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Fredholm modules over categories, Connes periodicity and classes in cyclic cohomology

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Abstract. We replace a ring with a small $\mathbb{C}$-linear category $\mathscr{C}$, seen as a ring with several objects in the sense of Mitchell. We introduce Fredholm modules over this category and construct a Chern character taking values in the cyclic cohomology of $\mathscr{C}$. We show that this categorified Chern character is homotopy invariant and is well-behaved with respect to the periodicity operator in cyclic cohomology. For this, we also obtain a description of cocycles and coboundaries in the cyclic cohomology of $\mathscr{C}$ (and more generally, in the Hopf cyclic cohomology of a Hopf-module category) by means of DG-semicategories equipped with a trace on endomorphism spaces.

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1. Introduction

In his celebrated work [12], Connes extended differential calculus beyond the framework of manifolds to include noncommutative spaces such as that of leaves of a foliation or the orbit space of the action of a group on a manifold. For this, he began by considering Fredholm modules over an algebra A which could in general be noncommutative. When A is commutative, such as the space of smooth functions on a manifold M, examples of Fredholm modules over A may be obtained by considering elliptic operators on M. More generally, by considering Schatten classes inside the collection of bounded operators on a Hilbert space, Connes studied the notion of p-summable Fredholm modules over A in [12]. The Fredholm modules over A lead to Chern characters taking values in the cyclic cohomology of A. Moreover, these cohomology classes are related by means of Connes' periodicity operator.

In this paper, we study Fredholm modules over linear categories, along with their Chern characters taking values in cyclic cohomology. Our idea is to have a counterpart of the algebraic notion of modules over a category, a subject which has been highly developed in the literature (see, for instance, [7, 17, 35, 36, 45, 46]). A small preadditive category is treated as a ring with several objects, following an idea first advanced by Mitchell [39]. We note that there is also a well-developed study of spaces in algebraic geometry over categories (see, for instance, [16, 17, 44]). It is also important to mention here the work of Baez [2] with the category of Hilbert spaces as well as the recent work of Henriques [22], Henriques and Penneys [23] with fusion categories with potential applications to physics.

Let $\mathscr{C}$ be a small linear category. Let $\mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}$ be the category of $\mathbb{Z}/2\mathbb{Z}$-graded separable Hilbert spaces and whose morphisms are bounded linear maps. We consider pairs $(\mathscr{H},\mathscr{F})$, where $\mathscr{H}$ is a linear functor

$$\mathscr{H}:\mathscr{C}\longrightarrow \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}} \tag{1}$$

and $\mathscr{F}=\{\mathscr{F}_X:\mathscr{H}(X)\to\mathscr{H}(X)\}_{X\in \mathrm{Ob}(\mathscr{C})}$ is a family of bounded and involutive linear operators each of degree 1. When the elements of $\mathscr{F}$ satisfy certain commutator conditions with respect to the operators $\{\mathscr{H}(f)\}_{f\in Mor(\mathscr{C})}$, we say that the pair $(\mathscr{H},\mathscr{F})$ is a Fredholm module over the category $\mathscr{C}$. Following the methods of Connes [12], we construct Chern characters of these Fredholm modules taking values in the cyclic cohomology of $\mathscr{C}$ and study how they are related by means of the periodicity operator. We hope this is the first step towards a larger program which mixes together the techniques in categorical algebra with those in differential geometry.

The paper consists of two parts. In the first part, we study cyclic cohomology. We work more generally with a small linear category $\mathscr{D}_H$ whose morphism spaces carry a well-behaved action of a Hopf algebra H. In other words, $\mathscr{D}_H$ is a small Hopf-module category (or H-category) in the sense of Cibils and Solotar [10], with which we can get H-linear categorical generalizations of several results in Hopf cyclic cohomology. We recall that in [13–15], Connes and Moscovici introduced Hopf cyclic cohomology as a generalization of Lie algebra cohomology adapted to noncommutative geometry. For an H-category $\mathscr{D}_H$, we describe the cocycles and coboundaries that determine its Hopf cyclic cohomology groups by extending Connes' original construction of cyclic cohomology from [11] and [12] in terms of cycles and closed graded traces on differential graded algebras. An important role in our paper is played by "semicategories," which are categories that may not contain identity maps. This notion, introduced by Mitchell [40], is precisely what we need in order to categorify non-unital algebras. We work with the Hopf cyclic cohomology groups $HC^\bullet_H(\mathscr{D}_H,M)$ having coefficients in M, where M is a stable anti-Yetter Drinfeld module in the sense of [18].

Let k be a field. After collecting some preliminaries in Section 2, we begin in Section 3 by considering the universal differential graded Hopf-module semicategory (or DGH-semicategory) associated to the H-category $\mathscr{D}_H$. For a DGH-semicategory $(\mathscr{S}_H,\widehat{\partial}_H)$ and $n\geq 0$, we let an n-dimensional closed graded (H,M)-trace on $\mathscr{S}_H$ be a collection of maps

$$\widehat{\mathscr{T}}^H:=\left\{\widehat{\mathscr{T}}^H_X: M\otimes \mathrm{Hom}^n_{\mathscr{S}_H}(X,X)\longrightarrow k\right\}_{X\in \mathrm{Ob}(\mathscr{S}_H)} \tag{2}$$

satisfying certain conditions (see Definition 12). A cycle over $\mathscr{D}_H$ consists of a tuple $(\mathscr{S}_H,\widehat{\partial}_H,M,\widehat{\mathscr{T}}^H)$ along with an H-linear semifunctor $\rho:\mathscr{D}_H\to\mathscr{S}^0_H$. In Theorem 14, we provide a description of the cocycles $Z^\bullet_H(\mathscr{D}_H,M)$ in Hopf cyclic cohomology in terms of characters of cycles over $\mathscr{D}_H$. This result is an H-linear categorical version of Connes' [12, Proposition 1, p. 98]. It also follows from Theorem 14 that there is a one-one correspondence between $Z^\bullet_H(\mathscr{D}_H,M)$ and the collection of n-dimensional closed graded (H,M)-traces on the universal DGH-semicategory $\Omega(\mathscr{D}_H)$ associated to $\mathscr{D}_H$.

In Sections 4 and 5, we provide a description of the space $B^\bullet_H(\mathscr{D}_H,M)$ of coboundaries. Throughout, we take $k=\mathbb{C}$. We consider families η of automorphisms $\eta=\{\eta(X)\in Aut_{\mathscr{D}_H}(X)\}_{X\in\mathrm{Ob}(\mathscr{D}_H)}$ such that $h(\eta(X))=\epsilon(h)\eta(X)$ for all $h\in H$ and $X\in\mathrm{Ob}(\mathscr{D}_H)$. We show that these families form a group, which we denote by $\mathbb{U}_H(\mathscr{D}_H)$. Further, we show that the inner automorphism of $\mathscr{D}_H$ induced by conjugating with an element $\eta\in\mathbb{U}_H(\mathscr{D}_H)$ induces the identity functor on $HC^\bullet_H(\mathscr{D}_H,M)$. Using this, we obtain in Proposition 23 a set of sufficient conditions for the Hopf cyclic cohomology of an H-category to be zero.

We say that a cycle $(\mathscr{S}_H,\widehat{\partial}_H,M,\widehat{\mathscr{T}}^H)$ is vanishing if $\mathscr{S}^0_H$ is an H-category and $\mathscr{S}^0_H$ satisfies the assumptions in Proposition 23. We describe the elements of $B^\bullet_H(\mathscr{D}_H,M)$ in Theorem 28 as the characters of vanishing cycles over $\mathscr{D}_H$. Finally, in Theorem 30, we use categorified cycles and vanishing cycles to construct a product in Hopf cyclic cohomologies

$$HC^p_H(\mathscr{D}_H, M) \otimes HC^q_H(\mathscr{D}'_H, M') \longrightarrow HC^{p+q}_H(\mathscr{D}_H \otimes \mathscr{D}'_H, M \square_H M') \qquad p, q \geq 0 \tag{3}$$

where $\mathscr{D}_H$ and $\mathscr{D}'_H$ are H-linear categories and M and M' are stable anti-Yetter Drinfeld modules over H satisfying certain conditions.

In the second part of the paper, we study Fredholm modules and Chern classes. For this, we assume $H = \mathbb{C} = M$ and consider a small $\mathbb{C}$-linear category $\mathscr{C}$. Let $p \geq 1$ be an integer. We will say that a pair $(\mathscr{H}, \mathscr{F})$ over $\mathscr{C}$ as in (1) is a p-summable Fredholm module if it satisfies

$$[\mathscr{F}, f] := \left(\mathscr{F}_Y \circ \mathscr{H}(f) - \mathscr{H}(f) \circ \mathscr{F}_X\right) \in \mathscr{B}^p\left(\mathscr{H}(X), \mathscr{H}(Y)\right) \tag{4}$$

for any morphism $f : X \to Y$ in $\mathscr{C}$ (see Definition 31). Here, $\mathscr{B}^p\left(\mathscr{H}(X), \mathscr{H}(Y)\right)$ is the p-th Schatten class inside the space of bounded linear operators from $\mathscr{H}(X)$ to $\mathscr{H}(Y)$. We mention here that in this paper, we will consider only even Fredholm modules. We hope to tackle the case of odd Fredholm modules over linear categories in a future paper [4].

Let $H^\bullet_\lambda(\mathscr{C}) := HC^\bullet_{\mathbb{C}}(\mathscr{C}, \mathbb{C})$ denote the cyclic cohomology groups of $\mathscr{C}$. Corresponding to a p-summable Fredholm module $(\mathscr{H}, \mathscr{F})$ and any $2m \geq p - 1$, we construct a DG-semicategory $(\Omega_{(\mathscr{H},\mathscr{F})}\mathscr{C}, \partial')$ along with a closed graded trace $\widehat{\mathrm{Tr}}_s = \{\mathrm{Tr}_s : \mathrm{Hom}^{2m}_{\Omega_{(\mathscr{H},\mathscr{F})}}(X, X) \to \mathbb{C}\}_{X \in \mathrm{Ob}(\mathscr{C})}$ of dimension $2m$. Let $CN_\bullet(\mathscr{C})$ denote the cyclic nerve of $\mathscr{C}$ and $CN^\bullet(\mathscr{C})$ its linear dual. By taking the character of the corresponding cycle over $\mathscr{C}$, we obtain $\phi^{2m} \in CN^{2m}(\mathscr{C})$ which is given by (see Theorem 34)

$$\phi^{2m}(f^0 \otimes f^1 \otimes \cdots \otimes f^{2m}) := \mathrm{Tr}_s\left(\mathscr{H}(f^0)[\mathscr{F}, f^1][\mathscr{F}, f^2] \ldots [\mathscr{F}, f^{2m}]\right) \tag{5}$$

for any $f^0 \otimes f^1 \otimes \cdots \otimes f^{2m} \in CN_{2m}(\mathscr{C})$. Then, ϕ^{2m} lies in the space $Z^{2m}_\lambda(\mathscr{C})$ of cocycles for the cyclic cohomology of $\mathscr{C}$. The Chern character $\mathrm{ch}^{2m}(\mathscr{H}, \mathscr{F})$ of the Fredholm module $(\mathscr{H}, \mathscr{F})$ will be the class of ϕ^{2m} in the cyclic cohomology $H^{2m}_\lambda(\mathscr{C})$ of $\mathscr{C}$.

We relate the Chern characters by means of the periodicity operator in Section 7. We know that the action of the periodicity operator $S : H^\bullet_\lambda(\mathscr{C}) \to H^{\bullet+2}_\lambda(\mathscr{C})$ is given by taking the product as in (3) with a certain class in the cohomology $H^2_\lambda(\mathbb{C})$. If $(\mathscr{H}, \mathscr{F})$ is a p-summable Fredholm module over $\mathscr{C}$ and $2m \geq p - 1$, we show in Theorem 38 that

$$S(\phi^{2m}) = -(m+1)\phi^{2m+2} \qquad \text{in } H^{2m+2}_\lambda(\mathscr{C}) \tag{6}$$

Finally, in Section 8, we describe the homotopy invariance of the Chern character. For this, we consider a family $\{(\rho_t, \mathscr{F}_t)\}_{t \in [0,1]}$ of p-summable Fredholm modules

$$\{\rho_t : \mathscr{C} \longrightarrow \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}\}_{t \in [0,1]} \qquad \mathscr{F}_t(X) : \rho_t(X) \longrightarrow \rho_t(X) \tag{7}$$

each having the same underlying Hilbert space and satisfying some conditions. Then, if the ρ_t and $\mathscr{F}_t$ vary in a strongly C^1 manner with respect to $t \in [0,1]$, we show in Theorem 45 that the $(p+2)$-dimensional character $\mathrm{ch}^{p+2}(\mathscr{H}_t, \mathscr{F}_t) \in H^{p+2}_\lambda(\mathscr{C})$ is independent of $t \in [0,1]$.

Notation. Throughout the paper, H is a Hopf algebra over the field k of characteristic zero, with comultiplication Δ, counit ε and bijective antipode S. We will use Sweedler's notation for the coproduct $\Delta(h) = h_1 \otimes h_2$ and for a left H-coaction $\rho : M \to H \otimes M$, $\rho(m) = m_{(-1)} \otimes m_{(0)}$ (with the summation sign suppressed). The small cyclic category of Connes [11] will be denoted by Λ. The Hochschild differential will always be denoted by b and the modified Hochschild differential (with the last face operator missing) will be denoted by b'.

