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Partial differential equations / *Équations aux dérivées partielles*

On optimal regularity estimates for finite-entropy solutions of scalar conservation laws

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Abstract. We consider finite-entropy solutions of scalar conservation laws $u_t + a(u)_x = 0$, that is, bounded weak solutions whose entropy productions are locally finite Radon measures. Under the assumptions that the flux function a is strictly convex (with possibly degenerate convexity) and a'' forms a doubling measure, we obtain a characterization of finite-entropy solutions in terms of an optimal regularity estimate involving a cost function first used by Golse and Perthame.

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1. Introduction

For any strictly convex C^1 flux function $a\colon \mathbb{R} \to \mathbb{R}$ we consider bounded weak solutions $u\colon [0,T] \times \mathbb{R} \to \mathbb{R}$ of the scalar conservation law

$$u_t + [a(u)]_x = 0 \text{ in } \mathscr{D}'((0,T) \times \mathbb{R}). \tag{1}$$

It is well known that, on the one hand, smooth initial data evolving according to (1) may develop singularities in finite time, but on the other hand, there can be infinitely many weak solutions

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corresponding to a single initial datum. One way to restore well-posedness is the concept of entropy solution [8]. For any convex or C^2 function $\eta\colon \mathbb{R}\to\mathbb{R}$, called entropy, and entropy flux $q\colon \mathbb{R}\to\mathbb{R}$ such that $q'=\eta' a'$, the associated entropy production is the distribution

$$\mu_\eta = [\eta(u)]_t + [q(u)]_x.$$

Entropy solutions are bounded weak solutions such that μ_η is a nonpositive measure for all convex entropies η, and for any bounded initial datum u_0 there exists a unique entropy solution defined for all positive times [8].

Here we are interested in the larger class of solutions with finite entropy production:

$$\mu_\eta \in \mathcal{M}_{loc}([0,T]\times\mathbb{R}) \qquad \text{for all } C^2 \text{ entropies } \eta. \tag{2}$$

This property does not ensure uniqueness of the initial value problem, but arises naturally in the study of some stochastic processes [3, 13, 14], where large deviation principles are open for want of a better understanding of finite-entropy solutions (despite major recent progress in [11, 12]).

In [3, Proposition 2.3] (see [10, Appendix B] for a more detailed proof in a slightly different context) it is shown that (2) implies the existence of a locally finite Radon measure $m \in \mathcal{M}_{loc}([0,T]\times\mathbb{R}\times\mathbb{R})$ such that

$$\mu_\eta = \int \eta''(v)\, m(\cdot,\cdot,\mathrm{d}v), \tag{3}$$

for all convex or C^2 entropy η. Then [7, Theorem 4.1] implies that u satisfies the regularity estimate

$$\begin{aligned}\sup_{|h|\le\epsilon}\frac{1}{|h|}\int_0^T\int_{-R}^R \chi(t,x)^2\Delta(u(t,x),u(t,x+h))\,\mathrm{d}x\,\mathrm{d}t&\\ \le C(\chi)\big(1+|m|([0,T]\times[-R-\epsilon,R+\epsilon]\times[\inf u,\sup u])\big),&\end{aligned} \tag{4}$$

for any smooth cut-off function χ with support in $[0,T]\times[-R,R]$, some constant $C(\chi)>0$, and the regularity cost Δ is given by

$$\begin{aligned}\Delta(u_1,u_2) &= \frac12\int_{u_1}^{u_2}\int_{u_1}^{u_2}\big|a'(v)-a'(w)\big|\,\mathrm{d}v\,\mathrm{d}w\\ &=\int_{[u_1,u_2]}(u_2-s)(s-u_1)\,a''(\mathrm{d}s).\end{aligned} \tag{5}$$

The last equality is obtained by writing $|a'(v)-a'(w)| = \int_{[u_1,u_2]}(\mathbf{1}_{v<s<w}+\mathbf{1}_{w<s<v})\,a''(\mathrm{d}s)$ and applying Fubini's theorem. Note that $[u_1,u_2]=[u_2,u_1]=\{tu_1+(1-t)u_2\}_{t\in[0,1]}$ and the integrand $(u_2-s)(s-u_1)$ is positive inside that segment, regardless of whether $u_1\le u_2$ or $u_2\le u_1$.

Remark 1. The explicit statement of [7, Theorem 4.1] is actually a corollary of (4), but its proof does contain (4), which corresponds to (4.10) in the proof of [7, Theorem 4.1]. The quantity Δ is defined in [7, Lemma 4.3] by the formula

$$\begin{aligned}&\Delta(u_1,u_2)=\iint \mathbf{1}_{v>w}(a'(v)-a'(w))(\mathcal{M}_{u_1}(v)-\mathcal{M}_{u_2}(v))(\mathcal{M}_{u_1}(w)-\mathcal{M}_{u_2}(w))\,\mathrm{d}v\,\mathrm{d}w,\\ &\mathcal{M}_u(v)=\mathbf{1}_{0\le v\le u}-\mathbf{1}_{u\le v<0}.\end{aligned}$$

To see that this coincides with (5), first note that both expressions are symmetric so it suffices to consider $u_1<u_2$. In the proof of [7, Lemma 4.3] it is shown that

$$\Delta(u_1,u_2)=\int_{u_1}^{u_2}\int_w^{u_2}\big|a'(v)-a'(w)\big|\,\mathrm{d}v\,\mathrm{d}w$$

which implies (5) by writing $a'(v)-a'(w)=\int_{[v,w]}a''(\mathrm{d}s)$ and applying Fubini's theorem.

