tau \in \mathscr{T}_N$, there exists a constant $C_P(K_\tau) > 0$ such that

$$\forall\, u \in T^1, \quad \|u - I_N u\|^2_{K_\tau,1/\omega} \le C_P(K_\tau)^2 \|u\|^2_{\Sigma_\tau,T^1}, \tag{19}$$

where Σ_τ is the union of the domains $K_{\tau'}$ such that τ' and τ are *neighbors*, i.e. share at least one vertex τ.

(A2) Some local *inverse inequalities* in $\mathscr{X}_N$: for all $\theta \in \mathscr{X}_N$ and for all $\tau \in \mathscr{T}_N$, there exists a constant $C_{\text{inv}}(K_\tau) > 0$ such that

$$\|\theta\|^2_{K_\tau,T^1} \le C_{\text{inv}}(K_\tau)^{-2} \|\theta\|^2_{K_\tau,1/\omega}. \tag{20}$$

(A3) A uniform estimate of the ratios $C_P(K_\tau)/C_{\text{inv}}(K_\tau)$, i.e. there exists a constant $C_{\text{rat}} > 0$ such that

$$\forall\, N \in \mathbb{N},\ \forall\, \tau \in \mathscr{T}_N, \quad C_P(K_\tau)/C_{\text{inv}}(K_\tau) \le C_{\text{rat}}. \tag{21}$$

Lemma 4. *If* (A1)–(A3) *hold, then the orthogonal projection $\pi_{N,\omega}$ satisfies Theorem 2 with*

$$C_\pi = \sqrt{2(K_\sharp C^2_{\text{rat}} + C_I^2)}, \tag{22}$$

$K_\sharp$ being an upper bound for all N on the maximal number of neighbors of $\tau \in \mathscr{T}_N$.

Proof. We adapt a well-known argument appearing for example in the proof of [2, Lem. 1]. Given $N \in \mathbb{N}$ and $u \in T^1$, we write

$$\begin{aligned}
\left\|\pi_{N,\omega} u\right\|^2_{T^1} &\le 2\left(\left\|\pi_{N,\omega}(u - I_N u)\right\|^2_{T^1} + \|I_N u\|^2_{T^1}\right)\\
&\le 2\left(\sum_{\tau\in\mathscr{T}_N} \left\|\pi_{N,\omega}(u - I_N u)\right\|^2_{K_\tau,T^1}\right) + 2C_I^2 \|u\|^2_{T^1}\\
&\le 2\left(\sum_{\tau\in\mathscr{T}_N} C_{\text{inv}}(K_\tau)^{-2} \left\|\pi_{N,\omega}(u - I_N u)\right\|^2_{K_\tau,1/\omega}\right) + 2C_I^2 \|u\|^2_{T^1}\\
&\le 2\left(\sum_{\tau\in\mathscr{T}_N} C_{\text{inv}}(K_\tau)^{-2} \|u - I_N u\|^2_{K_\tau,1/\omega}\right) + 2C_I^2 \|u\|^2_{T^1}\\
&\le 2\left(\sum_{\tau\in\mathscr{T}_N} C_{\text{inv}}(K_\tau)^{-2} C_P(K_\tau)^2 \|u\|^2_{U_K,T^1}\right) + 2C_I^2 \|u\|^2_{T^1}\\
&\le 2(K_\sharp C^2_{\text{rat}} + C_I^2) \|u\|^2_{T^1}.
\end{aligned}$$

We have applied, successively: the triangle inequality, the property that $\pi_{N,\omega}\theta = \theta$ for all $\theta \in \mathscr{X}_N$, the uniform continuity (i), the inverse inequalities (ii), the minimization properties of $\pi_{N,\omega}$ in $L^2_{1/\omega}$, the weighted Poincaré inequalities (i) eq.(19), the estimate of the ratio $C_P(K_\tau)/C_{\text{inv}}(K_\tau)$ (iii), and the definition of $K_\sharp$. □

In the next sections, we show that (A1)–(A3) hold, see Lemmas 10, 12 and 13, respectively.