On any cocyclic module $\mathscr{C}$, we will denote by τ_n the unsigned cyclic operator on $C^n(\mathscr{C})$ and by λ_n the signed cyclic operator $(-1)^n\tau_n$ on $C^n(\mathscr{C})$. The complex computing cyclic cohomology of $\mathscr{C}$ will be denoted by $C^\bullet_\lambda(\mathscr{C}) := \mathrm{Ker}(1-\lambda)$. Accordingly, the cyclic cocycles and cyclic coboundaries will be denoted by $Z^\bullet_\lambda(\mathscr{C})$ and $B^\bullet_\lambda(\mathscr{C})$ respectively.

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2. Preliminaries on H-categories and Hopf cyclic cohomology

A small Hopf-module category may be treated as a "Hopf-module algebra with several objects." In this section, we will collect some preliminaries on Hopf-module categories and on Hopf cyclic cohomology. We note that the Hopf cyclic cohomology introduced by Connes and Moscovici ([13–15]) has been developed extensively by a number of authors (see, for instance, [1, 3, 19–21, 26–28, 32, 33, 42]).

Definition 1 (see Cibils and Solotar [10]). *Let H be a Hopf algebra over a field k. A k-linear category $\mathscr{D}_H$ is said to be a left H-module category if*

- (i) $\mathrm{Hom}_{\mathscr{D}_H}(X,Y)$ *is a left H-module for all* $X, Y \in \mathrm{Ob}(\mathscr{D}_H)$
- (ii) $h1_X = \varepsilon(h)1_X$ *for all* $X \in \mathrm{Ob}(\mathscr{D}_H)$ *and* $h \in H$
- (iii) *the composition map is a morphism of H-modules, i.e.,* $h(gf) = (h_1g)(h_2f)$ *for any* $h \in H$, $f \in \mathrm{Hom}_{\mathscr{D}_H}(X,Y)$ *and* $g \in \mathrm{Hom}_{\mathscr{D}_H}(Y,Z)$.

A small left H-module category will be called a left H-category. We will denote by Cat_H *the category of all left H-categories with H-linear functors between them.*

For more on Hopf-module categories, we refer the reader, for instance, to [5, 6, 24, 29]. Let $\mathscr{D}_H$ be a left H-category. We set

$$CN_n(\mathscr{D}_H) := \bigoplus \mathrm{Hom}_{\mathscr{D}_H}(X_1,X_0) \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_2,X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_0,X_n) \tag{8}$$

where the direct sum runs over all $(X_0, X_1, \ldots, X_n) \in \mathrm{Ob}(\mathscr{D}_H)^{n+1}$. We observe that $CN_n(\mathscr{D}_H)$ carries the structure of a left H-module with action determined by

$$h(f^0 \otimes \cdots \otimes f^n) := h_1 f^0 \otimes h_2 f^1 \otimes \cdots \otimes h_{n+1} f^n \tag{9}$$

for any $f^0 \otimes \cdots \otimes f^n \in CN_n(\mathscr{D}_H)$ and $h \in H$.

Lemma 2. *Let M be a right H-module. For each $n \geq 0$, $M \otimes CN_n(\mathscr{D}_H)$ is a right H-module with action determined by*

$$(m \otimes f^0 \otimes \cdots \otimes f^n)h := mh_1 \otimes S(h_2)(f^0 \otimes \cdots \otimes f^n) = mh_1 \otimes S(h_{n+2})f^0 \otimes \cdots \otimes S(h_2)f^n \tag{10}$$

for any $m \in M$, $f^0 \otimes \cdots \otimes f^n \in CN_n(\mathscr{D}_H)$ and $h \in H$.

We now recall the notion of a stable anti-Yetter–Drinfeld module (SAYD) module from [19, Definition 2.1].

Definition 3. *Let H be a Hopf algebra with a bijective antipode S. A k-vector space M is said to be a right-left anti-Yetter–Drinfeld module over H if M is a right H-module and a left H-comodule such that*

$$\rho(mh) = (mh)_{(-1)} \otimes (mh)_{(0)} = S(h_3)m_{(-1)}h_1 \otimes m_{(0)}h_2 \tag{11}$$

for all $m \in M$ and $h \in H$, where $\rho : M \to H \otimes M$, $m \mapsto m_{(-1)} \otimes m_{(0)}$ is the coaction. Moreover, M is said to be stable if $m_{(0)}m_{(-1)} = m$.

We now take the Hopf cyclic cohomology $HC^\bullet_H(\mathscr{D}_H, M)$ of an H-category $\mathscr{D}_H$ with coefficients in a SAYD module M (see also [29]). This generalizes the construction of the Hopf cyclic cohomology for H-module algebras with coefficients in an SAYD module (see [18] and also [37]). For each $n \geq 0$, we set

$$C^n(\mathscr{D}_H, M) := \mathrm{Hom}_k(M \otimes CN_n(\mathscr{D}_H), k) \qquad C^n_H(\mathscr{D}_H, M) := \mathrm{Hom}_H(M \otimes CN_n(\mathscr{D}_H), k) \tag{12}$$

where $M \otimes CN_n(\mathscr{D}_H)$ is considered as a right H-module with the action described in Lemma 2 and k is considered as a right H-module via the counit. It is clear from the definition in (12) and the action described in (10) that an element in $C^n_H(\mathscr{D}_H, M)$ is a k-linear map $\phi : M \otimes CN_n(\mathscr{D}_H) \to k$ satisfying

$$\begin{aligned} \phi\big(mh_1 \otimes S(h_2)(f^0 \otimes \cdots \otimes f^n)\big) &= \phi\big(mh_1 \otimes S(h_{n+2}) f^0 \otimes \cdots \otimes S(h_2) f^n\big) \\ &= \varepsilon(h)\phi(m \otimes f^0 \otimes \cdots \otimes f^n) \end{aligned} \tag{13}$$

We recall that a (co)simplicial module is said to be para-(co)cyclic if all the relations for a (co)cyclic module are satisfied except $\tau_n^{n+1} = 1$ (see, for instance [29]). The following may be verified directly.

Proposition 4. *Let $\mathscr{D}_H$ be a left H-category and let M be a right-left SAYD module over H. Then,*

(i) *we have a para-cocyclic module $C^\bullet(\mathscr{D}_H, M) := \{C^n(\mathscr{D}_H, M)\}_{n\geq 0}$ with the following structure maps*

$$(\delta_i\phi)(m \otimes f^0 \otimes \cdots \otimes f^n) = \begin{cases} \phi(m \otimes f^0 \otimes \cdots \otimes f^i f^{i+1} \otimes \cdots \otimes f^n) & 0 \leq i \leq n-1 \\ \phi\big(m_{(0)} \otimes \big(S^{-1}(m_{(-1)}) f^n\big) f^0 \otimes \cdots \otimes f^{n-1}\big) & i = n \end{cases}$$

$$(\sigma_i\psi)(m \otimes f^0 \otimes \cdots \otimes f^n) = \begin{cases} \psi(m \otimes f^0 \otimes \cdots \otimes f^i \otimes 1_{X_{i+1}} \otimes f^{i+1} \otimes \cdots \otimes f^n) & 0 \leq i \leq n-1 \\ \psi(m \otimes f^0 \otimes \cdots \otimes f^n \otimes 1_{X_0}) & i = n \end{cases}$$

$$(\tau_n\varphi)(m \otimes f^0 \otimes \cdots \otimes f^n) = \varphi\big(m_{(0)} \otimes S^{-1}(m_{(-1)}) f^n \otimes f^0 \otimes \cdots \otimes f^{n-1}\big)$$

for any $\phi \in C^{n-1}(\mathscr{D}_H, M)$, $\psi \in C^{n+1}(\mathscr{D}_H, M)$, $\varphi \in C^n(\mathscr{D}_H, M)$, $m \in M$ and $f^0 \otimes \cdots \otimes f^n \in$ $\mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0) \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_0, X_n)$.

(ii) *By restricting to right H-linear morphisms $C^n_H(\mathscr{D}_H, M) = \mathrm{Hom}_H(M \otimes CN_n(\mathscr{D}_H), k)$, we obtain a cocyclic module $C^\bullet_H(\mathscr{D}_H, M) := \{C^n_H(\mathscr{D}_H, M)\}_{n\geq 0}$.*

The cohomology of the cocyclic module $C^\bullet_H(\mathscr{D}_H, M)$ is referred to as the Hopf cyclic cohomology of the H-category $\mathscr{D}_H$ with coefficients in the SAYD module M. The corresponding cohomology groups are denoted by $HC^\bullet_H(\mathscr{D}_H, M)$.

Remark 5. *As k contains $\mathbb{Q}$, we recall that the cohomology of a cocyclic module $\mathscr{C}$ can be expresed alternatively as the cohomology of the following complex (see, for instance [34, 2.5.9]):*

$$C^0_\lambda(\mathscr{C}) \xrightarrow{b} \cdots \xrightarrow{b} C^n_\lambda(\mathscr{C}) \xrightarrow{b} C^{n+1}_\lambda(\mathscr{C}) \xrightarrow{b} \cdots \tag{14}$$

where $C^n_\lambda(\mathscr{C}) = \mathrm{Ker}(1-\lambda) \subseteq C^n(\mathscr{C})$, $b = \sum_{i=0}^{n+1}(-1)^i\delta_i$ and $\lambda = (-1)^n\tau_n$. In particular, an element $\phi \in C^n_H(\mathscr{D}_H, M)$ is a cyclic cocycle if and only if

$$b(\phi) = 0 \quad \text{and} \quad (1-\lambda)(\phi) = 0 \tag{15}$$

In this paper, the cocycles and coboundaries of a cocyclic module will always refer to this complex.

Proposition 6. *Let $\mathscr{D}_H$ be a left H-category and let M be a right-left SAYD module. Then:*

(i) *We obtain a para-cyclic module $C_\bullet(\mathscr{D}_H, M) := \{C_n(\mathscr{D}_H, M) := M \otimes CN_n(\mathscr{D}_H)\}_{n\geq 0}$ with the following structure maps*

$$d_i(m \otimes f^0 \otimes \cdots \otimes f^n) = \begin{cases} m \otimes f^0 \otimes f^1 \otimes \cdots \otimes f^i f^{i+1} \otimes \cdots \otimes f^n & 0 \leq i \leq n-1 \\ m_{(0)} \otimes \big(S^{-1}(m_{(-1)}) f^n\big) f^0 \otimes f^1 \otimes \cdots \otimes f^{n-1} & i = n \end{cases}$$

$$s_i(m \otimes f^0 \otimes \cdots \otimes f^n) = \begin{cases} m \otimes f^0 \otimes f^1 \otimes \ldots f^i \otimes 1_{X_{i+1}} \otimes f^{i+1} \otimes \cdots \otimes f^n & 0 \leq i \leq n-1 \\ m \otimes f^0 \otimes f^1 \otimes \cdots \otimes f^n \otimes 1_{X_0} & i = n \end{cases}$$

$$t_n(m \otimes f^0 \otimes \cdots \otimes f^n) = m_{(0)} \otimes S^{-1}(m_{(-1)}) f^n \otimes f^0 \otimes \cdots \otimes f^{n-1}$$

for any $m \in M$ and $f^0 \otimes f^1 \otimes \cdots \otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0) \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1) \otimes \cdots \otimes$ $\mathrm{Hom}_{\mathscr{D}_H}(X_0, X_n)$.

(ii) *By passing to the tensor product over H, we obtain a cyclic module* $C^H_\bullet(\mathscr{D}_H, M) := \{C^H_n(\mathscr{D}_H, M) = M \otimes_H CN_n(\mathscr{D}_H)\}_{n\geq 0}$.

The cyclic homology groups corresponding to the cyclic module $C^H_\bullet(\mathscr{D}_H, M)$ will be denoted by $HC^H_\bullet(\mathscr{D}_H, M)$.

3. Traces, cocycles and DGH-semicategories

We continue with $\mathscr{D}_H$ being a left H-category and M a right-left SAYD module over H. Our purpose is to develop a formalism analogous to that of Connes [12] in order to interpret the cocycles $Z^\bullet_H(\mathscr{D}_H, M)$ of the complex $C^\bullet_H(\mathscr{D}_H, M)$ and its coboundaries $B^\bullet_H(\mathscr{D}_H, M)$ as characters of differential graded semicategories. In this section, we will describe $Z^\bullet_H(\mathscr{D}_H, M)$, for which we will need the framework of DG-semicategories. Let us first recall the notion of a semicategory introduced by Mitchell in [38] (for more on semicategories, see, for instance, [8]).

Definition 7 (see [38, Section 4]). *A semicategory $\mathscr{C}$ consists of a collection* $\mathrm{Ob}(\mathscr{C})$ *of objects together with a set of morphisms* $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ *for each* $X, Y \in \mathrm{Ob}(\mathscr{C})$ *and an associative composition. A semifunctor* $F : \mathscr{C} \to \mathscr{C}'$ *between semicategories assigns an object* $F(X) \in \mathrm{Ob}(\mathscr{C}')$ *to each* $X \in \mathrm{Ob}(\mathscr{C})$ *and a morphism* $F(f) \in \mathrm{Hom}_{\mathscr{C}'}(F(X), F(Y))$ *to each* $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ *and preserves composition.*

A left H-semicategory is a small k-linear semicategory $\mathscr{S}_H$ such that

(i) $\mathrm{Hom}_{\mathscr{S}_H}(X, Y)$ *is a left H-module for all* $X, Y \in \mathrm{Ob}(\mathscr{S}_H)$

(ii) $h(gf) = (h_1g)(h_2f)$ *for any* $h \in H$, $f \in \mathrm{Hom}_{\mathscr{S}_H}(X, Y)$ *and* $g \in \mathrm{Hom}_{\mathscr{S}_H}(Y, Z)$.