For instance, if $a(v) = |v|^{\beta+1}$ for some $\beta \geq 1$, then the regularity cost Δ admits the lower bound $\Delta(u_1, u_2) \gtrsim |u_1 - u_2|^{\beta+2}$. Hence in that case (4) implies a local $B^{1/p}_{p,\infty}$ bound for $p = \beta + 2$, in the x direction, that is, $(t, x) \mapsto |u(t, x + h) - u(t, x)|/|h|^{1/p}$ is locally bounded in L^p, uniformly with respect to h. In fact the same regularity is valid also in the t direction [7]. This local $B^{1/p}_{p,\infty}$ estimate is optimal in Besov regularity scales [5], but for $\beta > 1$ it is strictly weaker than (4) in regions where u stays away from the degenerate value $u = 0$. Loosely speaking, the regularity cost Δ takes into account that equation (1) regularizes more around values of u where a is more convex. Therefore one can hope (as similar estimates in our recent work [9] for a generalized eikonal equation) that (4) is optimal in the sense that a converse estimate is valid:

- If the left-hand side of (4) is finite, does it imply that all entropy productions are finite (2)?
- Moreover, are the entropy productions (2) controlled by the left-hand side of (4)?

The second question can be answered rather easily if a is C^2, thanks to the recent rectifiability result of [12]: under the *a priori* knowledge that all entropy productions are finite, they are concentrated on a 1-rectifiable jump set and can be explicitly computed in terms of the traces of u along that jump set. Elementary algebraic manipulation and a covering argument then provide the following estimate.

Theorem 2. *Assume that $a \in C^2(\mathbb{R})$ is strictly convex. Let $u \in L^\infty([0, T] \times \mathbb{R})$ be a weak solution of* (1) *such that u has finite entropy production* (2). *Then for any open set $U \subset [0, T] \times \mathbb{R}$ we have the estimate*

$$|\mu_\eta|(U) \leq C_0 \cdot \sup_I |\eta''| \cdot \limsup_{\epsilon \to 0} \sup_{|h|<\epsilon} \frac{1}{|h|} \iint_U \Delta(u(t, x), u(t, x + h))\, dx\, dt, \tag{6}$$

for some absolute constant $C_0 > 0$, where $I = [\inf u, \sup u]$.

Note that the *a priori* estimate (6) directly implies an estimate on $|m|(U \times \mathbb{R}) = |m|(U \times I)$ for the measure m satisfying (3). In light of Theorem 2, it is natural to reformulate the first question as follows: does finiteness of the right-hand side of (6) imply finiteness of the left-hand side, that is, finite entropy production (2)? We provide a positive answer under a doubling assumption on the nonnegative measure a''.

Theorem 3. *Assume that $a \in C^1(\mathbb{R})$ is strictly convex and that the nonnegative measure a'' is locally doubling, and let $u \in L^\infty([0, T] \times \mathbb{R})$ be a weak solution of* (1). *Assume that*

$$\limsup_{\epsilon \to 0} \sup_{|h|<\epsilon} \frac{1}{|h|} \int_0^T \int_{-R}^R \Delta(u(t, x), u(t, x + h))\, dx\, dt < \infty, \tag{7}$$

for all $R > 0$, then u has finite entropy production (2).

Theorem 3 provides a full converse to the estimate (4) proved in [7], under the assumption that a'' is locally doubling (this is satisfied in particular if a is analytic, see e.g. [9, Lemma 25]). The proof of Theorem 3 also provides the estimate (6) even when a is not C^2, but with a constant depending on the doubling property of a''. More precisely, in the proof of Theorem 3 we obtain

$$|\mu_\eta|(U) \leq C_0 \cdot \sup_I |\eta''| \cdot \limsup_{\epsilon \to 0} \sup_{|h|<\epsilon} \frac{1}{|h|} \iint_U \widehat{\Delta}(u(t, x), u(t, x + h))\, dx\, dt,$$

for some absolute constant C_0 and slightly different regularity cost $\widehat{\Delta}$ (22), and then check that $\widehat{\Delta} \leq C\Delta$ for some $C > 0$ depending on the doubling constant of a'' on I.

In the case of a uniformly convex flux function, $0 < c \leq a'' \leq C$, the condition (7) is equivalent to local $B^{1/3}_{3,\infty}$ regularity in the x direction, and Theorem 3 could be obtained by adapting the commutator estimates in [4, Proposition 2]. However, in the case $a(v) = |v|^{\beta+1}$ for some $\beta > 1$, the statement of Theorem 3 would not be valid with (7) replaced by a local $B^{1/p}_{p,\infty}$ bound for

$p = \beta+2 > 3$. Indeed, for a solution taking values for instance in $[1,2]$ where a is uniformly convex, $B^{1/3}_{3,\infty}$ regularity (in the x direction) would be needed to ensure (2) (see the examples in [5]).

