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A Note on Barker Sequences and the L_1-norm of Littlewood Polynomials

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Abstract. In this note, we investigate the L_1-norms of Barker polynomials and, more generally, Littlewood polynomials over the unit circle, and give improvements to some existing results.

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1. Introduction

For a sequence of complex numbers $a_0, a_1, \dots, a_{n-1}$, define its *aperiodic autocorrelation sequence* $\{c_j\}$ by

$$c_j = \sum_{l=0}^{n-1-j} a_{l+j}\overline{a_l}$$

for $0 \le j < n$, and $c_{-j} := \overline{c_j}$.

In 1953, Barker [2] asked to determine the ± 1 sequences $\{a_k\}$ satisfying $c_j \in \{0, -1\}$ for all $j \neq 0$. This condition was later relaxed in the followup researches, and now we call $a_0, a_1, \dots, a_{n-1} \in \{\pm 1\}$ a *generalized Barker sequence* (or simply a *Barker sequence*) of length n if $c_j \in \{0, \pm 1\}$ for all $j \neq 0$.

Ignoring the trivial case $n = 1$, one can assume $a_0 = a_1 = 1$ in a Barker sequence because negating every or every other term of $a_0, a_1, \dots, a_{n-1} \in \{\pm 1\}$ does not change the value of $|c_j|$. Under this normalization, the Barker sequences of length ≤ 13 are given in the following table. In fact, these are the only 8 Barker sequences that have been found so far.

$$\begin{array}{llll} n=2 & ++ & n=3 & ++- \\ n=4 & +++- & n=4 & ++-+ \\ n=5 & +++-+ & n=7 & +++--+- \\ n=11 & +++---+--+- & n=13 & +++++--++-+-+ \end{array} \tag{1}$$

In 1961, Turyn and Storer [16] proved that if the length n of a Barker sequence is odd, then $n \le 13$. (See also [14] for an alternative proof.) Thus all odd length Barker sequences are included in table (1). The even length case, on the other hand, is significantly harder. So far, with the efforts of many researchers, a lot of properties of Barker sequences of even length have been proved (cf. Borwein and Mossinghoff [3] for more references). Moreover, some computational results have been carried out since the 1990's (c.f. [4, 6, 8–10, 13]). In particular, it is shown in [9] that there is no Barker sequence of length $13 < n \le 4 \times 10^{33}$.

Although the even case is unsettled, it is generally believed that there are no more Barker sequences than those given in Table (1). This is the so-called *strong Barker sequence conjecture*. A weaker form of this conjecture, the *weak Barker sequence conjecture*, asserts that there are finitely many Barker sequences.

Barker sequences are closely related to the L_1-norm of *Littlewood polynomials*. A degree $n-1$ Littlewood polynomial is a polynomial $P(x) = s_0 + s_1 x + s_2 x^2 + \cdots + s_{n-1} x^{n-1}$ with $s_j = \pm 1$ for all $j \le n-1$. Here the L_p-norm of a polynomial refers to its L_p norm on the unit circle. In this note, just for convenience, we abuse the terminology and call the trigonometric polynomial

$$f(\alpha) = f_{\mathscr{S}_n}(\alpha) = P(e^{2\pi i\alpha}) = \sum_{j=0}^{n-1} s_j e(j\alpha), \quad \text{where} \quad e(t) = \exp(2\pi i t), \tag{2}$$

the *polynomial* corresponding to the complex sequence $\mathscr{S}_n = \{s_0, s_1, \ldots, s_{n-1}\}$ and, in particular, a *Barker Polynomial* if $\mathscr{S}_n$ is a Barker sequence. The L_p-norm of f is then accordingly defined by

$$\|f\|_p := \left(\int_0^1 |f(\alpha)|^p \mathrm{d}\alpha\right)^{\frac{1}{p}}.$$

There are numerous questions and conjectures concerning the L_p-norms of Littlewood polynomials. In particular, a conjecture attributed to Newman is as follows.