It is clear that any ordinary category may be treated as a semicategory. Conversely, to any k-semicategory $\mathscr{C}$, we can associate an ordinary k-category $\widetilde{\mathscr{C}}$ by setting $\mathrm{Ob}(\widetilde{\mathscr{C}}) = \mathrm{Ob}(\mathscr{C})$ and adjoining unit morphisms as follows:

$$\mathrm{Hom}_{\widetilde{\mathscr{C}}}(X, Y) := \begin{cases} \mathrm{Hom}_{\mathscr{C}}(X, X) \bigoplus k & \text{if } X = Y \\ \mathrm{Hom}_{\mathscr{C}}(X, Y) & \text{if } X \neq Y \end{cases}$$

A morphism in $\mathrm{Hom}_{\widetilde{\mathscr{C}}}(X, Y)$ will be denoted by $\widetilde{f} = f + \mu$, where $f \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $\mu \in k$. It is understood that $\mu = 0$ whenever $X \neq Y$. Any semifunctor $F : \mathscr{C} \to \mathscr{D}$ where $\mathscr{D}$ is an ordinary category may be extended to an ordinary functor $\widetilde{F} : \widetilde{\mathscr{C}} \to \mathscr{D}$. If $\mathscr{S}_H$ is a left H-semicategory, $\widetilde{\mathscr{S}}_H$ has a unique left H-category structure extending that of $\mathscr{S}_H$.

Definition 8. *A differential graded semicategory (DG-semicategory)* $(\mathscr{S}, \widehat{\partial})$ *is a k-linear semicategory $\mathscr{S}$ such that*

(i) $\mathrm{Hom}^\bullet_{\mathscr{S}}(X, Y) = \left(\mathrm{Hom}^n_{\mathscr{S}}(X, Y), \widehat{\partial}^n_{XY}\right)_{n\geq 0}$ *is a cochain complex of k-spaces for each* $X, Y \in \mathrm{Ob}(\mathscr{S})$.

(ii) *the composition map* $\mathrm{Hom}^\bullet_{\mathscr{S}}(Y, Z) \otimes \mathrm{Hom}^\bullet_{\mathscr{S}}(X, Y) \to \mathrm{Hom}^\bullet_{\mathscr{S}}(X, Z)$ *is a morphism of complexes. Equivalently, we have*

$$\widehat{\partial}^n_{XZ}(gf) = \widehat{\partial}^{n-r}_{YZ}(g)f + (-1)^{n-r} g\widehat{\partial}^r_{XY}(f) \tag{16}$$

for any $f \in \mathrm{Hom}_{\mathscr{S}}(X, Y)^r$ *and* $g \in \mathrm{Hom}_{\mathscr{S}}(Y, Z)^{n-r}$.

Whenever the meaning is clear from context, we will drop the subscript and simply write $\widehat{\partial}^\bullet$ *for the differential on any* $\mathrm{Hom}^\bullet_{\mathscr{S}}(X, Y)$.

A small DG-semicategory may be treated as a differential graded (but not necessarily unital) k-algebra with several objects. The DG-semicategories may be treated in a manner similar to DG-categories (see, for instance, [30, 31]). For instance, there is an obvious notion of DG-semifunctor between DG-semicategories. We also note that if $\mathscr{S}$ is a DG-semicategory, the morphisms in degree 0 determine a semicategory $\mathscr{S}^0$.

We now construct a "universal DG-semicategory" associated to a given k-linear semicategory $\mathscr{C}$, similar to the construction of the universal differential graded algebra associated to a (not necessarily unital) k-algebra (see, for instance, [12, p. 315]).

Let $\Omega\mathscr{C}$ be the semicategory with $Ob(\Omega\mathscr{C}) := Ob(\mathscr{C})$ and $Hom_{\Omega\mathscr{C}}(X,Y) = \bigoplus_{n\geq 0} Hom^n_{\Omega\mathscr{C}}(X,Y)$, where

$$Hom^n_{\Omega\mathscr{C}}(X,Y) := \begin{cases} Hom_{\mathscr{C}}(X,Y) & \text{if } n=0 \\ \bigoplus\limits_{\substack{(X_1,\dots,X_n)\\ \in Ob(\mathscr{C})^n}} Hom_{\tilde{\mathscr{C}}}(X_1,Y)\otimes Hom_{\mathscr{C}}(X_2,X_1)\otimes\cdots\otimes Hom_{\mathscr{C}}(X,X_n) & \text{if } n\geq 1 \end{cases} \tag{17}$$

Here the sum runs over the ordered tuples $(X_1,\dots,X_n)\in Ob(\mathscr{C})^n$. In particular, $(\Omega\mathscr{C})^0 = \mathscr{C}$. For $n\geq 1$, an element of the form $\tilde{f}^0\otimes f^1\otimes\cdots\otimes f^n$ in $Hom^n_{\Omega\mathscr{C}}(X,Y)$ will be denoted by $\tilde{f}^0 df^1\dots df^n = (f^0+\mu)df^1\dots df^n$ and said to be homogeneous of degree n. By abuse of notation, we will continue to use $\tilde{f}^0 df^1\dots df^n = (f^0+\mu)df^1\dots df^n$ to denote an element of $Hom^n_{\Omega\mathscr{C}}(X,Y)$ even when $n=0$. In that case, it will be understood that $\mu = 0$.

The composition in $\Omega\mathscr{C}$ is determined by

$$\begin{gathered} f^0\circ df^1\circ\cdots\circ df^n = f^0df^1\dots df^n \qquad (df^0)\circ f^1 = d(f^0f^1) - f^0(df^1) \\ df^1\circ\cdots\circ df^n = df^1\dots df^n \end{gathered} \tag{18}$$

In particular, it follows that

$$\begin{aligned} &((f^0+\mu)df^1\dots df^i)\cdot((g^0+\mu')dg^1\dots dg^j) \\ &= (f^0+\mu)\left(df^1\dots.df^{i-1}d(f^ig^0)dg^1\dots dg^j + \sum_{l=1}^{i-1}(-1)^{i-l}df^1\dots d(f^lf^{l+1})\dots df^idg^0dg^1\dots dg^j\right) \\ &\qquad + (-1)^i(f^0+\mu)f^1df^2\dots df^idg^0dg^1\dots dg^j + \mu'(f^0+\mu)df^1\dots df^idg^1\dots dg^j \end{aligned} \tag{19}$$

For each $X,Y\in Ob(\Omega\mathscr{C})$, the differential $\partial^n_{XY}: Hom^n_{\Omega\mathscr{C}}(X,Y)\to Hom^{n+1}_{\Omega\mathscr{C}}(X,Y)$ is determined by setting

$$\partial^n_{XY}((f^0+\mu)df^1\dots df^n) := df^0df^1\dots df^n$$

It follows from definition that $\partial^{n+1}_{XY}\circ\partial^n_{XY} = 0$. Therefore, $Hom^\bullet_{\Omega\mathscr{C}}(X,Y) := \left(Hom^n_{\Omega\mathscr{C}}(X,Y),\partial^n_{XY}\right)_{n\geq 0}$ is a cochain complex for each $X,Y\in Ob(\Omega\mathscr{C})$. It may also be verified that the composition in $\Omega\mathscr{C}$ is a morphism of complexes. Thus, $\Omega\mathscr{C}$ is a DG-semicategory.

Proposition 9. *Let $\mathscr{C}$ be a small k-linear semicategory. Then, the associated DG-semicategory $(\Omega\mathscr{C},\partial)$ is universal in the following sense: given*

(i) *any DG-semicategory $(\mathscr{S},\hat{\partial})$ and*
(ii) *a k-linear semifunctor $\rho:\mathscr{C}\to\mathscr{S}^0$,*

there exists a unique DG-semifunctor $\hat{\rho}:(\Omega\mathscr{C},\partial)\to(\mathscr{S},\hat{\partial})$ such that the restriction of $\hat{\rho}$ to the semicategory $\mathscr{C}$ is identical to $\rho:\mathscr{C}\to\mathscr{S}^0$.

Proof. We extend ρ to obtain a DG-semifunctor $\hat{\rho}:(\Omega\mathscr{C},\partial)\to(\mathscr{S},\hat{\partial})$ as follows:

$$\begin{gathered} \hat{\rho}(X) := \rho(X) \\ \hat{\rho}((f^0+\mu)df^1\dots df^n) := \rho(f^0)\circ\hat{\partial}^0(\rho(f^1))\circ\cdots\circ\hat{\partial}^0(\rho(f^n)) + \mu\hat{\partial}^0(\rho(f^1))\circ\cdots\circ\hat{\partial}^0(\rho(f^n)) \end{gathered} \tag{20}$$

for all $X\in Ob(\Omega\mathscr{C}) = Ob(\mathscr{C})$ and $(f^0+\mu)df^1\dots df^n\in Hom^n_{\Omega\mathscr{C}}(X,Y)$, $n\geq 1$. Since each $\rho(f^i)$ is a morphism of degree 0 in $\mathscr{S}$, it follows from (16) and (19) that

$$\begin{aligned} &\hat{\rho}(((f^0+\mu)df^1\dots df^n)\circ((f^{n+1}+\mu')df^{n+2}\dots df^m)) \\ &\qquad = \hat{\rho}((f^0+\mu)df^1\dots df^n)\circ\hat{\rho}((f^{n+1}+\mu')df^{n+2}\dots df^m) \end{aligned} \tag{21}$$

It is also clear by construction that $\widehat{\rho}|_{\mathscr{C}} = \rho$. Moreover, we have

$$\begin{aligned}\widehat{\partial}^n\left(\widehat{\rho}((f^0+\mu)\mathrm{d}f^1\ldots\mathrm{d}f^n)\right) &= \widehat{\partial}^n\left(\rho(f^0)\widehat{\partial}^0(\rho(f^1))\ldots\widehat{\partial}^0(\rho(f^n))\right) + \mu\widehat{\partial}^n\left(\widehat{\partial}^0(\rho(f^1))\ldots\widehat{\partial}^0(\rho(f^n))\right)\\ &= \widehat{\partial}^0(\rho(f^0))\widehat{\partial}^0(\rho(f^1))\ldots\widehat{\partial}^0(\rho(f^n)) + \rho(f^0)\widehat{\partial}^n\left(\widehat{\partial}^0(\rho(f^1))\ldots\widehat{\partial}^0(\rho(f^n))\right)\\ &= \widehat{\partial}^0(\rho(f^0))\widehat{\partial}^0(\rho(f^1))\ldots\widehat{\partial}^0(\rho(f^n)) = \widehat{\rho}\left(\partial^n((f^0+\mu)\mathrm{d}f^1\ldots\mathrm{d}f^n)\right)\end{aligned}$$

The uniqueness of $\widehat{\rho}$ is also clear from (18) and (19). □

Definition 10. *A left DGH-semicategory is a left H-semicategory $\mathscr{S}_H$ equipped with a DG-semicategory $(\mathscr{S}_H, \widehat{\partial}_H)$ structure such that for all $n \geq 0$:*

(i) *$\mathrm{Hom}^n_{\mathscr{S}_H}(X,Y)$ is a left H-module for $X, Y \in \mathrm{Ob}(\mathscr{S}_H)$.*

(ii) *$\widehat{\partial}^n_H : \mathrm{Hom}^n_{\mathscr{S}_H}(X,Y) \to \mathrm{Hom}^{n+1}_{\mathscr{S}_H}(X,Y)$ is H-linear for $X, Y \in \mathrm{Ob}(\mathscr{S}_H)$.*

We can similarly define the notion of a DGH-semifunctor between DGH-semicategories. If $(\mathscr{S}_H, \widehat{\partial}_H)$ is a left DGH-semicategory, we note that $\mathscr{S}^0_H$ is a left H-semicategory.

Proposition 11. *Let $\mathscr{D}_H$ be a left H-semicategory. Then, the universal DG-semicategory $(\Omega(\mathscr{D}_H), \partial_H)$ associated to $\mathscr{D}_H$ is a left DGH-semicategory with the H-action determined by*

$$h\left((f^0+\mu)\mathrm{d}f^1\ldots\mathrm{d}f^n\right) := (h_1f^0 + \mu\varepsilon(h_1))d(h_2f^1)\ldots d(h_{n+1}f^n) \tag{22}$$

for all $h \in H$ and $(f^0+\mu)\mathrm{d}f^1\ldots\mathrm{d}f^n \in \mathrm{Hom}_{\Omega(\mathscr{D}_H)}(X,Y)$.

Proof. This is immediate from the definitions in (19) and (22). □

Definition 12. *Let $(\mathscr{S}_H, \widehat{\partial}_H)$ be a left DGH-semicategory and M be a right-left SAYD module over H. A closed graded (H,M)-trace of dimension n on $\mathscr{S}_H$ is a collection of k-linear maps*

$$\widehat{\mathscr{T}}^H := \{\widehat{\mathscr{T}}^H_X : M \otimes \mathrm{Hom}^n_{\mathscr{S}_H}(X,X) \longrightarrow k\}_{X\in\mathrm{Ob}(\mathscr{S}_H)}$$

such that

$$\widehat{\mathscr{T}}^H_X\left(mh_1 \otimes S(h_2)f\right) = \varepsilon(h)\widehat{\mathscr{T}}^H_X(m\otimes f) \tag{23}$$

$$\widehat{\mathscr{T}}^H_X\left(m\otimes\widehat{\partial}^{n-1}_H(f')\right) = 0 \tag{24}$$

$$\widehat{\mathscr{T}}^H_X\left(m\otimes g'g\right) = (-1)^{ij}\ \widehat{\mathscr{T}}^H_Y\left(m_{(0)}\otimes\left(S^{-1}(m_{(-1)})g\right)g'\right) \tag{25}$$

for all $h \in H$, $m \in M$, $f \in \mathrm{Hom}^n_{\mathscr{S}_H}(X,X)$, $f' \in \mathrm{Hom}^{n-1}_{\mathscr{S}_H}(X,X)$, $g \in \mathrm{Hom}^i_{\mathscr{S}_H}(X,Y)$, $g' \in \mathrm{Hom}^j_{\mathscr{S}_H}(Y,X)$ and $i+j=n$.