It is also interesting to remark that, if the limit (7) is zero, then all entropy productions vanish. In our particular context this provides a very precise regularity threshold for Onsager-type statements in the spirit of [2], and a generalization of [4, Theorem 2] where $a(v) = v^2/2$ is considered.

The proof of Theorem 3 relies, as in [2,4], on good estimates of the commutator $[a(u)]_\epsilon - a(u_\epsilon)$, where the subscript ϵ denotes regularization at scale ϵ. However, if the convexity of a degenerates (e.g. $a(v) = |v|^{\beta+1}$ for some $\beta > 1$), our regularity requirement (7) is strictly weaker than the local $B^{1/3}_{3,\infty}$ regularity that is needed in order to directly use (as done e.g. in [6, Proposition 3.10]) the estimates of [4, Theorem 2]. As noted in [2] these estimates are valid for any C^2 function a and not related to its convexity. Here we take instead full advantage of the convexity of a in order to obtain finer bounds in terms of the regularity cost Δ. We do this by adapting ideas of [9], where a result analogous to Theorem 3 has been established for a class of generalized eikonal equations with degenerate convexity.

We do not know whether Theorem 3 is valid without the requirement that the nonnegative measure a'' is doubling, even though the *a priori* estimate of Theorem 2 suggests that this requirement is superfluous. In the next two sections we give the proofs of Theorems 2 and 3, respectively.

2. Proof of Theorem 2

Let u be a bounded weak solution to (1) with finite entropy production (2). The proof of [12, Theorem 1], where $a(v) = v^2/2$ is considered, actually uses only the facts that:

- u solves a kinetic formulation [12, (3)], which is a consequence of finite entropy production,
- the flux function a is C^2 (to construct a Lagrangian representation [11, Theorem 1.2]),
- and a' is an increasing function (see [12, Proposition 6] and Step 2 of [12, Theorem 10]).

Hence it applies in our setting: there exists an $\mathcal{H}^1$-rectifiable set J_u such that all entropy productions μ_η are absolutely continuous with respect to $\mathcal{H}^1_{\lfloor J_u}$. More precisely, u has strong traces on both sides of J_u and for any entropy η we have

$$\mu_\eta = \big((\eta(u^+) - \eta(u^-))\nu_t + (q(u^+) - q(u^-))\nu_x\big)\, \mathcal{H}^1_{\lfloor J_u},$$

where $\nu = (\nu_t, \nu_x)$ is the unit normal to J_u and $u^\pm$ are the traces. The equation (1) also provides the Rankine–Hugoniot condition

$$(u^+ - u^-)\nu_t + (a(u^+) - a(u^-))\nu_x = 0 \qquad \text{a.e. on } J_u,$$

so μ_η can be rewritten as

$$\begin{aligned} \mu_\eta &= c_\eta(u^+, u^-)\, \nu_x\, \mathcal{H}^1_{\lfloor J_u} \\ c_\eta(u^+, u^-) &= q(u^+) - q(u^-) - \frac{a(u^+) - a(u^-)}{u^+ - u^-}(\eta(u^+) - \eta(u^-)). \end{aligned} \tag{8}$$

The crucial fact here is that the entropy cost c_η is controlled by Δ.

Lemma 4. *For any $\eta \in C^2(\mathbb{R})$ and $u^\pm \in \mathbb{R}$ we have*

$$|c_\eta(u^+, u^-)| \le \frac{1}{2}\left(\sup_{[u^-, u^+]} |\eta''|\right)\Delta(u^+, u^-).$$

Proof of Lemma 4. Since both sides of the estimate are symmetric in (u^+, u^-) we may assume $u^- < u^+$. Using $q' = \eta' a'$ and Fubini's theorem we have the identities

$$\begin{aligned} c_\eta(u^+, u^-) &= \int_{u^-}^{u^+} \eta'(t) \left(a'(t) - \frac{a(u^+) - a(u^-)}{u^+ - u^-} \right) \mathrm{d}t \\ &= \frac{1}{u^+ - u^-} \int_{u^-}^{u^+} \eta'(t) \int_{u^-}^{u^+} (a'(t) - a'(s)) \, \mathrm{d}s \, \mathrm{d}t \\ &= \frac{1}{u^+ - u^-} \int_{[u^-, u^+]} w_\eta(\tau) \, a''(\mathrm{d}\tau), \end{aligned} \tag{9}$$

where

$$\begin{aligned} w_\eta(\tau) &= \int_{u^-}^{u^+} \eta'(t) \int_{u^-}^{u^+} (\mathbf{1}_{s<\tau<t} - \mathbf{1}_{t<\tau<s}) \, \mathrm{d}s \, \mathrm{d}t \\ &= \int_{u^-}^{u^+} \eta'(t) \left(\mathbf{1}_{t>\tau}(\tau - u^-) - \mathbf{1}_{t<\tau}(u^+ - \tau) \right) \mathrm{d}t. \end{aligned}$$