Conjecture. *There is a constant $c < 1$ such that $\|f\|_1 \le c\sqrt{n}$ for every Littlewood polynomial f of degree $n-1$.*

This conjecture has a lot of implications. For example, it yields a conjecture of Erdös on the maximum values of Littlewood polynomials, and the Turyn–Golay conjecture on the boundedness of the merit factor of any binary sequence. It is also closely related to Littlewood's flatness conjecture (which was solved by Körner [7] for unimodular polynomials and, most recently, by Balister et al. [1] for Littlewood polynomials). Although Newman's conjecture is generally believed to be true, it seems to be very difficult to prove at the moment.

The Barker sequence conjectures are related to some much weaker forms of Newman's conjecture. According to Turyn [15], if f is a Barker polynomial of degree $n-1$, then $\|f\|_1 > \sqrt{n-1}$. Thus even an upper bound as weak as $\|f\|_1 \lesssim \sqrt{n-1.1}$ for Littlewood polynomial of degree $n-1$ would at least settle the weak Barker sequence conjecture. It is unfortunate that the hitherto best upper bound for $\|f\|_1$ is not sufficient for this purpose. For any degree $n-1$ Littlewood polynomial f, Newman [11] proved that $\|f\|_1^2 < n - 0.03$; Habsieger [5] improved the bound to $\|f\|_1^2 < n - (3 - 2\sqrt{2}) + 0.18n^{-1}$ and $\|f\|_1^2 < n - 0.175 + o(1)$. Unaware of Habsieger's result[1], Borwein and Mossinghoff [3] improved Newman's bound to $\|f\|_1^2 < n - 0.09$.

In this note, we shall first improve Turyn's bound for L_1-norm of Barker polynomials.

Theorem 1. *Suppose $n > 13$ is an integer, and $f(\alpha)$ is a Barker polynomial of degree $n-1$. Then*

$$n - \frac{1}{2}\left(1 + \frac{1}{(\sqrt{2}+1)^2}\right) < \|f\|_1^2 \le n - \frac{1}{4} + \frac{1}{64n}. \tag{3}$$

[1] At the meeting of New Horizons in Additive Combinatorics(CRM, Montreal, October 2014), unaware of [5], the author reported a bound that is essentially the same as the first result in [5]. He thanks Professor Habsieger for bringing [5] to his attention.

Note that $\frac{1}{2}\left(1+\frac{1}{(\sqrt{2}+1)^2}\right) = 0.5857\ldots$. *Moreover, we have*

$$\|f\|_1^2 > n - \lambda(\theta) + O(n^{-1}) \tag{4}$$

where

$$\theta = \sup_{t>0} \frac{\sin^2 t}{t} = 0.72461135\ldots$$

and

$$\lambda(\theta) = \frac{1}{2}\left(\frac{1}{(\sqrt{1+\theta}+1)^2} + \frac{1}{(\sqrt{1-\theta}+1)^2}\right) = 0.3084981\ldots.$$

Our proof of Theorem 1 is quite simple and elementary. If f is a Barker polynomial of odd degree $n-1$, then $c_j = 0$ for all even $j \neq 0$. This implies that $|f(\alpha)|^2 + |f(\alpha+1/2)|^2 = 2n$ for any real α, and thus the sizes of $|f(\alpha)|$ and $|f(\alpha+1/2)|$ are mutually constrained. When bounding integrals involving $|f(\alpha)|$, we can gain some improvement by pairing up $|f(\alpha)|$ and $|f(\alpha+1/2)|$. The same idea applies to bounding the L_1-norm of Littlewood polynomials of odd degrees.

Theorem 2. *Suppose n is an even positive integer, and f is a Littlewood polynomial of degree $n-1$. Then*

$$\|f\|_1^2 \le n - \frac{1}{4(1+\gamma)} = n - 0.21588\ldots, \tag{5}$$

where $\gamma = 0.1580\ldots$ *is the only positive root of the polynomial* $64x^3 + 112x^2 + 25x - 7$.

If a degree $n-1$ Littlewood polynomial had L_1-norm close enough to $\sqrt{n}$, then the sum of $|c_j|$'s (including c_0) would tend to be close to $2n$. In particular, when $n \equiv 3 \pmod 4$, we would have $c_j = -1$ for most even $-n < j < n$. This observation enables us to better bound $\|f\|_\infty$ in such special case. Hence, while the approach used in Theorems 1 and 2 does not apply to Littlewood polynomials of even degrees, we are able to improve the existing upper bound for the L_1-norm in the case $n \equiv 3 \pmod 4$.