Definition 13. *An n-dimensional $\mathscr{S}_H$-cycle with coefficients in a SAYD module M is a tuple $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$ such that*

(i) *$(\mathscr{S}_H, \widehat{\partial}_H)$ is a left DGH-semicategory.*

(ii) *$\widehat{\mathscr{T}}^H$ is a closed graded (H,M)-trace of dimension n on $\mathscr{S}_H$.*

Let $\mathscr{D}_H$ be a left H-category. By an n-dimensional cycle over $\mathscr{D}_H$, we mean a tuple $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H, \rho)$ such that

(a) *$(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$ is an n-dimensional $\mathscr{S}_H$-cycle with coefficients in a SAYD module M.*

(b) *$\rho : \mathscr{D}_H \to \mathscr{S}^0_H$ is an H-linear semifunctor.*

We fix a left H-category $\mathscr{D}_H$. Given an n-dimensional cycle $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H, \rho)$ over $\mathscr{D}_H$, we define its character $\phi \in C^n_H(\mathscr{D}_H, M)$ by setting

$$\phi : M\otimes CN_n(\mathscr{D}_H) \longrightarrow k \qquad \phi(m\otimes f^0\otimes\cdots\otimes f^n) := \widehat{\mathscr{T}}^H_{X_0}\left(m\otimes\rho(f^0)\widehat{\partial}^0_H\left(\rho(f^1)\right)\ldots\widehat{\partial}^0_H\left(\rho(f^n)\right)\right)$$

for $m \in M$ and $f^0\otimes\cdots\otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1,X_0)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2,X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_0,X_n)$. We will often suppress the semifunctor ρ and refer to ϕ simply as the character of the n-dimensional cycle $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$.

We now have a characterization of the space $Z^n_H(\mathscr{D}_H, M)$ of n-cocycles in the Hopf cyclic cohomology of the category $\mathscr{D}_H$ with coefficients in the SAYD module M.

Theorem 14. *Let $\mathscr{D}_H$ be a left H-category and M be a right-left SAYD module over H. Let $\phi \in C^n_H(\mathscr{D}_H, M)$. Then, the following conditions are equivalent:*

(i) *ϕ is the character of an n-dimensional cycle over $\mathscr{D}_H$, i.e., there is an n-dimensional cycle $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$ with coefficients in M and an H-linear semifunctor $\rho : \mathscr{D}_H \to \mathscr{S}^0_H$ such that*

$$\begin{aligned}\phi(m\otimes f^0\otimes\cdots\otimes f^n) &= \widehat{\mathscr{T}}^H_{X_0}((1_M\otimes\widehat{\rho})(m\otimes f^0\mathrm{d}f^1\dots\mathrm{d}f^n))\\ &= \widehat{\mathscr{T}}^H_{X_0}\big(m\otimes\rho(f^0)\widehat{\partial}^0_H\big(\rho(f^1)\big)\dots\widehat{\partial}^0_H\big(\rho(f^n)\big)\big)\end{aligned}\tag{26}$$

for any $m \in M$ and $f^0\otimes\cdots\otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_0, X_n)$.

(ii) *There exists a closed graded (H, M)-trace $\mathscr{T}^H$ of dimension n on $(\Omega(\mathscr{D}_H), \partial_H)$ such that*

$$\phi(m\otimes f^0\otimes\cdots\otimes f^n) = \mathscr{T}^H_{X_0}(m\otimes f^0\mathrm{d}f^1\dots\mathrm{d}f^n)\tag{27}$$

for any $m \in M$ and $f^0\otimes\cdots\otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_0, X_n)$.

(iii) *$\phi \in Z^n_H(\mathscr{D}_H, M)$.*

Proof. **(i)⇒(ii).** By the universal property of $\Omega(\mathscr{D}_H)$, the H-linear semifunctor $\rho : \mathscr{D}_H \to \mathscr{S}^0_H$ can be extended to a DGH-semifunctor $\widehat{\rho} : \Omega(\mathscr{D}_H) \to \mathscr{S}_H$ as in (20). We define a collection $\mathscr{T}^H := \{\mathscr{T}^H_X : M\otimes\mathrm{Hom}^n_{\Omega(\mathscr{D}_H)}(X, X) \to k\}_{X\in\mathrm{Ob}(\Omega(\mathscr{D}_H))}$ of k-linear maps given by

$$\mathscr{T}^H_X(m\otimes(f^0+\mu)\mathrm{d}f^1\dots\mathrm{d}f^n) := \widehat{\mathscr{T}}^H_X\big(m\otimes\widehat{\rho}((f^0+\mu)\mathrm{d}f^1\dots\mathrm{d}f^n)\big)\tag{28}$$

for any $m \in M$ and $f^0\otimes\cdots\otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X, X_n)$. In particular, it follows from (28) that

$$\phi(m\otimes f^0\otimes\cdots\otimes f^n) = \widehat{\mathscr{T}}^H_X\big(m\otimes\rho(f^0)\widehat{\partial}^0_H\big(\rho(f^1)\big)\dots\widehat{\partial}^0_H\big(\rho(f^n)\big)\big) = \mathscr{T}^H_X(m\otimes f^0\mathrm{d}f^1\dots\mathrm{d}f^n)\tag{29}$$

It may be verified that the collection $\mathscr{T}^H$ is an n-dimensional closed graded (H, M)-trace on $\Omega(\mathscr{D}_H)$.

(ii)⇒(i). Suppose that we have a closed graded (H, M)-trace $\mathscr{T}^H$ of dimension n on $\Omega(\mathscr{D}_H)$ satisfying (27). Then, the tuple $(\Omega(\mathscr{D}_H), \partial_H, M, \mathscr{T}^H)$ forms an n-dimensional cycle over $\mathscr{D}_H$ with coefficients in M. Further, by observing that $\partial^0_H(f) = \mathrm{d}f$ for any $f \in \mathrm{Hom}_{\mathscr{D}_H}(X, Y)$, we get (26).

(i)⇒(iii). Let $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$ be an n-dimensional cycle over $\mathscr{D}_H$ with coefficients in M and $\rho : \mathscr{D}_H \to \mathscr{S}^0_H$ be an H-linear semifunctor satisfying

$$\phi(m\otimes f^0\otimes\cdots\otimes f^n) = \widehat{\mathscr{T}}^H_{X_0}\big(m\otimes\rho(f^0)\widehat{\partial}^0_H\big(\rho(f^1)\big)\dots\widehat{\partial}^0_H\big(\rho(f^n)\big)\big)$$

for any $m \in M$ and $f^0\otimes\cdots\otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_0, X_n)$. For simplicity of notation, we will drop the functor ρ. To show that ϕ is an n-cocycle, it suffices to check that (see (15))

$$b(\phi) = 0 \quad\text{and}\quad (1-\lambda)(\phi) = 0$$

where $b = \sum_{i=0}^{n+1}(-1)^i\delta_i$ and $\lambda = (-1)^n\tau_n$. For any $p^0\otimes\cdots\otimes p^{n+1} \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_0, X_{n+1})$, we have

$$\begin{aligned}&\sum_{i=0}^{n+1}(-1)^i\delta_i(\phi)(m\otimes p^0\otimes\cdots\otimes p^{n+1})\\ &= \sum_{i=0}^{n}(-1)^i\phi(m\otimes p^0\otimes\cdots\otimes p^ip^{i+1}\otimes\cdots\otimes p^{n+1})\\ &\quad + (-1)^{n+1}\phi\big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})p^{n+1}\big)p^0\otimes p^1\otimes\cdots\otimes p^n\big)\\ &= \widehat{\mathscr{T}}^H_{X_0}\big(m\otimes p^0p^1\widehat{\partial}^0_H(p^2)\dots\widehat{\partial}^0_H(p^{n+1})\big) + \sum_{i=1}^{n}(-1)^i\widehat{\mathscr{T}}^H_{X_0}\big(m\otimes p^0\widehat{\partial}^0_H(p^1)\dots\widehat{\partial}^0_H(p^ip^{i+1})\dots\widehat{\partial}^0_H(p^{n+1})\big)\\ &\quad + (-1)^{n+1}\widehat{\mathscr{T}}^H_{X_{n+1}}\big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})p^{n+1}\big)p^0\widehat{\partial}^0_H(p^1)\cdots\otimes\widehat{\partial}^0_H(p^n)\big)\end{aligned}$$

Now using the equality $\widehat{\partial}^0_H(fg)=\widehat{\partial}^0_H(f)g+f\widehat{\partial}^0_H(g)$ for any f and g of degree 0, we have

$$\big(p^0\widehat{\partial}^0_H(p^1)\dots\widehat{\partial}^0_H(p^n)\big)p^{n+1} = \sum_{i=1}^{n}(-1)^{n-i}p^0\widehat{\partial}^0_H(p^1)\dots\widehat{\partial}^0_H(p^ip^{i+1})\dots\widehat{\partial}^0_H(p^{n+1})+(-1)^np^0p^1\widehat{\partial}^0_H(p^2)\dots\widehat{\partial}^0_H(p^{n+1})$$

Thus, using the condition in (25), we obtain

$$\begin{aligned}\sum_{i=0}^{n+1}&(-1)^i\delta_i(\phi)(m\otimes p^0\otimes\cdots\otimes p^{n+1})\\ &=(-1)^n\widehat{\mathscr{T}}^H_{X_0}\big(m\otimes\big(p^0\widehat{\partial}^0_H(p^1)\dots\widehat{\partial}^0_H(p^n)\big)p^{n+1}\big)\\ &\quad+(-1)^{n+1}\widehat{\mathscr{T}}^H_{X_{n+1}}\big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})p^{n+1}\big)p^0\widehat{\partial}^0_H(p^1)\dots\widehat{\partial}^0_H(p^n)\big)=0\end{aligned}$$

Next, using (24), (25), and the H-linearity of $\widehat{\partial}_H$, we have

$$\begin{aligned}&\big(\big(1-(-1)^n\tau_n\big)\phi\big)(m\otimes f^0\otimes\cdots\otimes f^n)\\ &=\phi(m\otimes f^0\otimes\cdots\otimes f^n)-(-1)^n\phi\big(m_{(0)}\otimes S^{-1}(m_{(-1)})f^n\otimes f^0\otimes\cdots\otimes f^{n-1}\big)\\ &=\widehat{\mathscr{T}}^H_{X_0}(m\otimes f^0\widehat{\partial}^0_H(f^1)\dots\widehat{\partial}^0_H(f^n))-(-1)^n\widehat{\mathscr{T}}^H_{X_n}\big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})f^n\big)\widehat{\partial}^0_H(f^0)\widehat{\partial}^0_H(f^1)\dots\widehat{\partial}^0_H(f^{n-1})\big)\\ &=(-1)^{n-1}\widehat{\mathscr{T}}^H_{X_n}\big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})\widehat{\partial}^0_H(f^n)\big)f^0\widehat{\partial}^0_H(f^1)\dots\widehat{\partial}^0_H(f^{n-1})\big)\\ &\quad+(-1)^{n-1}\widehat{\mathscr{T}}^H_{X_n}\big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})f^n\big)\widehat{\partial}^0_H(f^0)\widehat{\partial}^0_H(f^1)\dots\widehat{\partial}^0_H(f^{n-1})\big)\\ &=(-1)^{n-1}\widehat{\mathscr{T}}^H_{X_n}\big(m_{(0)}\otimes\widehat{\partial}^{n-1}_H\big((S^{-1}(m_{(-1)})f^n)f^0\widehat{\partial}^0_H(f^1)\dots\widehat{\partial}^0_H(f^{n-1})\big)\big)=0\end{aligned}$$

(iii)⇒(ii). Let $\phi\in Z^n_H(\mathscr{D}_H,M)$. For each $X\in Ob(\Omega(\mathscr{D}_H))$, we define an H-linear map $\mathscr{T}^H_X : M\otimes \mathrm{Hom}^n_{\Omega(\mathscr{D}_H)}(X,X)\to k$ given by

$$\mathscr{T}^H_X(m\otimes(f^0+\mu)\mathrm{d}f^1\dots\mathrm{d}f^n):=\phi(m\otimes f^0\otimes\cdots\otimes f^n)$$

for $f^0\otimes\cdots\otimes f^n\in \mathrm{Hom}_{\mathscr{D}_H}(X_1,X)\otimes\mathrm{Hom}_{\mathscr{D}_H}(X_2,X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{D}_H}(X,X_n)$. We now verify that the collection $\{\mathscr{T}^n_X : M\otimes\mathrm{Hom}^n_{\Omega(\mathscr{D}_H)}(X,X)\to k\}_{X\in Ob(\Omega(\mathscr{D}_H))}$ is a closed graded (H,M)-trace on $(\Omega(\mathscr{D}_H),\partial_H)$. For any $(p^0+\mu)\mathrm{d}p^1\dots\mathrm{d}p^{n-1}\in\mathrm{Hom}^{n-1}_{\Omega(\mathscr{D}_H)}(X,X)$, we have

$$\begin{aligned}\mathscr{T}^H_X\big(m\otimes\partial^{n-1}_H((p^0+\mu)\mathrm{d}p^1\dots\mathrm{d}p^{n-1})\big)&=\mathscr{T}^H_X\big(m\otimes 1\mathrm{d}p^0\mathrm{d}p^1\dots\mathrm{d}p^{n-1}\big)\\ &=\phi(m\otimes 0\otimes p^0\otimes\cdots\otimes p^{n-1})=0\end{aligned}$$