4. Weighted Poincaré inequalities

In what follows, for any open region $U \subset \mathbb{D}$, we write

$$\langle u \rangle_U := \left(\int_U 1/\omega(x) \right)^{-1} \int_U \frac{u(x)}{\omega(x)} \, \mathrm{d}x. \tag{23}$$

It is proved in [1, Thm. 1] that

$$\forall\, u \in T^1, \quad \| u - \langle u \rangle_{\mathbb{D}} \|^2_{1/\omega} \leq \int_{\mathbb{D}} \omega(x) \, |\nabla u(x)|^2 \, \mathrm{d}x. \tag{24}$$

The goal of this section is to prove similar inequalities when the domain of integration is replaced by a subset of $\mathbb{D}$. We start with two technical lemmas.

Lemma 5. *Let $N \in \mathbb{N}$, Q a vertex of the triangulation $\mathcal{T}_N$ and $\varphi_{N,Q}$ the element of $\mathcal{X}_N$ such that*

$$\varphi_{N,Q}(Q') = \begin{cases} 1 & \text{if } Q = Q', \\ 0 & \text{otherwise}, \end{cases} \tag{25}$$

for all vertices Q' of $\mathcal{T}_N$. Let $S_{N,Q} = \operatorname{supp} \varphi_{N,Q}$. Then, there exists a bilipschitz application $\kappa_{N,Q}$, mapping $\mathbb{D}$ to $S_{N,Q}$ such that

$$\forall\, x, y \in \mathbb{D}, \quad c_3 h_N \left| x - y \right| \leq \left| \kappa_{N,Q}(x) - \kappa_{N,Q}(y) \right| \leq C_3 h_N \left| x - y \right|, \tag{26}$$

where the constants c_3 and C_3 do not depend on N nor on Q.

The proof can be done by introducing polar coordinates in $S_{N,Q}$, centered at the vertex Q, and using the shape-regularity of $(\mathcal{T}_N)_{N \in \mathbb{N}}$.

Lemma 6. *Let A and B be two bounded open sets and $\kappa : A \to B$ such that*

$$\forall\, x, y \in A, \quad l \left\| x - y \right\| \leq \left\| \kappa(x) - \kappa(y) \right\| \leq L \left\| x - y \right\|. \tag{27}$$

Then there holds

$$\forall\, x \in A, \quad l \leq \frac{d(\kappa(x), \partial B)}{d(x, \partial A)} \leq L. \tag{28}$$

Proof. Let $x \in A$. For any $y \in \partial A$, $\kappa(y) \in \partial B$, so

$$d(\kappa(x), \partial B) \leq \left\| \kappa(x) - \kappa(y) \right\| \leq L \left\| x - y \right\|.$$

Taking the infimum over $y \in \partial A$, we deduce

$$d(\kappa(x), \partial B) \leq L d(x, \partial A).$$

The left inequality is obtained by a similar reasoning. □

Let us point out that for all $x \in \mathbb{D}$,

$$1 \leq \frac{\omega(x)}{\sqrt{d(x, \partial \mathbb{D})}} \leq 2. \tag{29}$$

These remarks being made, we can prove the following result:

Theorem 7. *Let $N \in \mathbb{N}$ and Q be a vertex of $\mathcal{T}_N$. Let $S = S_{N,Q}$ be defined as in Lemma 5.*

$$\forall\, u \in T^1, \quad \| u - \langle u \rangle_S \|^2_{S, 1/\omega} \leq C_4 \gamma(S) h_N \| u \|^2_{S, T^1} \tag{30}$$

where $C_4 > 0$ does not depend on N nor on Q and where

$$\gamma(S) := \sup_{x \in S} \frac{d(x, \partial S)}{d(x, \partial \mathbb{D})}. \tag{31}$$