Since the second factor in the integrand has zero average on $[u^-, u^+]$ we deduce

$$\begin{aligned} |w_\eta(\tau)| &= \left| \int_{u^-}^{u^+} (\eta'(t) - \eta'(\tau)) \left(\mathbf{1}_{t>\tau}(\tau - u^-) - \mathbf{1}_{t<\tau}(u^+ - \tau) \right) \mathrm{d}t \right| \\ &\leq \left(\sup_{[u^-, u^+]} |\eta''| \right) \int_{u^-}^{u^+} |t - \tau| \left(\mathbf{1}_{t>\tau}(\tau - u^-) + \mathbf{1}_{t<\tau}(u^+ - \tau) \right) \mathrm{d}t \\ &= \left(\sup_{[u^-, u^+]} |\eta''| \right) \frac{1}{2} (u^+ - u^-)(\tau - u^-)(u^+ - \tau). \end{aligned}$$

The last equality is obtained by directly calculating the integral. Plugging this into (9) we deduce

$$|c_\eta(u^+, u^-)| \leq \frac{1}{2} \left(\sup_{[u^-, u^+]} |\eta''| \right) \int_{[u^-, u^+]} (\tau - u^-)(u^+ - \tau) \, a''(\mathrm{d}\tau),$$

and we recognize the definition (5) of $\Delta(u^+, u^-)$ in the right-hand side. □

Theorem 2 follows from Lemma 4 and the rectifiability of J_u in (8) by a covering argument similar to [9, Lemma 32]. We assume without loss of generality that $\sup_I |\eta''| \leq 1$ and $U \subset\subset [0, T] \times \mathbb{R}$. For general open $U \subset [0, T] \times \mathbb{R}$ we may approximate it by open sets $U_k \subset\subset [0, T] \times \mathbb{R}$. Thanks to (8) we have

$$|\mu_\eta|(U) = \int_{J_u \cap U} |c_\eta(u^+, u^-)| \, |\nu_x| \, \mathrm{d}\mathcal{H}^1. \tag{10}$$

Further, for any $J' \subset J_u$ such that $\mathcal{H}^1(J') < \infty$ we have, on the one hand, thanks to Lemma 4,

$$\int_{J' \cap U} |c_\eta(u^+, u^-)| \, |\nu_x| \, \mathrm{d}\mathcal{H}^1 \leq \frac{1}{2} \int_{J' \cap U} \Delta(u^+, u^-) \, |\nu_x| \, \mathrm{d}\mathcal{H}^1, \tag{11}$$

and, on the other hand, we will show

$$\int_{J' \cap U} \Delta(u^+, u^-) \, |\nu_x| \, \mathrm{d}\mathcal{H}^1 \leq C_0 \limsup_{\epsilon \to 0} \sup_{|h| < \epsilon} \frac{1}{|h|} \iint_U \Delta(u(t, x), u(t, x + h)) \, \mathrm{d}x \, \mathrm{d}t. \tag{12}$$

The proof of (12) will follow as a consequence of the rectifiability of J_u and the trace properties of u. Applying (12) to $J'_\delta = U \cap J_u \cap \{|c_\eta(u^+, u^-)\nu_x| > \delta\} \subset J_u$ and noting from (10) that $\mathcal{H}^1(J'_\delta) \leq \delta^{-1} |\mu_\eta|(U) < \infty$, we deduce, thanks to (11),

$$\int_{J'_\delta \cap U} |c_\eta(u^+, u^-)| \, |\nu_x| \, \mathrm{d}\mathcal{H}^1 \leq C_0 \limsup_{\epsilon \to 0} \sup_{|h| < \epsilon} \frac{1}{|h|} \iint_U \Delta(u(t, x), u(t, x + h)) \, \mathrm{d}x \, \mathrm{d}t.$$

Letting $\delta \to 0$, the left-hand side converges to (10), and this proves the *a priori* estimate (6).

To conclude the proof of Theorem 2 it remains to justify (12). The elementary building block is that (12) is valid if u is a pure jump: for constant values $u^\pm \in \mathbb{R}$ and a unit vector ν, let $\psi^{u^\pm,\nu}\colon \mathbb{R}^2 \to \mathbb{R}$ denote the pure jump from u^- to u^+ across a line with unit normal ν, namely

$$\psi^{u^\pm,\nu}(t,x) = u^- \mathbf{1}_{(t,x)\cdot\nu<0} + u^+ \mathbf{1}_{(t,x)\cdot\nu>0},$$

then for any $r \ge |h| > 0$ we claim

$$\int_{J_{\psi^{u^\pm,\nu}}\cap B_r} \Delta(u^+,u^-)\,|\nu_x|\,\mathrm{d}\mathscr{H}^1 \le \frac{2}{|h|}\iint_{B_r} \Delta(\psi^{u^\pm,\nu}(t,x+h),\psi^{u^\pm,\nu}(t,x))\,\mathrm{d}x\,\mathrm{d}t. \tag{13}$$

To check (13), simply use that the left-hand side is equal to $2r|\nu_x|\Delta(u^+,u^-)$, that $\Delta \ge 0$ and that the integrand in the right-hand side is equal to $\Delta(u^+,u^-)$ in a region of two-dimensional measure $\ge r|h\nu_x|$ (the intersection of B_r with a straight band of width $|h\nu_x|$).