This proves the condition in (24). Using (13), it is also clear that $\{\mathscr{T}^n_X : M\otimes\mathrm{Hom}^n_{\Omega(\mathscr{D}_H)}(X,X)\to k\}_{X\in Ob(\Omega(\mathscr{D}_H))}$ satisfies condition (23). Finally, for any $g'=(g^0+\mu')\mathrm{d}g^1\dots\mathrm{d}g^r\in\mathrm{Hom}^r_{\Omega(\mathscr{D}_H)}(Y,X)$ and $g=(g^{r+1}+\mu)\mathrm{d}g^{r+2}\dots\mathrm{d}g^{n+1}\in\mathrm{Hom}^{n-r}_{\Omega(\mathscr{D}_H)}(X,Y)$, we have

$$\begin{aligned}&\mathscr{T}^H_X\big(m\otimes g'g\big)\\ &=\sum_{j=1}^{r}(-1)^{r-j}\mathscr{T}^H_X\big(m\otimes(g^0+\mu')\mathrm{d}g^1\dots\mathrm{d}(g^jg^{j+1})\dots\mathrm{d}g^{n+1}\big)\\ &\quad+(-1)^r\mathscr{T}^H_X\big(m\otimes(g^0+\mu')g^1\mathrm{d}g^2\dots\mathrm{d}g^{n+1}\big)+\mathscr{T}^H_X\big(m\otimes\mu(g^0+\mu')\mathrm{d}g^1\dots\mathrm{d}g^r\mathrm{d}g^{r+2}\dots\mathrm{d}g^{n+1}\big)\\ &=\sum_{j=1}^{r}(-1)^{r-j}\phi(m\otimes g^0\otimes\cdots\otimes g^jg^{j+1}\otimes\cdots\otimes g^{n+1})+(-1)^r\,\phi(m\otimes g^0g^1\otimes g^2\otimes\cdots\otimes g^{n+1})\\ &\quad+(-1)^r\,\mu'\phi(m\otimes g^1\otimes g^2\otimes\cdots\otimes g^{n+1})+\mu\phi(m\otimes g^0\otimes g^1\otimes\cdots\otimes g^r\otimes g^{r+2}\otimes\cdots\otimes g^{n+1})\\ &=\sum_{j=0}^{r}(-1)^{r+j}\phi(m\otimes g^0\otimes\cdots\otimes g^jg^{j+1}\otimes\cdots\otimes g^{n+1})+(-1)^r\,\mu'\phi(m\otimes g^1\otimes g^2\otimes\cdots\otimes g^{n+1})\\ &\quad+\mu\phi(m\otimes g^0\otimes g^1\otimes\cdots\otimes g^r\otimes g^{r+2}\otimes\cdots\otimes g^{n+1})\end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
&(-1)^{r(n-r)}\mathscr{T}_Y^H\Big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})g\big)g'\Big)\\
&=(-1)^{r(n-r)}\mathscr{T}_Y^H\Big(m_{(0)}\otimes\Big(\big[S^{-1}((m_{(-1)})_{n-r+1})(g^{r+1}+\mu)\big]\big[d(S^{-1}((m_{(-1)})_{n-r})g^{r+2})\big]\ldots\\
&\qquad\qquad\big[d(S^{-1}((m_{(-1)})_1)g^{n+1})\big]\Big)\circ\big((g^0+\mu')\mathrm{d}g^1\ldots\mathrm{d}g^r\big)\Big)\\
&=(-1)^{r(n-r)}\sum_{j=r+2}^{n}(-1)^{n-j+1}\mathscr{T}_Y^H\Big(m_{(0)}\otimes\big[S^{-1}((m_{(-1)})_{n-r})(g^{r+1}+\mu)\big]\ldots\\
&\qquad\qquad d\big[S^{-1}((m_{(-1)})_{n-j+1})(g^jg^{j+1})\big]\ldots\mathrm{d}g^r\Big)\\
&\quad+(-1)^{r(n-r)}\mathscr{T}_Y^H\Big(m_{(0)}\otimes\big[S^{-1}((m_{(-1)})_{n-r+1})(g^{r+1}+\mu)\big]\ldots d\big[\big(S^{-1}((m_{(-1)})_1)g^{n+1}\big)g^0\big]\ldots\mathrm{d}g^r\Big)\\
&\quad+(-1)^{r(n-r)}(-1)^{n-r}\mathscr{T}_Y^H\Big(m_{(0)}\otimes\big(\big[S^{-1}((m_{(-1)})_{n-r})((g^{r+1}+\mu)g^{r+2})\big]\big)\ldots\\
&\qquad\qquad\big[d\big(S^{-1}((m_{(-1)})_1)g^{n+1}\big)\big]\big(\mathrm{d}g^0\mathrm{d}g^1\ldots\mathrm{d}g^r\big)\Big)\\
&\quad+(-1)^{r(n-r)}\mu'\mathscr{T}_Y^H\Big(m_{(0)}\otimes.\big[S^{-1}((m_{(-1)})_{n-r+1})(g^{r+1}+\mu)\big]\big[d\big(S^{-1}((m_{(-1)})_{n-r})g^{r+2}\big)\big]\ldots\\
&\qquad\qquad\big[d\big(S^{-1}((m_{(-1)})_1)g^{n+1}\big)\big].\mathrm{d}g^1\ldots\mathrm{d}g^r\Big)\\
&=(-1)^{r(n-r)}\sum_{j=r+2}^{n}(-1)^{n-j+1}\phi\Big(m_{(0)}\otimes S^{-1}((m_{(-1)})_{n-r})g^{r+1}\otimes\ldots\\
&\qquad\qquad\otimes S^{-1}((m_{(-1)})_{n-j+1})(g^jg^{j+1})\otimes\cdots\otimes g^r\Big)\\
&\quad+(-1)^{r(n-r)}\phi\Big(m_{(0)}\otimes S^{-1}((m_{(-1)})_{n-r+1})g^{r+1}\otimes\cdots\otimes\big(S^{-1}((m_{(-1)})_1)g^{n+1}\big)g^0\otimes\cdots\otimes g^r\Big)\\
&\quad+(-1)^{r(n-r)}(-1)^{n-r}\phi\Big(m_{(0)}\otimes S^{-1}((m_{(-1)})_{n-r})(g^{r+1}g^{r+2})\otimes\ldots\\
&\qquad\qquad\otimes(S^{-1}((m_{(-1)})_1)g^{n+1})\otimes g^0\otimes g^1\otimes\cdots\otimes g^r\Big)\\
&\quad+(-1)^{r(n-r)}(-1)^{n-r}\mu\phi\Big(m_{(0)}\otimes S^{-1}((m_{(-1)})_{n-r})g^{r+2}\otimes\ldots\\
&\qquad\qquad\otimes(S^{-1}((m_{(-1)})_1)g^{n+1})\otimes g^0\otimes g^1\otimes\cdots\otimes g^r\Big)\\
&\quad+(-1)^{r(n-r)}\mu'\phi\Big(m_{(0)}\otimes.S^{-1}((m_{(-1)})_{n-r+1})g^{r+1}\otimes S^{-1}((m_{(-1)})_{n-r})g^{r+2}\otimes\ldots\\
&\qquad\qquad\otimes(S^{-1}((m_{(-1)})_1)g^{n+1}\otimes.g^1\otimes\cdots\otimes g^r\Big)
\end{aligned}
$$

Using repeatedly the fact that $\phi=(-1)^n\tau_n\phi$, we get

$$
\begin{aligned}
&(-1)^{r(n-r)}\mathscr{T}_Y^n\Big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})g\big)g'\Big)\\
&\quad=-\sum_{j=r+1}^{n}(-1)^{r+j}\phi\big(m\otimes g^0\otimes\cdots\otimes g^jg^{j+1}\otimes\cdots\otimes g^{n+1}\big)\\
&\qquad-(-1)^{n+r+1}\phi\Big(m_{(0)}\otimes\big(S^{-1}(m_{(-1)})g^{n+1}\big)g^0\otimes g^1\otimes\cdots\otimes g^n\Big)\\
&\qquad+(-1)^r\,\mu'\phi\big(m\otimes g^1\otimes g^2\otimes\cdots\otimes g^{n+1}\big)+\mu\phi\big(m\otimes g^0\otimes g^1\otimes\cdots\otimes g^r\otimes g^{r+2}\otimes\cdots\otimes g^{n+1}\big)
\end{aligned}
$$

The condition (25) now follows using the fact that $b(\phi)=0$. This proves the result. □

Remark 15. From the statement and proof of Theorem 14, it is clear that there is a one to one correspondence between n-dimensional closed graded (H,M)-traces on $\Omega(\mathscr{D}_H)$ and $Z^n_H(\mathscr{D}_H,M)$.

4. Linearization by matrices and Hopf cyclic cohomology

We continue with $\mathscr{D}_H$ being a left H-category. In the previous section, we described the spaces $Z_H^\bullet(\mathscr{D}_H, M)$. The next aim is to find a characterization of $B_H^\bullet(\mathscr{D}_H, M)$ which will be done in several steps. Let $M_r(k)$ denote the algebra of $r\times r$-matrices with entries in the field k. The linearization $\mathscr{D}_H\otimes M_r(k)$ of $\mathscr{D}_H$ by the algebra $M_r(k)$ is the k-linear category defined as follows:

$$\mathrm{Ob}(\mathscr{D}_H\otimes M_r(k)) := \mathrm{Ob}(\mathscr{D}_H) \qquad \mathrm{Hom}_{\mathscr{D}_H\otimes M_r(k)}(X,Y) := \mathrm{Hom}_{\mathscr{D}_H}(X,Y)\otimes_k M_r(k) \tag{30}$$

for any $X, Y\in \mathrm{Ob}(\mathscr{D}_H\otimes M_r(k))$. The composition in $\mathscr{D}_H\otimes M_r(k)$ is determined by setting

$$(f'\otimes A')\circ(f\otimes A) := (f'\circ f)\otimes A'A \qquad \forall\, f\in \mathrm{Hom}_{\mathscr{D}_H}(X,Y), f'\in \mathrm{Hom}_{\mathscr{D}_H}(Y,Z), A, A'\in M_r(k) \tag{31}$$

for any $X, Y, Z\in \mathrm{Ob}(\mathscr{D}_H\otimes M_r(k)) = \mathrm{Ob}(\mathscr{D}_H)$. We observe that $\mathscr{D}_H\otimes M_r(k)$ is also a left H-category with left H-module structure on any $\mathrm{Hom}_{\mathscr{D}_H\otimes M_r(k)}(X,Y)$ determined by setting

$$h(f\otimes A) := hf\otimes A \qquad \forall\, h\in H, f\in \mathrm{Hom}_{\mathscr{D}_H}(X,Y), A\in M_r(k) \tag{32}$$

We denote by $\overline{\mathrm{Cat}}_H$ the category whose objects are left H-categories and whose morphisms are H-linear semifunctors.

We denote by Vect_k the category of all k-vector spaces and by H-Mod the category of all left H-modules. Let $\mathrm{Hom}_H(\cdot,k) : H\text{-Mod}\to \mathrm{Vect}_k$ be the functor that takes $N\mapsto \mathrm{Hom}_H(N,k)$.

We now make some conventions. If $\mathscr{P}_\bullet = \{\mathscr{P}_n\}_{n\geq 0}$ is a simplicial module with face maps $\{d_i : \mathscr{P}_n\to\mathscr{P}_{n-1}\}_{0\leq i\leq n}$ (see, for instance, [34, Section 1.6.1]) we let $\mathscr{P}_\bullet^{hoc}$ denote the associated Hochschild complex whose terms and differentials are given as follows

$$\mathscr{P}_n^{hoc} := \mathscr{P}_n \qquad b := \sum_{i=0}^{n}(-1)^i d_i : \mathscr{P}_n \longrightarrow \mathscr{P}_{n-1} \tag{33}$$

Further, if $\mathscr{Q}^\bullet = \{\mathscr{Q}^n\}_{n\geq 0}$ is a cocyclic module with coface maps $\{\delta_i : \mathscr{Q}^{n-1}\to\mathscr{Q}^n\}_{0\leq i\leq n}$, codegeneracy maps $\{\sigma_i : \mathscr{Q}^{n+1}\to\mathscr{Q}^n\}_{0\leq i\leq n}$ and cocyclic operators $\{\tau_n : \mathscr{Q}^n\to\mathscr{Q}^n\}_{n\geq 0}$, we let $\mathscr{Q}_{cy}^{\bullet\bullet}$ denote the bicomplex with terms and differentials given as follows

$$\mathscr{Q}_{cy}^{m,n} := \mathscr{Q}^{m-n} \quad b := \sum_{i=0}^{l+1}(-1)^i\delta_i : \mathscr{Q}^{m,n}\longrightarrow \mathscr{Q}^{m+1,n}$$
$$B := \left(\sum_{i=0}^{l-1}(-1)^{i(l-1)}\tau_{l-1}^i\right)\sigma_l\tau_l(1-(-1)^l\tau_l) : \mathscr{Q}^{m,n}\longrightarrow \mathscr{Q}^{m,n+1}$$

where we have set $l := m-n$. Then, $\mathscr{Q}_{cy}^{\bullet\bullet}$ is a bicomplex whose total cohomology computes the cyclic cohomology of $\mathscr{Q}^\bullet$ (see, for instance, [34, Section 2.5]). Additionally, we let $\mathscr{Q}_{hoc}^\bullet$ denote the complex with differentials $b := \sum_{i=0}^{n+1}(-1)^i\delta_i : \mathscr{Q}^n\to\mathscr{Q}^{n+1}$ computing the Hochschild cohomology of the cosimplicial module underlying $\mathscr{Q}^\bullet$. The cohomology groups of this complex will be denoted by $H^\bullet(\mathscr{Q}_{hoc}^\bullet)$.