Proof. To begin with, we observe that for any $\alpha \in \mathbb{C}$,

$$\|u - \langle u\rangle_S\|^2_{S,1/\omega} \le \|u-\alpha\|^2_{S,1/\omega}\,.$$

Let $\alpha \in \mathbb{C}$ and $v = u - \alpha$. The main idea is the following estimate:

$$\int_S \frac{|v(x)|^2}{\omega(x)}\,\mathrm{d}x \le \int_S \frac{|v(x)|^2\,\mathrm{d}x}{\sqrt{d(x,\partial\mathbb{D})}} \le \sqrt{\gamma(S)}\int_S \frac{|v(x)|^2\,\mathrm{d}x}{\sqrt{d(x,\partial S)}}\,.$$

Now, the singularity of the integrand is on ∂S, and by mapping S to the disk, we will be able to use the Poincaré inequality (24). To see this, let us introduce the change of variables $x = \kappa(y)$, where $\kappa : \mathbb{D} \to S$ is a bilipschitz map as in Lemma 5. This leads to

$$\int_S \frac{|v(x)|^2}{\omega(x)}\,\mathrm{d}x \le C\sqrt{\gamma(S)}\,h_N^2 \int_{\mathbb{D}} \frac{\left|v\circ\kappa(y)\right|^2\mathrm{d}y}{\sqrt{d(\kappa(y),\partial S)}}\,\mathrm{d}y.$$

By Lemmas 5 and 6, there holds

$$d(\kappa(y),\partial S) \ge C h_N d(y,\partial\mathbb{D}) \ge C h_N \omega^2\,.$$

We deduce that

$$\int_S \frac{|v(x)|^2}{\omega(x)}\,\mathrm{d}x \le C\sqrt{\gamma(S)}\,\frac{h_N^2}{\sqrt{h_N}}\int_{\mathbb{D}} \frac{\left|f(y)-\alpha\right|^2}{\omega(y)}\mathrm{d}y \tag{32}$$

$$\le C\sqrt{\gamma(S)}\,h_N^{3/2}\int_{\mathbb{D}} \frac{\left|f(y)-\alpha\right|^2}{\omega(y)}\mathrm{d}y \tag{33}$$

where $f(y) := u(\kappa(y))$. Taking $\alpha = \langle f\rangle_{\mathbb{D}}$, we can now apply the inequality (24) to f:

$$\int_{\mathbb{D}} \frac{\left|f(y)-\alpha\right|^2}{\omega(y)}\mathrm{d}y \le \int_{\mathbb{D}} \omega(y)\left|\nabla f(y)\right|^2\mathrm{d}y.$$

Injecting this inequality in what precedes, we obtain

$$\|u - \langle u\rangle_S\|^2_{S,1/\omega} \le C\sqrt{\gamma(S)}\,h_N^{3/2}\int_{\mathbb{D}} \omega(y)\left|\nabla f(y)\right|^2\mathrm{d}y.$$

It remains to return to the domain S by applying the inverse change of variables, while keeping track of the powers of h_N. We have, again by Lemma 5, $\left|\nabla f(y)\right| \le C h_N \left|[\nabla u](\kappa(y))\right|$, hence

$$\|u - \langle u\rangle_S\|^2_{S,1/\omega} \le C\sqrt{\gamma(S)}\,h_N^{7/2}\int_{\mathbb{D}} \sqrt{d(y,\partial\mathbb{D})}\left|[\nabla u](\kappa(y))\right|^2\mathrm{d}y.$$

We now reuse Lemma 6:

$$\|u - \langle u\rangle_S\|^2_{S,1/\omega} \le C\sqrt{\gamma(S)}\,h_N^{3}\int_{\mathbb{D}} \sqrt{d(\kappa(y),\partial S)}\left|[\nabla u](\kappa(y))\right|^2\mathrm{d}y. \tag{34}$$