Theorem 3. *Suppose $n \equiv 3 \pmod 4$ is a positive integer. Then for any Littlewood polynomial f of degree $n-1$, we have*

$$\|f\|_1^2 \le n - \lambda + o(1) \tag{6}$$

where

$$\lambda = \frac{1}{\left(\sqrt{2+2\sqrt{4-2\sqrt{2}}-2\sqrt{2}}+1\right)^2} = 0.2151\ldots.$$

We can make the bound in Theorem 3 uniform in n, but we shall not do so just to avoid complicated calculations. Our primary purpose is to present our approach in this particular case.

2. Some Lemmas and Proof of Theorem 1

Before proving Theorem 1, we record two results about Barker sequences.

Lemma 4. *Suppose $a_0, a_1, \ldots, a_{n-1}$ is a $\{\pm 1\}$ sequence, and let $\{c_k\}$ denote the aperiodic autocorrelations. Then for every $1 \le j \le n-1$,*

$$c_j + c_{n-j} \equiv n \pmod 4. \tag{7}$$

If in addition n is even and $\{a_j\}$ is a Barker sequence, then $n = 4m^2$ for some integer m.

Proof. This is from [15, Theorem 2.1]. One can also refer to [3, Theorem 2.1]. □

Lemma 5. *Suppose f is a Barker polynomial of degree $n-1$. Then*

$$\frac{\left||f(\alpha)|^2-n\right|}{n}\le\theta+O(n^{-1}),\tag{8}$$

where

$$\theta=\sup_{t>0}\frac{\sin^2 t}{t}=0.7246113537\ldots.$$

Proof. This essentially belongs to Saffari [12], with an oversight corrected in [3, Theorem 3.1]. □

Now we prove Theorem 1. Suppose n is even, and f is a Barker polynomial of degree $n-1$. First we note that $c_j=0$ for all non-zero even j. Thus, for any $\alpha\in\mathbb{R}$,

$$|f(\alpha)|^2+|f(\alpha+1/2)|^2=2c_0=2n.\tag{9}$$

From

$$\int_0^1(|f(\alpha)|-\sqrt{n})^2\mathrm{d}\alpha=2n-2\sqrt{n}\|f\|_1,$$

we get

$$\|f\|_1=\sqrt{n}-\frac{1}{2\sqrt{n}}\int_0^1(|f(\alpha)|-\sqrt{n})^2\mathrm{d}\alpha.$$

By the periodicity of f, we then have

$$\begin{aligned}\|f\|_1&=\sqrt{n}-\frac{1}{4\sqrt{n}}\int_0^1\left[(|f(\alpha)|-\sqrt{n})^2+(|f(\alpha+1/2)|-\sqrt{n})^2\right]\mathrm{d}\alpha\\&=\sqrt{n}-\frac{1}{4\sqrt{n}}\int_0^1(|f(\alpha)|^2-n)^2\left[(|f(\alpha)|+\sqrt{n})^{-2}+(|f(\alpha+1/2)|+\sqrt{n})^{-2}\right]\mathrm{d}\alpha\end{aligned}\tag{10}$$

where the last step follows from (9). Writing $x=x(\alpha)=\frac{|f(\alpha)|^2}{n}$, we get

$$(|f(\alpha)|+\sqrt{n})^{-2}+(|f(\alpha+1/2)|+\sqrt{n})^{-2}=\frac{((\sqrt{x}+1)^{-2}+(\sqrt{2-x}+1)^{-2})}{n}=\frac{g(x(\alpha))}{n},\quad\text{say.}$$

From

$$\int_0^1\left(|f(\alpha)|^2-n\right)^2\mathrm{d}\alpha=n$$

and (10), then we have

$$\sqrt{n}-\frac{\max\{g(x)\}}{4\sqrt{n}}\le\|f\|_1\le\sqrt{n}-\frac{\min\{g(x)\}}{4\sqrt{n}}.\tag{11}$$

On $[0,2]$, $g(x)$ is convex, symmetric about $x=1$, and with only one critical number $x=1$. Hence, trivially, we have

$$\max\{g(x(\alpha))\}\le g(0),\quad\text{and}\quad\min\{g(x(\alpha))\}\ge g(1).$$

Taking this into (11) and squaring all sides, we get (3).