We deduce (12) from (13) via a covering argument similar to [9, Lemma 32], making use of the rectifiability of J', the trace properties of u and the Lipschitz quality of Δ. We provide the details for the reader's convenience.

Let $\delta \in (0,1)$. There exists $\epsilon_0 > 0$ and a subset $\widetilde{J} \subset J'$ with $\mathscr{H}^1\left(J' \cap U \setminus \widetilde{J}\right) < \delta$ and $\widetilde{J} + B_{\epsilon_0} \subset U$, such that for any $(t_0,x_0) \in \widetilde{J}$ and $0 < r < \epsilon_0$, denoting $u_0^\pm = u^\pm(t_0,x_0)$ and $\nu_0 = \nu(t_0,x_0)$, we have

$$\begin{aligned} \fint_{B_r(t_0,x_0)\cap J'} \left(\left|u^\pm - u_0^\pm\right| + |\nu-\nu_0|\right)\mathrm{d}\mathscr{H}^1 &< \delta, \\ \left|\mathscr{H}^1\left(B_r(t_0,x_0)\cap J'\right) - 2r\right| &< \delta\, r, \\ \text{and}\quad \frac{1}{\pi r^2}\iint_{B_r(t_0,x_0)} \left|u - \psi_{t_0,x_0}^{u_0^\pm,\nu_0}\right| \mathrm{d}x\,\mathrm{d}t &< \delta, \end{aligned} \tag{14}$$

where $\psi_{t_0,x_0}^{u_0^\pm,\nu_0}(t,x) = \psi^{u_0^\pm,\nu_0}(t-t_0,x-x_0)$ is the pure jump centered at (t_0,x_0). Let $\epsilon \in (0,\epsilon_0/2)$. By Besicovitch's covering theorem [1, Theorem 2.18] there exists an absolute constant $Q \in \mathbb{N}$ and families $\mathscr{B}_1,\mathscr{B}_2,\dots,\mathscr{B}_Q$ of pairwise disjoint balls in the set $\left\{B_\epsilon(t,x)\colon (t,x)\in\widetilde{J}\right\}$ such that

$$\widetilde{J} \subset \bigcup_{k=1}^{Q}\bigcup_{B\in\mathscr{B}_k} B.$$

We fix $k \in \{1,\dots,Q\}$ and denote $\mathscr{B}_k = \{B_\epsilon(t_j,x_j)\}_{j=1,\dots,p}$ for some $(t_j,x_j) \in \widetilde{J}$. We also write $u_j^\pm = u^\pm(t_j,x_j)$ and $\nu_j = \nu(t_j,x_j)$.

Note that Δ is Lipschitz on $I \times I$, with Lipschitz constant $L \lesssim |I|a''(I)$ thanks to its definition (5). Using the first two properties (14) of $\widetilde{J}$, we find

$$\begin{aligned} \int_{J'\cap B_\epsilon(t_j,x_j)} \Delta\left(u^+,u^-\right)|\nu_x|\,\mathrm{d}\mathscr{H}^1 &\le L\int_{J'\cap B_\epsilon(t_j,x_j)} \left(|u^+-u_j^+| + |u^- - u_j^-|\right)|\nu_x|\,\mathrm{d}\mathscr{H}^1 \\ &\quad + \Delta\left(u_j^+,u_j^-\right)\int_{J'\cap B_\epsilon(t_j,x_j)} |\nu_x - \nu_{j,x}|\,\mathrm{d}\mathscr{H}^1 \\ &\quad + \Delta\left(u_j^+,u_j^-\right)|\nu_{j,x}|\mathscr{H}^1(J'\cap B_\epsilon(t_j,x_j)) \\ &\le 4\epsilon\Delta\left(u_j^+,u_j^-\right)|\nu_{j,x}| + C\delta\epsilon, \end{aligned}$$

for some constant $C = C(|I|,a''(I))$ depending on $|I|$ and $a''(I)$. Applying the elementary estimate (13) for pure jumps with $r = h = \epsilon$, we deduce

$$\int_{J'\cap B_\epsilon(t_j,x_j)} \Delta\left(u^+,u^-\right)|\nu_x|\,\mathrm{d}\mathscr{H}^1 \le \frac{4}{\epsilon}\iint_{B_\epsilon(t_j,x_j)} \Delta(\psi_{t_j,x_j}^{u_j^\pm,\nu_j}(t,x+\epsilon),\psi_{t_j,x_j}^{u_j^\pm,\nu_j}(t,x))\,\mathrm{d}x\,\mathrm{d}t + C\delta\epsilon.$$