Finally, with the change of variables $x = \kappa(y)$ and using Lemma 5, this leads to

$$\|u - \langle u\rangle_S\|^2_{S,1/\omega} \le C\sqrt{\gamma(S)}\,\frac{h_N^3}{h_N^2}\int_S \sqrt{d(x,\partial S)}\,|\nabla u(x)|^2\,\mathrm{d}x \tag{35}$$

$$\le C\sqrt{\gamma(S)}\,h_N\int_S \sqrt{d(x,\partial S)}\,|\nabla u(x)|^2\,\mathrm{d}x. \tag{36}$$

With the simple estimate

$$\sqrt{d(x,\partial S)} \le \sqrt{\gamma(S)}\sqrt{d(x,\partial\mathbb{D})} \le \sqrt{\gamma(S)}\,\omega\,,$$

we easily obtain the claimed inequality. □

Remark 8. There is a large corpus of works devoted to weighted Poincaré-type inequalities, but to the best of our knowledge, the kind of inequalities treated in other references (see e.g. [9, 11]) do not quite have the form of the one we deal with here.

5. Inverse inequalities

First, we have inverse inequalities *without weights*:

Lemma 9. *There exists a constant $C_5 > 0$ such that, for all $N \in \mathbb{N}$, $\theta \in \mathscr{X}_N$ and $\tau \in \mathscr{T}_N$, there holds*

$$\int_{K_\tau} |\nabla\theta(x)|^2 \,\mathrm{d}x \le C_5 h_N^{-2} \int_{K_\tau} |\theta(x)|^2 \,\mathrm{d}x. \tag{37}$$

This is well-known when $K_\tau = \tau$ (i.e. when K_τ is a triangle). The only "difficulty" is to extend this to the case where τ has two vertices in the boundary. But in that case, we may enclose K_τ between two triangles of uniformly comparable areas, and the proof merely becomes a technical formality. We spare the readers with the details.

Corresponding weighted inverse inequalities can be deduced in the following manner:

Lemma 10. *Condition* (A2) *is satisfied with the constant*

$$C_{\mathrm{inv}}(K_\tau)^{-2} = 1 + C_5 h_N^{-2} \rho_\omega(K_\tau) M_\omega(K_\tau)$$

where $\rho_\omega(K_\tau)$ and $M_\omega(K_\tau)$ are the average and the maximum of ω on K_τ, respectively.

Proof. Let $N \in \mathbb{N}$, $\tau \in \mathscr{T}_N$ and $\theta \in \mathscr{X}_N$. Since $\nabla\theta$ is constant on K_τ, one has

$$\int_{K_\tau} \omega(x) |\nabla\theta(x)|^2 \,\mathrm{d}x = \rho_\omega(K) \int_{K_\tau} |\nabla\theta|^2 \,\mathrm{d}x.$$

Applying the previous lemma, we get

$$\int_{K_\tau} \omega(x) |\nabla\theta|^2 \,\mathrm{d}x \le C_5 h_N^{-2} \rho_\omega(K_\tau) \int_{K_\tau} |\theta(x)|^2 \,\mathrm{d}x \tag{38}$$

$$\le C_5 h_N^{-2} \rho_\omega(K_\tau) M_\omega(K_\tau) \int_{K_\tau} \frac{|\theta(x)|^2}{\omega(x)} \,\mathrm{d}x. \tag{39}$$

The result follows immediately. □

Lemma 11. *There exists a constant $C_6 > 0$ independent on N such that for all $\tau \in \mathscr{T}_N$ and for any vertex Q of τ,*

$$h_N \gamma(S_{N,Q}) C_{\mathrm{inv}}(K_\tau)^{-2} \le C_6,$$

where $S_{N,Q}$ is the support of the basis function of $\mathscr{X}_N$ attached to Q, as defined in Lemma 5.

Proof. Let us rewrite $S = S_{N,Q}$. We have

$$h_N \gamma(S) C_i(K_\tau)^{-2} = h_N \gamma(S) + C_5 h_N^{-1} \gamma(S) \rho_\omega(K_\tau) M_\omega(K_\tau) =: T_1 + T_2.$$

We can write $T_1 \le C$, since this term tends to 0 when $N \to \infty$. The main task is thus to estimate T_2.