We now fix $r\geq 1$. For $1\leq i,j\leq r$ and $\alpha\in k$, we let $E_{ij}(\alpha)$ denote the elementary matrix in $M_r(k)$ having α at (i,j)-th position and 0 everywhere else. We will often use E_{ij} for $E_{ij}(1)$. For each $1\leq p\leq r$, we have an inclusion $\mathrm{inc}_p : \mathscr{D}_H\to\mathscr{D}_H\otimes M_r(k)$ in $\overline{\mathrm{Cat}}_H$ which fixes the objects and $\mathrm{inc}_p(f) = f\otimes E_{pp} = f\otimes E_{pp}(1)$ for any morphism $f\in\mathscr{D}_H$.

For any right-left SAYD module M, the inclusion $\mathrm{inc}_p : \mathscr{D}_H\to\mathscr{D}_H\otimes M_r(k)$ induces an inclusion map

$$\begin{aligned}(\mathrm{inc}_p, M) : M\otimes CN_n(\mathscr{D}_H) &\longrightarrow M\otimes CN_n(\mathscr{D}_H\otimes M_r(k))\\ m\otimes f^0\otimes\cdots\otimes f^n &\longmapsto m\otimes (f^0\otimes E_{pp})\otimes\cdots\otimes(f^n\otimes E_{pp})\end{aligned} \tag{34}$$

If we consider the para-cyclic modules $C_\bullet(\mathscr{D}_H, M)$ and $C_\bullet(\mathscr{D}_H\otimes M_r(k), M)$ as in the notation of Proposition 6, we see that the morphisms in (34) induce a morphism $C_\bullet(\mathrm{inc}_p, M) : C_\bullet(\mathscr{D}_H, M)\to$

$C_\bullet(\mathscr{D}_H \otimes M_r(k), M)$ of para-cyclic modules. Accordingly, we have an induced morphism of Hochschild complexes

$$C_\bullet(\mathrm{inc}_p, M)^{hoc} : C_\bullet(\mathscr{D}_H, M)^{hoc} \longrightarrow C_\bullet(\mathscr{D}_H \otimes M_r(k), M)^{hoc} \tag{35}$$

Applying the functor $\mathrm{Hom}_H(\cdot, k)$ and considering the cocyclic modules $C^\bullet_H(\mathscr{D}_H, M)$ and $C^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)$ as in the notation of Proposition 4, we obtain morphisms of Hochschild cohomology complexes

$$C^\bullet_H(\mathrm{inc}_p, M)_{hoc} : C^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)_{hoc} \longrightarrow C^\bullet_H(\mathscr{D}_H, M)_{hoc} \tag{36}$$

Similarly, we obtain a morphism of bicomplexes computing cyclic cohomology

$$C^{\bullet\bullet}_H(\mathrm{inc}_p, M)_{cy} : C^{\bullet\bullet}_H(\mathscr{D}_H \otimes M_r(k), M)_{cy} \longrightarrow C^{\bullet\bullet}_H(\mathscr{D}_H, M)_{cy} \tag{37}$$

For each $n \geq 0$, there is an H-linear trace map $\mathrm{tr}^M : M \otimes CN_n(\mathscr{D}_H \otimes M_r(k)) \to M \otimes CN_n(\mathscr{D}_H)$ given by

$$\mathrm{tr}^M\left(m \otimes (f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n)\right) := (m \otimes f^0 \otimes \cdots \otimes f^n)\,\mathrm{trace}(B^0 \dots B^n) \tag{38}$$

for any $m \in M$ and $(f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n) \in CN_n(\mathscr{D}_H \otimes M_r(k))$. It may be verified easily that the trace map as in (38) defines a morphism $C_\bullet(\mathrm{tr}^M) : C_\bullet(\mathscr{D}_H \otimes M_r(k), M) \to C_\bullet(\mathscr{D}_H, M)$ of para-cyclic modules. In particular, we have an induced morphism between underlying Hochschild complexes

$$C_\bullet(\mathrm{tr}^M)^{hoc} : C_\bullet(\mathscr{D}_H \otimes M_r(k), M)^{hoc} \longrightarrow C_\bullet(\mathscr{D}_H, M)^{hoc} \tag{39}$$

Applying the functor $\mathrm{Hom}_H(\cdot, k)$, we see that we have a morphism $C^\bullet_H(\mathrm{tr}^M)_{hoc} : C^\bullet_H(\mathscr{D}_H, M)_{hoc} \to C^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)_{hoc}$ of complexes computing Hochschild cohomologies.

Proposition 16. *The maps $C_\bullet(\mathrm{inc}_1, M)^{hoc}$ and $C_\bullet(\mathrm{tr}^M)^{hoc}$ are homotopy inverses of each other.*

Proof. It may be easily verified that $C_\bullet(\mathrm{tr}^M)^{hoc} \circ C_\bullet(\mathrm{inc}_1, M)^{hoc} = 1$. To show that $C_\bullet(\mathrm{inc}_1, M)^{hoc} \circ C_\bullet(\mathrm{tr}^M)^{hoc} \sim 1$, we define k-linear maps $\{\hbar_i : C_n(\mathscr{D}_H \otimes M_r(k), M) \to C_{n+1}(\mathscr{D}_H \otimes M_r(k), M)\}_{0\leq i\leq n}$ by setting:

$$\begin{aligned}
&\hbar_i\left(m \otimes (f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n)\right)\\
&\qquad := m \otimes \sum_{1\leq j,k,l,\dots,p,q\leq r} (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \dots\\
&\qquad\qquad \otimes (f^i \otimes E_{11}(B^i_{pq})) \otimes (1_{X_{i+1}} \otimes E_{1q}(1)) \otimes (f^{i+1} \otimes B^{i+1}) \otimes \cdots \otimes (f^n \otimes B^n)
\end{aligned}$$

for $0 \leq i < n$ and

$$\begin{aligned}
&\hbar_n\left(m \otimes (f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n)\right)\\
&:= m \otimes \sum_{1\leq j,k,m,\dots,p,q\leq r} (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{km})) \otimes \cdots \otimes (f^n \otimes E_{11}(B^n_{pq})) \otimes (1_{X_0} \otimes E_{1q}(1))
\end{aligned}$$

We now verify that $\hbar^n := \sum_{i=0}^n (-1)^i \hbar_i$ is a pre-simplicial homotopy (see, for instance, [34, Section 1.0.8]) between $C_\bullet(\mathrm{inc}_1, M)^{hoc} \circ C_\bullet(\mathrm{tr}^M)^{hoc}$ and $1_{C_\bullet(\mathscr{D}_H \otimes M_r(k), M)^{hoc}}$. For this, we need to verify the following identities:

$$\begin{aligned}
d_i\hbar_{i'} &= \hbar_{i'-1}d_i && \text{for } i < i'\\
d_i\hbar_i &= d_i\hbar_{i-1} && \text{for } 0 < i \leq n\\
d_i\hbar_{i'} &= \hbar_{i'}d_{i-1} && \text{for } i > i'+1\\
d_0\hbar_0 &= 1_{C_\bullet(\mathscr{D}_H \otimes M_r(k), M)^{hoc}} && \text{and } d_{n+1}\hbar_n = C_\bullet(\mathrm{inc}_1, M)^{hoc} \circ C_\bullet(\mathrm{tr}^M)^{hoc}
\end{aligned} \tag{40}$$

where $d_i : C_{n+1}(\mathscr{D}_H \otimes M_r(k), M) \to C_n(\mathscr{D}_H \otimes M_r(k), M)$, $0 \le i \le n+1$ are the face maps. We only verify the last one in (40) because the others follow similarly. Using the fact that $E_{1q}(1)E_{j1}(B_{jk}) = 0$ unless $q = j$, we have

$$
\begin{aligned}
&d_{n+1}\hbar_n\big(m \otimes (f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n)\big)\\
&= d_{n+1}\Big(m \otimes \sum_{1\le j,k,l,\dots,p,q\le r} (f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \cdots \otimes (f^n \otimes E_{11}(B^n_{pq})) \otimes (1_{X_0} \otimes E_{1q}(1))\Big)\\
&= m_{(0)} \otimes \sum_{1\le j,k,l,\dots,p,q\le r} \big(S^{-1}(m_{(-1)})(1_{X_0} \otimes E_{1q}(1))\big)(f^0 \otimes E_{j1}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \dots\\
&\quad \cdots \otimes (f^n \otimes E_{11}(B^n_{pq}))\\
&= m \otimes \sum_{1\le j,k,l,\dots,p,q\le r} \Big(f^0 \otimes E_{1q}(1)E_{j1}(B^0_{jk})\Big) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \cdots \otimes (f^n \otimes E_{11}(B^n_{pq}))\\
&= m \otimes \sum_{1\le j,k,l,\dots,p\le r} \Big(f^0 \otimes E_{1j}(1)E_{j1}(B^0_{jk})\Big) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \cdots \otimes (f^n \otimes E_{11}(B^n_{pj}))\\
&= m \otimes \sum_{1\le j,k,l,\dots,p\le r} (f^0 \otimes E_{11}(B^0_{jk})) \otimes (f^1 \otimes E_{11}(B^1_{kl})) \otimes \cdots \otimes (f^n \otimes E_{11}(B^n_{pj}))\\
&= \big(m \otimes (f^0 \otimes E_{11}) \otimes \cdots \otimes (f^n \otimes E_{11})\big) \sum_{1\le j,k,l,\dots,p\le r} (B^0_{jk}B^1_{kl}\dots B^n_{pj})\\
&= \big(m \otimes (f^0 \otimes E_{11}) \otimes \cdots \otimes (f^n \otimes E_{11})\big) \sum_{1\le j\le r} (B^0B^1\dots B^n)_{jj}\\
&= \big(m \otimes (f^0 \otimes E_{11}) \otimes \cdots \otimes (f^n \otimes E_{11})\big)\,\mathrm{trace}(B^0B^1\dots B^n)\\
&= \Big(C_\bullet(\mathrm{inc}_1, M)^{hoc} \circ C_\bullet(\mathrm{tr}^M)^{hoc}\Big)\big(m \otimes (f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n)\big)
\end{aligned}
$$

This proves the result. □

Proposition 17. *Let $\mathscr{D}_H$ be a left H-category and M be a right-left SAYD module. Then,*

(i) *The morphisms*

$$
\begin{aligned}
H^\bullet(C^\bullet_H(\mathrm{inc}_1, M)_{hoc}) &: H^\bullet(C^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)_{hoc}) \longrightarrow H^\bullet(C^\bullet_H(\mathscr{D}_H, M)_{hoc})\\
H^\bullet(C^\bullet_H(\mathrm{tr}^M)_{hoc}) &: H^\bullet(C^\bullet_H(\mathscr{D}_H, M)_{hoc}) \longrightarrow H^\bullet(C^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)_{hoc})
\end{aligned}
$$

induced by $C^\bullet_H(\mathrm{inc}_1, M)_{hoc}$ and $C^\bullet_H(\mathrm{tr}^M)_{hoc}$ are mutually inverse isomorphisms of Hochschild cohomologies.

(ii) *We have isomorphisms*

$$
HC^\bullet_H(\mathscr{D}_H, M) \quad \underset{HC^\bullet_H(\mathrm{inc}_1, M)}{\overset{HC^\bullet_H(\mathrm{tr}^M)}{\rightleftarrows}} \quad HC^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)
$$

Proof.

(i). By Proposition 16, we know that $C_\bullet(\mathrm{tr}^M)^{hoc} \circ C_\bullet(\mathrm{inc}_1, M)^{hoc} = 1_{C_\bullet(\mathscr{D}_H, M)^{hoc}}$ and $C_\bullet(\mathrm{inc}_1, M)^{hoc} \circ C_\bullet(\mathrm{tr}^M)^{hoc} \sim 1_{C_\bullet(\mathscr{D}_H \otimes M_r(k), M)^{hoc}}$. Thus, applying the functor $\mathrm{Hom}_H(\cdot, k)$, we obtain

$$
C^\bullet_H(\mathrm{inc}_1, M)_{hoc} \circ C^\bullet_H(\mathrm{tr}^M)_{hoc} = 1_{C^\bullet_H(\mathscr{D}_H, M)_{hoc}} \quad C^\bullet_H(\mathrm{tr}^M)_{hoc} \circ C^\bullet_H(\mathrm{inc}_1, M)_{hoc} \sim 1_{C^\bullet_H(\mathscr{D}_H \otimes M_r(k), M)_{hoc}}
$$

Therefore, $C^\bullet_H(\mathrm{inc}_1, M)_{hoc}$ and $C^\bullet_H(\mathrm{tr}^M)_{hoc}$ are homotopy inverses of each other.

(ii). This follows immediately from (i) and the Hochschild to cyclic spectral sequence. □

Corollary 18. *For an n-cocycle $\phi \in Z^n_H(\mathscr{D}_H, M)$, the n-cocycle $\widetilde{\phi} = \mathrm{Hom}_H(\mathrm{tr}^M, k)(\phi) = \phi \circ \mathrm{tr}^M \in Z^n_H(\mathscr{D}_H \otimes M_r(k), M)$ may be described as follows*

$$
\widetilde{\phi}\big(m \otimes (f^0 \otimes B^0) \otimes \cdots \otimes (f^n \otimes B^n)\big) = \phi(m \otimes f^0 \otimes \cdots \otimes f^n)\,\mathrm{trace}(B^0 \dots B^n)
$$

5. Vanishing cycles on an H-category and coboundaries

From now onwards, we will always assume that $k=\mathbb{C}$. In this section, we will describe the spaces $B^\bullet_H(\mathscr{D}_H, M)$. We will then use the formalism of categorified cycles and vanishing cycles developed in this paper to obtain a product on Hopf cyclic cohomologies of H-categories. We begin by recalling the notion of an inner automorphism of a category.