From Lemma 5, we have $1-\theta+O(n^{-1})\le x\le1+\theta+O(n^{-1})$. By applying this to (11), we have the lower bound (4).

3. Proof of Theorem 2

Let

$$O:=\sum_{j\text{ odd}}|c_j|,\quad\text{and}\quad E:=\sum_{\substack{j\text{ even}\\ j\ne0}}|c_j|.$$

When n is even, c_j has the same parity as j. Thus $O \geq n$. First, we have

$$\begin{aligned}\|f\|_1^2 &= \|f\|_2{}^2 - \int_0^1 (|f(\alpha)| - \|f\|_1)^2 \mathrm{d}\alpha \\ &= n - \int_0^1 \frac{(|f(\alpha)|^2 - \|f\|_1^2)^2}{(|f(\alpha)| + \|f\|_1)^2} \mathrm{d}\alpha \\ &\leq n - \frac{\int_0^1 (|f(\alpha)|^2 - \|f\|_1^2)^2 \mathrm{d}\alpha}{(\|f\|_\infty + \|f\|_1)^2} \\ &\leq n - \frac{\sum_{j\neq 0} c_j^2}{(\|f\|_\infty + \|f\|_1)^2}. \end{aligned} \tag{12}$$

A trivial bound

$$\|f\|_\infty \leq \sqrt{n + \sum_{j\neq 0} |c_j|} = \sqrt{n+O+E}$$

together with (12) gives

$$\|f\|_1^2 \leq n - \frac{\sum_{j\neq 0} c_j^2}{(\sqrt{n+O+E}+\sqrt{n})^2} \leq n - \frac{O+2E}{(\sqrt{n+O+E}+\sqrt{n})^2} \tag{13}$$

where the last inequality in (13) follows from the fact $|c_j|^2 \geq 2|c_j|$ for even j. On the other hand, from the periodicity of $f(\alpha)$ we have

$$\begin{aligned}\|f\|_1^2 &= \|f\|_2{}^2 - \frac{1}{2}\left(\int_0^1 (|f(\alpha)| - \|f\|_1)^2 \mathrm{d}\alpha + \int_0^1 \left(\left|f\left(\alpha+\frac{1}{2}\right)\right| - \|f\|_1\right)^2 \mathrm{d}\alpha\right) \\ &\leq n - \frac{1}{4}\int_0^1 \left(|f(\alpha)| - \left|f\left(\alpha+\frac{1}{2}\right)\right|\right)^2 \mathrm{d}\alpha \\ &\leq n - \frac{\int_0^1 \left(|f(\alpha)|^2 - \left|f\left(\alpha+\frac{1}{2}\right)\right|^2\right)^2 \mathrm{d}\alpha}{4\sup\left\{\left(|f(\alpha)| + \left|f\left(\alpha+\frac{1}{2}\right)\right|\right)^2\right\}}. \end{aligned} \tag{14}$$

Note that

$$\int_0^1 \left(|f(\alpha)|^2 - \left|f\left(\alpha+\frac{1}{2}\right)\right|^2\right)^2 \mathrm{d}\alpha \geq 4O, \tag{15}$$

and

$$\left(|f(\alpha)| + \left|f\left(\alpha+\frac{1}{2}\right)\right|\right)^2 \leq 2\left(|f(\alpha)|^2 + \left|f\left(\alpha+\frac{1}{2}\right)\right|^2\right) \leq 4(n+E). \tag{16}$$

Thus from (14)–(16), and the fact that $O \geq n$, we get another bound for $\|f\|_1^2$,

$$\|f\|_1^2 \leq n - \frac{O}{4(n+E)} \leq n - \frac{n}{4(n+E)}. \tag{17}$$

Next we prove (5) by using (13) and (17) in accordance with the size of E.