On the one hand, assume that $d(S, \partial\mathbb{D}) \le h_N$. Then we use the simple estimate $\gamma(S) \le 1$. Moreover, for all $x \in K_\tau$, there holds $d(x, \partial\mathbb{D}) \le d(x, \partial S) + d(S, \partial\mathbb{D}) \le C h_N$. Using (29), we deduce $\rho_\omega(K) \le C\sqrt{h_j}$ and $M_\omega(K) \le C\sqrt{h_N}$ and thus $T_2 \le C$.

On the other hand, if $d(S, \partial\mathbb{D}) \ge h_N$, we estimate $\gamma(S)$ as follows. First, we have $d(x, \partial\mathbb{D}) \ge \omega(x)^2$ hence

$$\gamma(S) \le \frac{d(x, \partial S)}{m_\omega(S)^2}, \tag{40}$$

where $m_\omega(S)$ is the minimum of ω on S. Note that $d(S, \partial\mathbb{D}) \ge h_N$ implies that

$$h_N \le C m_\omega(S)^2. \tag{41}$$

By the quasi-uniformity assumption (11) the diameter d_S of S satisfies

$$d_S \le C h_N. \tag{42}$$

Therefore, there holds $d(x,\partial S) \leq d_S \leq C h_N$. This shows that $\gamma(S) \leq C \dfrac{h_N}{m_\omega(S)^2}$, which, injected in the expression of T_2, leads to

$$T_2 \leq C \frac{\rho_\omega(K_\tau)}{m_\omega(S)} \frac{M_\omega(K_\tau)}{m_\omega(S)}.$$

Observing that $\nabla\omega = x/\omega$, a Taylor–Lagrange inequality combined with the estimates (41) and (42) gives

$$\left|\rho_\omega(K_\tau) - m_\omega(S)\right| \leq \frac{d_S}{m_\omega(S)} \leq C\sqrt{h_N}.$$

Hence,

$$\frac{\rho_\omega(K_\tau)}{m_\omega(S)} \leq 1 + \frac{\left|\rho_\omega(K_\tau) - m_\omega(K_\tau)\right|}{m_\omega(S)} \leq C,$$

using again (41). For similar reasons, there holds $\dfrac{M_\omega(K_\tau)}{m_\omega(S)} \leq C$ and so $T_2 \leq C$ also in this case. This concludes the proof of the lemma. □

6. Clément type quasi-interpolant

Fix $N \in \mathbb{N}$ and denote by $\{Q_1, \ldots, Q_n\}$ the vertices of $\mathscr{T}_N$. Let us rewrite φ_{N,Q_i}, defined in (25), as φ_i. Similarly, we write S_i instead of S_{N,Q_i}. For the quasi-interpolant I_N, we put

$$\forall\, u \in L^2_{1/\omega}, \quad I_N u := \sum_{i=1}^{n} \langle u \rangle_{S_i} \varphi_i. \tag{43}$$

Lemma 12. *The quasi-interpolant* (43) *satisfies (A1) with*

$$C_P(K_\tau)^2 = C_7 h_N \sum_{i \in I(\tau)} \gamma(S_i) \tag{44}$$

where $C_7 > 0$ is a constant independent on N and τ and $I(\tau)$ is the set of indices i such that Q_i is a vertex of τ.

Proof. We adapt the proof of [6, Thm. 1]. Let $\tau \in \mathscr{T}_N$ and fix some $j \in I(\tau)$. On K_τ, we have

$$I_N u = \sum_{i \in I(\tau)} c_i \varphi_i = c_j \sum_{i \in I(\tau)} \varphi_i + \sum_{i \in I(\tau)\setminus\{j\}} (c_i - c_j)\varphi_i. \tag{45}$$

where $c_i = \langle u \rangle_{S_i}$. Since $\sum_{i \in I(\tau)} \varphi_i = 1$, we deduce

$$\|u - I_N u\|_{K_\tau,1/\omega} \leq \left\|u - c_j\right\|_{K_\tau,1/\omega} + \sum_{i \in I(\tau)\setminus\{j\}}^{3} \left|c_i - c_j\right| \left\|\varphi_i\right\|_{K_\tau,1/\omega} \tag{46}$$

$$\leq \left\|u - c_j\right\|_{S_j,1/\omega} + \sum_{i \in I(\tau)\setminus\{j\}} \left|c_i - c_j\right| \left\|\varphi_i\right\|_{K_\tau,1/\omega}. \tag{47}$$