Definition 19 ([43, p. 24]). *Let $\mathscr{D}_H$ be a left H-category. An automorphism $\Phi \in \mathrm{Hom}_{\mathrm{Cat}_H}(\mathscr{D}_H, \mathscr{D}_H)$ is said to be inner if Φ is isomorphic to the identity functor $1_{\mathscr{D}_H}$. In particular, there exist isomorphisms $\{\eta(X) : X \to \Phi(X)\}_{X\in \mathrm{Ob}(\mathscr{D}_H)}$ such that $\Phi(f) = \eta(Y)\circ f \circ (\eta(X))^{-1}$ for any $f \in \mathrm{Hom}_{\mathscr{D}_H}(X,Y)$.*

We now set

$$\mathbb{G}(\mathscr{D}_H) := \prod_{X\in \mathrm{Ob}(\mathscr{D}_H)} \mathrm{Aut}_{\mathscr{D}_H}(X) \tag{41}$$

By definition, an element $\eta \in \mathbb{G}(\mathscr{D}_H)$ corresponds to a family of automorphisms $\{\eta(X) : X \to X\}_{X\in \mathrm{Ob}(\mathscr{D}_H)}$. We now set

$$\mathbb{U}_H(\mathscr{D}_H) := \left\{\eta \in \mathbb{G}(\mathscr{D}_H) \,\middle|\, h(\eta(X)) = \varepsilon(h)\eta(X) \text{ for every } h\in H \text{ and } X \in \mathrm{Ob}(\mathscr{D}_H)\right\} \tag{42}$$

Lemma 20. *$\mathbb{U}_H(\mathscr{D}_H)$ is a subgroup of $\mathbb{G}(\mathscr{D}_H)$.*

Proof. The element $\mathbf{e} = \prod_{X\in \mathrm{Ob}(\mathscr{D}_H)} 1_X$ is the identity of the group $\mathbb{G}(\mathscr{D}_H)$. By definition of an H-category, we know that $h\cdot 1_X = \varepsilon(h)\cdot 1_X$ for each $X\in \mathrm{Ob}(\mathscr{D}_H)$ and $h\in H$. Thus, $\mathbf{e}\in \mathbb{U}_H(\mathscr{D}_H)$. Now, suppose that $\eta,\eta' \in \mathbb{U}_H(\mathscr{D}_H)$. Then, for each $X\in \mathrm{Ob}(\mathscr{D}_H)$ and $h \in H$,

$$\begin{aligned} h\big((\eta\circ\eta')(X)\big) = h(\eta(X)\circ \eta'(X)) &= (h_1\eta(X))\circ(h_2\eta'(X)) \\ &= (\varepsilon(h_1)\eta(X))\circ(\varepsilon(h_2)\eta'(X)) = \varepsilon(h)(\eta(X)\circ\eta'(X)) \end{aligned}$$

Hence, $\eta\circ\eta' \in \mathbb{U}_H(\mathscr{D}_H)$. Also, $\eta^{-1}\in \mathbb{G}(\mathscr{D}_H)$ corresponds to a family of morphisms $\{\eta^{-1}(X) := \eta(X)^{-1} : X \to X\}_{X\in \mathrm{Ob}(\mathscr{D}_H)}$. Then, for each $h\in H$ and $X \in \mathrm{Ob}(\mathscr{D}_H)$,

$$\varepsilon(h)1_X = h(\eta(X)\circ\eta^{-1}(X)) = (\varepsilon(h_1)\eta(X))\circ(h_2\eta^{-1}(X)) = \eta(X)\circ(h\eta^{-1}(X))$$

which gives $\varepsilon(h)\eta^{-1}(X) = h\eta^{-1}(X)$. Therefore, $\eta^{-1}\in \mathbb{U}_H(\mathscr{D}_H)$. □

Lemma 21. *Let $\mathscr{D}_H$ be a left H-category and let $\eta \in \mathbb{U}_H(\mathscr{D}_H)$.*

(i) *Consider $\Phi_\eta : \mathscr{D}_H \to \mathscr{D}_H$ defined by*

$$\Phi_\eta(X) = X \qquad \Phi_\eta(f) := \eta(Y)\circ f\circ \eta(X)^{-1}$$

for every $X \in \mathrm{Ob}(\mathscr{D}_H)$ and $f\in \mathrm{Hom}_{\mathscr{D}_H}(X,Y)$. Then, $\Phi_\eta : \mathscr{D}_H \to \mathscr{D}_H$ is an inner automorphism of $\mathscr{D}_H$.

(ii) *Consider $\widetilde{\Phi}_\eta : \mathscr{D}_H\otimes M_2(k) \to \mathscr{D}_H \otimes M_2(k)$ defined by*

$$\widetilde{\Phi}_\eta(X) = X \qquad \widetilde{\Phi}_\eta(f\otimes B) = (1_Y\otimes E_{11} + \eta(Y)\otimes E_{22})\circ(f\otimes B)\circ(1_X\otimes E_{11} + \eta(X)^{-1}\otimes E_{22})$$

for every $X \in \mathrm{Ob}(\mathscr{D}_H\otimes M_2(k)) = \mathrm{Ob}(\mathscr{D}_H)$ and $f\otimes B \in \mathrm{Hom}_{\mathscr{D}_H\otimes M_2(k)}(X,Y)$. Then, $\widetilde{\Phi}_\eta$: $\mathscr{D}_H\otimes M_2(k) \to \mathscr{D}_H\otimes M_2(k)$ is an inner automorphism.

Proof.

(i). Using the fact that $\eta,\eta^{-1} \in \mathbb{U}_H(\mathscr{D}_H)$, we have

$$\begin{aligned} h(\Phi_\eta(f)) = (h_1\eta(Y))\circ(h_2 f)\circ(h_3\eta(X)^{-1}) &= (\varepsilon(h_1)\eta(Y))\circ(h_2 f)\circ(\varepsilon(h_3)\eta(X)^{-1}) \\ &= \eta(Y)\circ(h_1 f)\circ(\varepsilon(h_2)\eta(X)^{-1}) \\ &= \eta(Y)\circ(hf)\circ\eta(X)^{-1} \end{aligned}$$

for any $h\in H$ and $f \in \mathrm{Hom}_{\mathscr{D}_H}(X,Y)$. By Definition 19, we now see that Φ_η is an inner automorphism.

(ii). Setting $\widetilde{\eta}(X): X \to X$ in $\mathscr{D}_H \otimes M_2(k)$ as $\widetilde{\eta}(X) = 1_X \otimes E_{11} + \eta(X) \otimes E_{22}$, we see that

$$\begin{aligned}\widetilde{\Phi}_\eta(f \otimes B) &= (1_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ (f \otimes B) \circ (1_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) \\ &= \widetilde{\eta}(Y) \circ (f \otimes B) \circ \widetilde{\eta}(X)^{-1}\end{aligned}$$

for any $f \otimes B \in \mathrm{Hom}_{\mathscr{D}_H \otimes M_2(k)}(X, Y)$. Considering the H-action on the category $\mathscr{D}_H \otimes M_2(k)$, we have

$$\begin{aligned}h\big(\widetilde{\Phi}_\eta((f \otimes B)\big) &= h_1(1_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ h_3(1_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) \\ &= (h_1 1_Y \otimes E_{11} + h_1\eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ (h_3 1_X \otimes E_{11} + h_3\eta(X)^{-1} \otimes E_{22}) \\ &= \varepsilon(h_1)(1_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h_2(f \otimes B) \circ \varepsilon(h_3)(1_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) \\ &= (1_Y \otimes E_{11} + \eta(Y) \otimes E_{22}) \circ h(f \otimes B) \circ (1_X \otimes E_{11} + \eta(X)^{-1} \otimes E_{22}) \\ &= \widetilde{\Phi}_\eta(h(f \otimes B))\end{aligned}$$

for any $h \in H$ and $f \otimes B \in \mathrm{Hom}_{\mathscr{D}_H \otimes M_2(k)}(X, Y)$. By Definition 19, we now see that $\Phi_{\widetilde{\eta}}$ is an inner automorphism. □

For any $\eta \in \mathbb{U}_H(\mathscr{D}_H)$, we will always denote by Φ_η and $\widetilde{\Phi}_\eta$ the inner automorphisms defined in Lemma 21.

Lemma 22. *Let M be a right-left SAYD module over H. Then,*

(i) *A semifunctor $\alpha \in \mathrm{Hom}_{\overline{\mathrm{Cat}}_H}(\mathscr{D}_H, \mathscr{D}'_H)$ induces a morphism (for all $n \geq 0$)*

$$C^n_H(\alpha, M) : C^n_H(\mathscr{D}'_H, M) = \mathrm{Hom}_H(M \otimes CN_n(\mathscr{D}'_H), k) \longrightarrow C^n_H(\mathscr{D}_H, M) = \mathrm{Hom}_H(M \otimes CN_n(\mathscr{D}_H), k)$$

determined by

$$C^n_H(\alpha, M)(\phi)(m \otimes f^0 \otimes \cdots \otimes f^n) = \phi\big(m \otimes \alpha(f^0) \otimes \cdots \otimes \alpha(f^n)\big)$$

for any $\phi \in C^n_H(\mathscr{D}'_H, M)$, $m \in M$ and $f^0 \otimes \cdots \otimes f^n \in CN_n(\mathscr{D}_H)$. This leads to a morphism $C^{\bullet\bullet}_H(\alpha, M)^{cy} : C^{\bullet\bullet}_H(\mathscr{D}'_H, M)^{cy} \to C^{\bullet\bullet}_H(\mathscr{D}_H, M)^{cy}$ of double complexes and induces a functor $HC^\bullet_H(\cdot, M) : \overline{\mathrm{Cat}}^{op}_H \to \mathrm{Vect}_k$.

(ii) *Let $\eta \in \mathbb{U}_H(\mathscr{D}_H)$. Then, Φ_η induces the identity map on $HC^\bullet_H(\mathscr{D}_H, M)$.*

Proof.

(i). Since ϕ and α are H-linear, the morphisms $C^n_H(\alpha, M)$ are well-defined and well behaved with respect to the maps appearing in the Hochschild and cyclic complexes. The result follows.

(ii). Let $\eta \in \mathbb{U}_H(\mathscr{D}_H)$ and $\Phi_\eta \in \mathrm{Hom}_{\mathrm{Cat}_H}(\mathscr{D}_H, \mathscr{D}_H)$ be the corresponding inner automorphism. By Proposition 17, the maps $HC^\bullet_H(\mathrm{inc}_1, M)$ and $HC^\bullet_H(\mathrm{tr}^M)$ are mutually inverse isomorphisms of Hopf cyclic cohomology groups. Thus, we have

$$HC^\bullet_H(\mathrm{inc}_2, M) \circ \big(HC^\bullet_H(\mathrm{inc}_1, M)\big)^{-1} = HC^\bullet_H(\mathrm{inc}_2, M) \circ HC^\bullet_H(\mathrm{tr}^M) = HC^\bullet_H\big(\mathrm{tr}^M \circ (\mathrm{inc}_2, M)\big) = 1 \quad (43)$$

Further, we have the following commutative diagram in the category $\overline{\mathrm{Cat}}_H$:

$$\begin{array}{ccccc} \mathscr{D}_H & \xrightarrow{\mathrm{inc}_1} & \mathscr{D}_H \otimes M_2(k) & \xleftarrow{\mathrm{inc}_2} & \mathscr{D}_H \\ {\scriptstyle 1_{\mathscr{D}_H}}\downarrow & & \downarrow{\scriptstyle \tilde{\Phi}_\eta} & & \downarrow{\scriptstyle \Phi_\eta V} \\ \mathscr{D}_H & \xrightarrow{\mathrm{inc}_1} & \mathscr{D}_H \otimes M_2(k) & \xleftarrow{\mathrm{inc}_2} & \mathscr{D}_H \end{array} \quad (44)$$

Thus, by applying the functor $HC^\bullet_H(\cdot, M)$ to the commutative diagram (44) and using (43), we obtain

$$\begin{aligned}&HC^\bullet_H(\Phi_\eta, M) \\ &= \big(HC^\bullet_H(\mathrm{inc}_2, M)\big) \circ HC^\bullet_H(\mathrm{inc}_1, M)^{-1} \circ HC^\bullet_H(1_{\mathscr{D}_H}, M) \circ \big(HC^\bullet_H(\mathrm{inc}_1, M)\big) \circ HC^\bullet_H(\mathrm{inc}_2, M)^{-1} \\ &= 1_{HC^\bullet_H(\mathscr{D}_H, M)}\end{aligned}$$

□

Proposition 23. *Let $\mathscr{D}_H$ be a left H-category. Suppose that there is a semifunctor $v \in \mathrm{Hom}_{\overline{\mathrm{Cat}}_H}(\mathscr{D}_H, \mathscr{D}_H)$ and an $\eta \in \mathbb{U}_H(\mathscr{D}_H \otimes M_2(k))$ such that*

(i) $v(X) = X \qquad \forall\ X \in \mathrm{Ob}(\mathscr{D}_H)$

(ii) $\Phi_\eta(f \otimes E_{11} + v(f) \otimes E_{22}) = v(f) \otimes E_{22}$

for all $f \in \mathrm{Hom}_{\mathscr{D}_H}(X, Y)$ and $X, Y \in \mathrm{Ob}(\mathscr{D}_H)$. Then, $HC^\bullet_H(\mathscr{D}_H, M) = 0$.