And using the last property (14) of $\widetilde{J}$ we infer

$$\int_{J'\cap B_\epsilon(t_j,x_j)} \Delta\left(u^+,u^-\right)|\nu_x|\,\mathrm{d}\mathscr{H}^1 \le \frac{4}{\epsilon}\iint_{B_\epsilon(t_j,x_j)} \Delta(u(t,x+\epsilon),u(t,x))\,\mathrm{d}x\,\mathrm{d}t + (20\pi L + C)\delta\epsilon.$$

Summing over $j=1,\dots,p$ and over the families $\mathscr{B}_1,\dots,\mathscr{B}_Q$ we obtain

$$\int_{\widetilde{J}} \Delta\left(u^+,u^-\right)|\nu_x|\,\mathrm{d}\mathscr{H}^1 \le 4Q\frac{1}{\epsilon}\iint_U \Delta\left(u(t,x+\epsilon),u(t,x)\right)\mathrm{d}x\,\mathrm{d}t + Q(20\pi L+C)\delta\, p\epsilon.$$

Noting from the properties (14) of $\widetilde{J}$ that

$$\mathscr{H}^1\left(J'\cap U\right) \ge \sum_{j=1}^{p}\mathscr{H}^1\left(B_\epsilon(t_j,x_j)\cap J_u\right) \ge p\epsilon,$$

this implies

$$\int_{\widetilde{J}} \Delta\left(u^+,u^-\right)|\nu_x|\,\mathrm{d}\mathscr{H}^1 \le 4Q\frac{1}{\epsilon}\iint_U \Delta\left(u(t,x+\epsilon),u(t,x)\right)\mathrm{d}x\,\mathrm{d}t + Q(20\pi L+C)\delta\,\mathscr{H}^1\left(J'\cap U\right).$$

Taking the limits $\epsilon\to 0$ and then $\delta\to 0$, we obtain (12). □

3. Proof of Theorem 3

We fix an entropy $\eta\in C^2(\mathbb{R})$ and an entropy flux q with $q'=\eta' a'$. The start of the proof is as in [4, Theorem 2], we recall the argument for the reader's convenience.

We denote by a subscript ϵ convolution at scale ϵ in the x variable:

$$u_\epsilon(t,x)=\int u(t,z)\rho_\epsilon(x-z)\,\mathrm{d}z,$$

where $\rho_\epsilon(x)=\epsilon^{-1}\rho(x/\epsilon)$ for some smooth kernel $\rho\ge 0$ with $\operatorname{supp}\rho\subset[-1,1]$ and $\int\rho=1$. We let

$$\mu_\eta^\epsilon=[\eta(u_\epsilon)]_t+[q(u_\epsilon)]_x,$$

and prove Theorem 3 by appropriately estimating μ_η^ϵ. The regularized function u_ϵ is pointwise differentiable with respect to t and satisfies

$$u_{\epsilon,t}=-[a(u)]_{\epsilon,x},$$

so we have

$$\begin{aligned}\mu_\eta^\epsilon &= \eta'(u_\epsilon)u_{\epsilon,t}+q'(u_\epsilon)u_{\epsilon,x}\\ &= -\eta'(u_\epsilon)[a(u)]_{\epsilon,x}+\eta'(u_\epsilon)[a(u_\epsilon)]_x\\ &= \eta'(u_\epsilon)\left[a(u_\epsilon)-[a(u)]_\epsilon\right]_x\\ &= \left[\eta'(u_\epsilon)(a(u_\epsilon)-[a(u)]_\epsilon)\right]_x-\eta''(u_\epsilon)u_{\epsilon,x}(a(u_\epsilon)-[a(u)]_\epsilon).\end{aligned}$$

Testing this with a function $\psi\in C_c^\infty((0,T)\times\mathbb{R})$ we obtain

$$\begin{aligned}\left\langle\mu_\eta^\epsilon,\psi\right\rangle &= -\iint \eta'(u_\epsilon)(a(u_\epsilon)-[a(u)]_\epsilon)\psi_x\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\iint \eta''(u_\epsilon)u_{\epsilon,x}([a(u)]_\epsilon-a(u_\epsilon))\psi\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}\tag{15}$$

We have the convergences $u_\epsilon\to u$ and $[a(u)]_\epsilon\to a(u)$ a.e. and u_ϵ is uniformly bounded, so by dominated convergence the left-hand side of (15) converges to $\langle\mu_\eta,\psi\rangle$, and the first integral in the right-hand side of (15) converges to 0. Hence we deduce

$$\langle\mu_\eta,\psi\rangle \le \|\psi\|_\infty \sup_I|\eta''|\cdot\limsup_{\epsilon\to 0}\iint_{\operatorname{supp}\psi}|u_{\epsilon,x}|([a(u)]_\epsilon-a(u_\epsilon))\,\mathrm{d}x\,\mathrm{d}t.\tag{16}$$

Here recall that $I=[\inf u,\sup u]$, and note that

$$[a(u)]_\epsilon-a(u_\epsilon)\ge 0,\tag{17}$$

by convexity of a thanks to Jensen's inequality. Therefore it all boils down to estimating the right-hand side of (16), and this is where our proof needs to depart from [4].