Case 1. $E \geq 2n$**.** Note that the function

$$g(x) := \frac{x}{(1+\sqrt{1+x})^2}$$

is increasing on $[0,\infty)$, so if $O+E \geq 3n$, then from (13) we have

$$\|f\|_1^2 \leq n - \frac{O+E}{(\sqrt{n+O+E}+\sqrt{n})^2} \leq n - \frac{3n}{(\sqrt{3n+n}+\sqrt{n})^2} = n - \frac{1}{3} \tag{18}$$

which is admissible.

Case 2. $E \le 2n$**.** Let

$$g_1(O,E) = n - \frac{O+2E}{(\sqrt{n+O+E}+\sqrt{n})^2}.$$

When $E \le 2n$, we have

$$\frac{\partial g_1(O,E)}{\partial O} = \frac{E-(n+\sqrt{n(n+O+E)})}{\sqrt{n+O+E}(\sqrt{n+O+E}+\sqrt{n})^3} < 0.$$

Thus from (13), we have

$$\|f\|_1^2 \le n - \frac{n+2E}{(\sqrt{2n+E}+\sqrt{n})^2}. \tag{19}$$

Let $E = xn$, and

$$g_2(x) = n - \frac{1+2x}{(\sqrt{2+x}+1)^2}, \quad g_3(x) = n - \frac{1}{4(1+x)}.$$

Note that on $[0,2]$, $g_2(x)$ is decreasing and $g_3(x)$ is increasing, and they intersect at $x = \gamma = 0.15802\ldots$ where γ is the only positive root of $64x^3+112x^2+25x-7$. Hence from this and (17), (19), we get

$$\|f\|_1^2 \le n - \frac{1}{4(1+\gamma)} = n - 0.21588\ldots. \qquad \square$$

4. Proof of Theorem 3

Let

$$O := \sum_{j \text{ odd}} |c_j|, \quad \text{and} \quad e := \sum_{\substack{j \text{ even} \\ j\ne 0}} |c_j| - (n-1).$$

Since n is odd, we note that c_j is even when j is odd, and odd when j is even. In particular, we have $e \ge 0$, $4 \mid O$, $4 \mid e$ and, moreover,

$$\sum_{\substack{j \text{ even} \\ j\ne 0}} |c_j|^2 = \sum_{|c_j|=1} |c_j| + 3\sum_{|c_j|=3} |c_j| + 5\sum_{|c_j|=5} |c_j| + \cdots \ge 4e+n-1. \tag{20}$$

From this, and an argument similar to (12), we get

$$\|f\|_1^2 \le n - \frac{2O+4e+n-1}{(\|f\|_\infty+\sqrt{n})^2} \le n - \frac{2O+2e+n}{(\|f\|_\infty+\sqrt{n})^2} + O(n^{-1}). \tag{21}$$

We note that, trivially, $\|f\|_\infty \le \sqrt{2n+O+e}$.

If $O+e \ge \frac{n}{5}$, then

$$\begin{aligned} \|f\|_1^2 &\le n - \frac{2O+2e+n}{(\sqrt{2n+O+e}+\sqrt{n})^2} + O(n^{-1}) \le n - \frac{7/5}{(\sqrt{11/5}+1)^2} + O(n^{-1}) \\ &= n - \frac{7}{16+2\sqrt{55}} + O(n^{-1}) \le n - 0.22 + O(n^{-1}), \end{aligned} \tag{22}$$

which is admissible.

We suppose $O+e \le \frac{n}{5}$ henceforth. Note that there are at most $\frac{e}{2}$ even non-zero j's with $|c_j| \ne 1$, also there are at most $\frac{O}{2}$ odd j's such that $c_j \ne 0$. Thus, from

$$c_j + c_{n-j} \equiv n \equiv -1 \pmod 4,$$

we conclude that there are at least $n-1-\frac{O}{2}$ even non-zero j's satisfying $c_j \equiv -1 \pmod 4$, among which at least $n-1-\frac{O+e}{2}$ even j's satisfy $c_j = -1$.