By Theorem 7, the first term can be estimated by

$$\left\|u - c_j\right\|_{S_j,1/\omega} \leq \sqrt{C_P \gamma(S_j) h_N}\, \|u\|_{S_j,T^1}.$$

On the other hand for $i \in I(\tau) \setminus \{j\}$, we may write

$$\left|c_i - c_j\right|^2 \left\|\varphi_i\right\|^2_{K_\tau,1/\omega} = \left(\int_{K_\tau} 1/\omega\right)^{-1} \left\|c_i - c_j\right\|^2_{K_\tau,1/\omega} \left\|\varphi_i\right\|^2_{K_\tau,1/\omega} \tag{48}$$

$$\leq \left\|c_i - c_j\right\|^2_{K_\tau,1/\omega} \tag{49}$$

$$\leq 2\left(\|u - c_i\|^2_{S_i,1/\omega} + \left\|u - c_j\right\|^2_{S_j,1/\omega}\right), \tag{50}$$

since $\varphi_i \leq 1$ on K_τ. Applying again Theorem 7 leads to

$$\|u - I_N u\|^2_{K_\tau,1/\omega} \leq C h_N \left(\sum_{i \in I(\tau)}^{3} \gamma(S_i)\right) \|u\|^2_{\Sigma_\tau,T^1},$$

where Σ_τ is defined below Eq. (19), and we used that $S_i \subset \Sigma_\tau$ whenever $i \in I(\tau)$.

To show that the T^1-continuity (18) holds, we can write, using again (45),

$$\|u - I_N u\|_{K_\tau,T^1} \leq \|u\|_{K_\tau,T^1} + \sum_{i\in I(\tau)\setminus\{j\}} \left|c_i - c_j\right| \left\|\varphi_i\right\|_{K_\tau,T^1}.$$

Using the inverse inequality shown in Lemma 10 and using similar arguments as above, we find

$$\begin{aligned}\|u - I_N u\|^2_{K_\tau,T^1} &\leq \left(1 + C \sum_{i\in I(\tau)\setminus\{j\}} \frac{h_N(\gamma(S_i)+\gamma(S_j))}{C_{\mathrm{inv}}(K_\tau)^2 \int_{K_\tau} \dfrac{1}{\omega(x)}\mathrm{d}x} \left\|\varphi_i\right\|^2_{1/\omega,K_\tau}\right) \|u\|^2_{\Sigma_\tau,T^1}\\ &\leq \left(1 + C \sum_{i\in I(\tau)} h_N \gamma(S_i) C_{\mathrm{inv}}(K_\tau)^{-2}\right) \|u\|^2_{\Sigma_\tau,T^1}.\end{aligned}$$

Thanks to Lemma 11, we conclude that

$$\|u - I_N u\|_{K_\tau,T^1} \leq C\,\|u\|^2_{\Sigma_\tau,T^1}.$$

The continuity (18) follows easily. □

Combining Lemma 11 and Lemma 12, we deduce that

Lemma 13. *Condition* (A3) *is satisfied.*

This concludes the proof of Theorem 2.

7. Conclusions

We have shown Theorem 2 by combining some inverse inequalities with a weighted Poincaré inequality. Our proof relies essentially on the fact that the constants appearing in both inequalities have a uniformly bounded ratio. Identical arguments can be used to treat quasi-uniform and shape-regular family of triangulations of more general domains, but we have restricted our attention to the disk $\mathbb{D}$ for conciseness. We do not know whether the result extends to locally refined triangulations.

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