We start by writing

$$
\begin{aligned}
[a(u)]_\epsilon(t,x) - a(u(t,x)) &= \int (a(u(t,z)) - a(u(t,x)))\, \rho_\epsilon(x-z)\,\mathrm{d}z \\
&= \int \left(\int_{u(t,x)}^{u(t,z)} a'(\tau)\,\mathrm{d}\tau \right) \rho_\epsilon(x-z)\,\mathrm{d}z \\
&= \int \left(\int_{u(t,x)}^{u(t,z)} (a'(\tau) - a'(u(t,x)))\,\mathrm{d}\tau \right) \rho_\epsilon(x-z)\,\mathrm{d}z + a'(u(t,x)) \int (u(t,z) - u(t,x))\, \rho_\epsilon(x-z)\,\mathrm{d}z \\
&= \int \left(\int_{u(t,x)}^{u(t,z)} (a'(\tau) - a'(u(t,x)))\,\mathrm{d}\tau \right) \rho_\epsilon(x-z)\,\mathrm{d}z + a'(u(t,x))\,(u_\epsilon(t,x) - u(t,x)),
\end{aligned}
$$

hence

$$
\begin{aligned}
[a(u)]_\epsilon(t,x) - a(u_\epsilon(t,x)) &= [a(u)]_\epsilon(t,x) - a(u(t,x)) + a(u(t,x)) - a(u_\epsilon(t,x)) \\
&= \int \left(\int_{u(t,x)}^{u(t,z)} (a'(\tau) - a'(u(t,x)))\,\mathrm{d}\tau \right) \rho_\epsilon(x-z)\,\mathrm{d}z \\
&\quad + a(u(t,x)) - a(u_\epsilon(t,x)) + a'(u(t,x))\,(u_\epsilon(t,x) - u(t,x)). \qquad (18)
\end{aligned}
$$

By convexity of a, we have

$$
a(u(t,x)) - a(u_\epsilon(t,x)) + a'(u(t,x))\,(u_\epsilon(t,x) - u(t,x)) \le 0,
$$

and applying this to (18) we deduce

$$
[a(u)]_\epsilon(t,x) - a(u_\epsilon(t,x)) \le \int \left(\int_{u(t,x)}^{u(t,z)} (a'(\tau) - a'(u(t,x)))\,\mathrm{d}\tau \right) \rho_\epsilon(x-z)\,\mathrm{d}z. \qquad (19)
$$

To estimate this further, we define, for all $v \in \mathbb{R}$ and $r \ge 0$,

$$
\mathscr{G}_v(r) = \int_{v-r}^{v+r} \int_{v-r}^{v+r} |a'(\sigma) - a'(\tau)|\,\mathrm{d}\sigma\,\mathrm{d}\tau,
$$

which satisfies

$$
\mathscr{G}_v'(r) = 2\int_{v-r}^{v+r} \left(|a'(v+r) - a'(\tau)| + |a'(v-r) - a'(\tau)| \right) \mathrm{d}\tau. \qquad (20)
$$

As a' is strictly increasing, so is $\mathscr{G}_v'$, and thus $\mathscr{G}_v$ is strictly convex. Further, denoting by $g(x,z,t) = |u(t,z) - u(t,x)|$, we have

$$
\int_{u(t,x)}^{u(t,z)} |a'(\tau) - a'(u(t,x))|\,\mathrm{d}\tau \le \frac{1}{2} \mathscr{G}'_{u(t,x)}(g(x,z,t)),
$$

and thus from (19), and recalling also (17), we infer

$$
0 \le [a(u)]_\epsilon(t,x) - a(u_\epsilon((t,x)) \le \frac{1}{2} \int_{B_\epsilon} \mathscr{G}'_{u(t,x)}(g(x,z,t))\rho_\epsilon(x-z)\,\mathrm{d}z.
$$

Moreover we have

$$
\left|u_{\epsilon,x}(t,x)\right| \lesssim \epsilon^{-1} \fint_{[x-\epsilon,x+\epsilon]} |u(t,z) - u(t,x)|\,\mathrm{d}z = \epsilon^{-1} \fint_{[x-\epsilon,x+\epsilon]} g(x,z,t)\,\mathrm{d}z.
$$

Multiplying the last two estimates, we obtain

$$
\begin{aligned}
&\epsilon\,([a(u)]_\epsilon(t,x) - a(u_\epsilon(t,x)))\left|u_{\epsilon,x}(t,x)\right| \\
&\quad \lesssim \fint_{[x-\epsilon,x+\epsilon]} \fint_{[x-\epsilon,x+\epsilon]} \mathscr{G}'_{u(t,x)}(g(x,z,t))\,g(x,y,t)\,\mathrm{d}y\,\mathrm{d}z \\
&\quad \le \fint_{[x-\epsilon,x+\epsilon]} \mathscr{H}_{u(t,x)}\left(\mathscr{G}'_{u(t,x)}(g(x,z,t))\right)\mathrm{d}z + \fint_{[x-\epsilon,x+\epsilon]} \mathscr{G}_{u(t,x)}\left(g(x,y,t)\right)\mathrm{d}y, \qquad (21)
\end{aligned}
$$