For a positive integer $K \ge \frac{n-1}{2}$, let

$$h_K(\alpha) := \sum_{0\le|j|\le K} \left(1-\frac{|j|}{K}\right) e(j\alpha) = \frac{1}{K}\left|\sum_{j=1}^{K} e(j\alpha)\right|^2.$$

Then for any number $\delta > 0$, we have

$$|f(\alpha)|^2 \le |f(\alpha)|^2 + \delta h_K(2\alpha) = \sum_{-2K<n<2K} b_j e(j\alpha), \tag{23}$$

where

$$b_j = \begin{cases} c_j, & \text{if } j \text{ is odd} \\ c_j + \delta\left(1 - \frac{|j|}{2K}\right), & \text{if } j \text{ is even.} \end{cases}$$

Hence

$$|f(\alpha)|^2 \le \sum_{-2K<j<2K} |b_j| = O + \sum_{-K<j<K} \left| c_{2j} + \delta\left(1 - \frac{|j|}{K}\right)\right|. \tag{24}$$

We suppose $\delta > 1$ and

$$\frac{O+e}{4} < (1-\delta^{-1})K < \frac{n-1}{2}. \tag{25}$$

The purpose of introducing $\delta h_K(2\alpha)$ in (23) is to reduce the contribution from $|c_{2j}|$ for most negative c_{2j}. We observe that the largest value of the last sum in (24) occurs when we have $c_{2j} = -1$ for $(1-\delta^{-1})K < |j| \le \frac{n-1}{2}$ and, other than the $2[(1-\delta^{-1})K] - \frac{O+e}{2}$ such $j \in [-(1-\delta^{-1})K, (1-\delta^{-1})K]$, it holds that $c_{2j} > 0$ for all other j in the range. With this observation, we get from (24) that

$$|f(\alpha)|^2 \le 2n + O + e + \delta K - 4\left((1-\delta^{-1})K - \frac{O+e}{4}\right) - 4\delta \sum_{(1-\delta^{-1})K<j\le\frac{n-1}{2}} (1 - j/K). \tag{26}$$

For convenience, we let $O + e = \epsilon n$, $K = \kappa n$. Then (26) gives

$$|f(\alpha)|^2 \le g(\epsilon;\delta,\kappa)n + O(1), \tag{27}$$

where

$$g(\epsilon;\delta,\kappa) = 2 + 2\epsilon + \delta\kappa - \delta(1 - 2(1-\delta^{-1})\kappa)\left(1 + \delta^{-1} - \frac{1}{2\kappa}\right) - 4(1-\delta^{-1})\kappa,$$

with (25) and the other conditions on the variable ϵ and parameters δ, κ translated into

$$0 \le \epsilon \le \frac{1}{5}, \quad \kappa > \frac{1}{2}, \quad \delta > 1, \quad \text{and} \quad \frac{\epsilon}{4} \le (1-\delta^{-1})\kappa < \frac{1}{2}. \tag{28}$$

Now by (21), we have

$$\|f\|_1^2 \le n - \frac{1+2\epsilon}{(\sqrt{g(\epsilon;\delta,\kappa)}+1)^2} + O(n^{-1}). \tag{29}$$

It turns out that the optimal choices for δ and κ are

$$\kappa = \kappa_0 = 2^{-1/2}, \quad \text{and} \quad \delta = \delta_0 = \sqrt{1 + 2^{-1/2}}.$$

It is easy to check that all conditions in (28) are satisfied under these choices and $0 \le \epsilon \le \frac{1}{5}$. Moreover, the function $\frac{1+2\epsilon}{(\sqrt{g(\epsilon;\delta_0,\kappa_0)}+1)^2}$ is strictly increasing on $[0, 1/5]$. Hence from (29) we have

$$\|f\|_1^2 \le n - \frac{1}{(\sqrt{g(0;\delta_0,\kappa_0)}+1)^2} + O(n^{-1}) = n - \frac{1}{\left(\sqrt{2 + 2\sqrt{4-2\sqrt{2}} - 2\sqrt{2}} + 1\right)^2} + O(n^{-1})$$

and thus the theorem is proved. □

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