where $\mathscr{H}_v(p) = \sup_{r\in\mathbb{R}}\{pr - \mathscr{G}_v(r)\}$ is the Legendre transform of $\mathscr{G}_v$. Using $\mathscr{H}_v(p) = pr^* - \mathscr{G}_v(r^*)$ where r^* is characterized by $\mathscr{G}_v'(r^*) = p$, we find that

$$\begin{aligned}\mathscr{H}_v\left(\mathscr{G}_v'(r)\right) &= r\mathscr{G}_v'(r) - \mathscr{G}_v(r)\\ &\le r\mathscr{G}_v'(r) \le 8r^2 a''\left([v-r, v+r]\right).\end{aligned}$$

The last inequality follows from writing $a'(v+r) - a'(\tau) = a''([\tau, v+r])$ and $a'(\tau) - a'(v-r) = a''([v-r,\tau])$ in the explicit expression (20) of $\mathscr{G}_v'$, and applying Fubini's theorem. Similarly we have

$$\mathscr{G}_v(r) \le 4r^2 a''\left([v-r, v+r]\right),$$

and plugging these bounds for $\mathscr{H}_v(\mathscr{G}_v'(r))$ and $\mathscr{G}_v(r)$ into (21) gives

$$\begin{aligned}&([a(u)]_\epsilon(t,x) - a(u_\epsilon(t,x)))\left|u_{\epsilon,x}(t,x)\right|\\ &\qquad\lesssim \frac{1}{\epsilon}\fint_{[x-\epsilon,x+\epsilon]} g(x,z,t)^2 a''\left([u(t,x) - g(x,z,t), u(t,x) + g(x,z,t)]\right)\mathrm{d}z,\end{aligned}$$

where we recall that $g(x,z,t) = |u(t,z) - u(t,x)|$. This implies

$$\begin{aligned}([a(u)]_\epsilon(t,x) - a(u_\epsilon(t,x)))\left|u_{\epsilon,x}(t,x)\right| &\lesssim \frac{1}{\epsilon}\fint_{[x-\epsilon,x+\epsilon]}\widehat{\Delta}(u(t,x),u(t,z))\,\mathrm{d}z\\ &= \frac{1}{\epsilon}\fint_{[-\epsilon,\epsilon]}\widehat{\Delta}(u(t,x),u(t,x+h))\,\mathrm{d}h,\end{aligned}$$

where

$$\widehat{\Delta}(u_1,u_2) = |u_1-u_2|^2 a''([\min(u_1,u_2) - |u_1-u_2|, \max(u_1,u_2) + |u_1-u_2|]). \tag{22}$$

Integrating, we deduce

$$\iint_{\operatorname{supp}\psi}|u_{\epsilon,x}|([a(u)]_\epsilon - a(u_\epsilon))\,\mathrm{d}x\,\mathrm{d}t \lesssim \frac{1}{\epsilon}\sup_{|h|<\epsilon}\iint_{\operatorname{supp}\psi}\widehat{\Delta}(u(t,x),u(t,x+h))\,\mathrm{d}x\,\mathrm{d}t.$$

Plugging this estimate into the bound (16) for $\langle\mu_\eta,\psi\rangle$, we find

$$\langle\mu_\eta,\psi\rangle \lesssim \|\psi\|_\infty \sup_I|\eta''|\cdot\limsup_{\epsilon\to0}\sup_{|h|<\epsilon}\frac{1}{|h|}\iint_{\operatorname{supp}\psi}\widehat{\Delta}(u(t,x),u(t,x+h))\,\mathrm{d}x\,\mathrm{d}t.$$

This is valid for any test function ψ and implies in particular that μ_η is a locally finite Radon measure if the lim sup in the right-hand side is finite. It remains to show that, under the doubling assumption on a'', this lim sup is controlled by (7), thus concluding the proof of Theorem 3.

Specifically, we claim

$$\Delta(u_1,u_2) \ge C\widehat{\Delta}(u_1,u_2) \qquad \forall\, u_1,u_2\in I, \tag{23}$$

for some constant C depending on the doubling constant of a''. To prove (23) we may assume $u_1 < u_2$. Letting $u_0 = (u_1+u_2)/2$ and $r = |u_1-u_2|$, and recalling the explicit expression (5) of Δ, we have

$$\Delta(u_1,u_2) = \int_{[u_1,u_2]}(s-u_1)(u_2-s)a''(\mathrm{d}s) \ge \frac{r^2}{9}\int_{[u_0-r/6,u_0+r/6]}a''(\mathrm{d}s).$$

Thanks to the doubling property of a'' we deduce the lower bound

$$\Delta(u_1,u_2) \ge Cr^2 a''([u_0-2r,u_0+2r]),$$

which implies (23) thanks to the explicit expression (22) of $\widehat{\Delta}$, since $[u_0-2r,u_0+2r]$ contains $[\min(u_1,u_2)-r,\max(u_1,u_2)+r]$. □

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