Proof. Let $\alpha, \alpha' \in \mathrm{Hom}_{\overline{\mathrm{Cat}}_H}(\mathscr{D}_H, \mathscr{D}_H \otimes M_2(k))$ be the semifunctors defined by

$$\alpha(X) := X \qquad \alpha(f) := f \otimes E_{11} + v(f) \otimes E_{22}$$
$$\alpha'(X) := X \qquad \alpha'(f) := v(f) \otimes E_{22}$$

for all $X \in \mathrm{Ob}(\mathscr{D}_H)$ and $f \in \mathrm{Hom}_{\mathscr{D}_H}(X, Y)$. Then, by assumption, $\alpha' = \Phi_\eta \circ \alpha$. Therefore, applying the functor $HC^\bullet_H(\cdot, M)$ and using Lemma 22 (2), we get

$$\begin{aligned} HC^\bullet_H(\alpha', M) &= HC^\bullet_H(\alpha, M) \circ HC^\bullet_H(\Phi_\eta, M) \\ &= HC^\bullet_H(\alpha, M) : HC^\bullet_H(\mathscr{D}_H \otimes M_2(k), M) \longrightarrow HC^\bullet_H(\mathscr{D}_H, M) \end{aligned} \tag{45}$$

Let $\phi \in Z^n_H(\mathscr{D}_H, M)$ and $\widetilde{\phi} = \mathrm{Hom}_H(\mathrm{tr}^M, k)(\phi) = \phi \circ \mathrm{tr}^M \in Z^n_H(\mathscr{D}_H \otimes M_2(k), M)$ as in Corollary 18. Let $[\widetilde{\phi}]$ denote the cohomology class of $\widetilde{\phi}$. Then, by (45), we have $HC^\bullet_H(\alpha, M)([\widetilde{\phi}]) = HC^\bullet_H(\alpha', M)([\widetilde{\phi}])$, i.e.,

$$\widetilde{\phi} \circ (1_M \otimes CN_n(\alpha)) + B^n_H(\mathscr{D}_H, M) = \widetilde{\phi} \circ (1_M \otimes CN_n(\alpha')) + B^n_H(\mathscr{D}_H, M)$$

so that $\widetilde{\phi} \circ (1_M \otimes CN_n(\alpha)) - \widetilde{\phi} \circ (1_M \otimes CN_n(\alpha')) \in B^n_H(\mathscr{D}_H, M)$. Applying the definition of $\widetilde{\phi}$, we now have

$$\begin{aligned} (\widetilde{\phi} \circ (1_M \otimes CN_n(\alpha)))(m \otimes f^0 \otimes \cdots \otimes f^n) \\ &= \widetilde{\phi}\big(m \otimes \alpha(f^0) \otimes \cdots \otimes \alpha(f^n)\big) \\ &= \widetilde{\phi}\big(m \otimes (f^0 \otimes E_{11} + v(f^0) \otimes E_{22}) \otimes \cdots \otimes (f^n \otimes E_{11} + v(f^n) \otimes E_{22})\big) \\ &= \phi(m \otimes f^0 \otimes \cdots \otimes f^n) + \phi(m \otimes v(f^0) \otimes \cdots \otimes v(f^n)) \end{aligned}$$

Similarly, $(\widetilde{\phi} \circ (1_M \otimes CN_n(\alpha')))(m \otimes f^0 \otimes \cdots \otimes f^n) = \phi(m \otimes v(f^0) \otimes \cdots \otimes v(f^n))$. Thus, $\phi = \widetilde{\phi} \circ (1_M \otimes CN_n(\alpha)) - \widetilde{\phi} \circ (1_M \otimes CN_n(\alpha')) \in B^n_H(\mathscr{D}_H, M)$. This proves the result. □

Definition 24. *Let $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$ be an n-dimensional $\mathscr{S}_H$-cycle with coefficients in a SAYD module M over H (see, Definition 13). Then, we say that the cycle $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H)$ is vanishing if $\mathscr{S}^0_H$ is a left H-category and $\mathscr{S}^0_H$ satisfies the assumptions in Proposition 23.*

We now recall from Connes [12, p. 103] the algebra $\mathbf{C}$ of infinite matrices $(a_{ij})_{i,j\in\mathbb{N}}$ with entries from $\mathbb{C}$ satisfying the following conditions (see also Karoubi–Villamayor [25])

(i) the set $\{a_{ij} \mid i, j \in \mathbb{N}\}$ is finite,

(ii) the number of non-zero entries in a row or a column is bounded.

Explicitly, if $A = (a_{ij})_{i,j\in\mathbb{N}}$ and $A' = (a'_{ij})_{i,j\in\mathbb{N}}$ are elements of $\mathbf{C}$, their product $B := AA'$ in $\mathbf{C}$ is the matrix $B = (b_{ij})_{i,j\in\mathbb{N}}$ whose entries are given by

$$b_{ij} := \sum_{k\in\mathbb{N}} a_{ik} a'_{kj} \tag{46}$$

for any $i, j \in \mathbb{N}$. The unit $\mathbf{1} \in \mathbf{C}$ is given by the infinite matrix whose diagonal entries are all 1, with zero entries everywhere else.

Identifying $M_2(\mathbf{C}) = \mathbf{C} \otimes M_2(\mathbb{C})$, we now recall the following result from [12, p. 104]:

Lemma 25. *There exists an algebra homomorphism $\omega : \mathbf{C} \to \mathbf{C}$ and an invertible element $\widetilde{U} \in M_2(\mathbf{C})$ such that the corresponding inner automorphism $\Xi : M_2(\mathbf{C}) \to M_2(\mathbf{C})$ satisfies*

$$\Xi(B \otimes E_{11} + \omega(B) \otimes E_{22}) = \omega(B) \otimes E_{22} \qquad \forall\ B \in \mathbf{C} \tag{47}$$

Then, $HC^\bullet(\mathbf{C}) = 0$.

Remark 26. We note that the condition in (47) ensures that $\omega(\mathbf{1}) \neq \mathbf{1}$, where $\mathbf{1}$ is the unit element of $\mathbf{C}$.

For any k-algebra $\mathscr{A}$, we may define a k-linear category $\mathscr{A} \otimes \mathscr{D}_H$ by setting $Ob(\mathscr{A} \otimes \mathscr{D}_H) = Ob(\mathscr{D}_H)$ and $Hom_{\mathscr{A} \otimes \mathscr{D}_H}(X, Y) = \mathscr{A} \otimes Hom_{\mathscr{D}_H}(X, Y)$. The category $\mathscr{A} \otimes \mathscr{D}_H$ is a left H-category via the action $h(a \otimes f) := a \otimes hf$ for any $h \in H$, $a \otimes f \in \mathscr{A} \otimes Hom_{\mathscr{D}_H}(X, Y)$.

Lemma 27. *We have $HC_H^\bullet(\mathbf{C} \otimes \mathscr{D}_H, M) = 0$.*

Proof. We will verify that the category $\mathbf{C} \otimes \mathscr{D}_H$ satisfies the assumptions of Proposition 23. Let ω and $\widetilde{U}$ be as in Lemma 25. We now define $v : \mathbf{C} \otimes \mathscr{D}_H \to \mathbf{C} \otimes \mathscr{D}_H$ given by

$$v(X) := X \qquad v(B \otimes f) := \omega(B) \otimes f$$

for any $X \in Ob(\mathbf{C} \otimes \mathscr{D}_H)$ and $B \otimes f \in Hom_{\mathbf{C} \otimes \mathscr{D}_H}(X, Y)$. Since $\omega : \mathbf{C} \to \mathbf{C}$ is an algebra homomorphism, it follows that v is a semifunctor. By the definition of the H-action on $\mathbf{C} \otimes \mathscr{D}_H$, it is also clear that v is H-linear.

Using the identification $\mathbf{C} \otimes \mathscr{D}_H \otimes M_2(\mathbb{C}) = M_2(\mathbf{C}) \otimes \mathscr{D}_H$, we now define an element $\eta \in \mathbb{G}(\mathbf{C} \otimes \mathscr{D}_H \otimes M_2(\mathbb{C})) = \mathbb{G}(M_2(\mathbf{C}) \otimes \mathscr{D}_H)$ given by the family of morphims

$$\{\eta(X) := \widetilde{U} \otimes 1_X \in Hom_{M_2(\mathbf{C}) \otimes \mathscr{D}_H}(X, X) = M_2(\mathbf{C}) \otimes Hom_{\mathscr{D}_H}(X, X)\}_{X \in Ob(\mathscr{D}_H)} \tag{48}$$

Since $\widetilde{U}$ is a unit in $M_2(\mathbf{C})$, it follows that each $\eta(X)$ in (48) is an automorphism. Since H acts trivially on $M_2(\mathbf{C})$, we see that $\eta \in \mathbb{U}_H(\mathbf{C} \otimes \mathscr{D}_H \otimes M_2(\mathbb{C}))$. Moreover, for any $\widetilde{B} \otimes f \in Hom_{M_2(\mathbf{C}) \otimes \mathscr{D}_H}(X, Y) = M_2(\mathbf{C}) \otimes Hom_{\mathscr{D}_H}(X, Y)$, we have

$$\Phi_\eta(\widetilde{B} \otimes f) = \eta(Y) \circ (\widetilde{B} \otimes f) \circ \eta(X)^{-1} = (\widetilde{U} \otimes 1_Y) \circ (\widetilde{B} \otimes f) \circ (\widetilde{U}^{-1} \otimes 1_X) = \widetilde{U}\widetilde{B}\widetilde{U}^{-1} \otimes f = \Xi(\widetilde{B}) \otimes f$$

Therefore, for any $B \otimes f \in \mathbf{C} \otimes Hom_{\mathscr{D}_H}(X, Y)$, we have

$$\begin{aligned}
\Phi_\eta((B \otimes f) \otimes E_{11} + v(B \otimes f) \otimes E_{22}) &= \Phi_\eta(B \otimes f \otimes E_{11} + \omega(B) \otimes f \otimes E_{22}) \\
&= \Phi_\eta(B \otimes E_{11} \otimes f + \omega(B) \otimes E_{22} \otimes f) \\
&= \Xi(B \otimes E_{11} + \omega(B) \otimes E_{22}) \otimes f \\
&= \omega(B) \otimes E_{22} \otimes f = v(B \otimes f) \otimes E_{22}
\end{aligned}$$

This proves the result. □

We are now ready to describe elements in the space $B_H^n(\mathscr{D}_H, M)$.

Theorem 28. *An element $\phi \in C_H^n(\mathscr{D}_H, M)$ is a coboundary iff ϕ is the character of an n-dimensional vanishing $\mathscr{S}_H$-cycle $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H, \rho)$ over $\mathscr{D}_H$.*

Proof. Let ϕ be the character of an n-dimensional vanishing $\mathscr{S}_H$-cycle $(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H, \rho)$. By definition, $\widehat{\mathscr{T}}^H$ is an n-dimensional closed graded (H, M)-trace on the H-semicategory $\mathscr{S}_H$ and that $\mathscr{S}_H^0$ is an ordinary H-category. We now define $\psi \in C_H^n(\mathscr{S}_H^0, M)$ by setting

$$\psi(m \otimes g^0 \otimes \cdots \otimes g^n) := \widehat{\mathscr{T}}_{X_0}^H\big(m \otimes g^0 \widehat{\partial}_H^0(g^1) \ldots \widehat{\partial}_H^0(g^n)\big)$$

for $m \in M$ and $g^0 \otimes \cdots \otimes g^n \in Hom_{\mathscr{S}_H^0}(X_1, X_0) \otimes Hom_{\mathscr{S}_H^0}(X_2, X_1) \otimes \cdots \otimes Hom_{\mathscr{S}_H^0}(X_0, X_n)$. Then, by the implication (1) ⇒ (3) in Theorem 14, we have that $\psi \in Z_H^n(\mathscr{S}_H^0, M)$. Since $HC_H^n(\mathscr{S}_H^0, M) = 0$, we have that $\psi = b\psi'$ for some $\psi' \in C_H^{n-1}(\mathscr{S}_H^0, M)$.

By Lemma 22, the semifunctor $\rho \in \mathrm{Hom}_{\overline{\mathrm{Cat}_H}}(\mathscr{D}_H, \mathscr{S}^0_H)$ induces a map $C^{n-1}_H(\rho, M)$: $C^{n-1}_H(\mathscr{S}^0_H, M) \to C^{n-1}_H(\mathscr{D}_H, M)$. Setting $\psi'' := C^{n-1}_H(\rho, M)(\psi')$, we have

$$\left(\psi''\right)(m \otimes p^0 \otimes \cdots \otimes p^{n-1}) = \psi'\left(m \otimes \rho(p^0) \otimes \cdots \otimes \rho(p^{n-1})\right)$$

for any $m \in M$ and $p^0 \otimes \cdots \otimes p^{n-1} \in CN_{n-1}(\mathscr{D}_H)$. Therefore,

$$\begin{aligned}\phi(m \otimes f^0 \otimes \cdots \otimes f^n) &= \widehat{\mathscr{T}}^H_{X_0}\left(m \otimes \rho(f^0)\hat{\partial}^0_H\left(\rho(f^1)\right)\dots\hat{\partial}^0_H\left(\rho(f^n)\right)\right) = \psi\left(m \otimes \rho(f^0) \otimes \cdots \otimes \rho(f^n)\right)\\ &= (b\psi')\left(m \otimes \rho(f^0) \otimes \cdots \otimes \rho(f^n)\right) = (b\psi'')(m \otimes f^0 \otimes \cdots \otimes f^n)\end{aligned}$$

for any $m \in M$ and $f^0 \otimes \cdots \otimes f^n \in \mathrm{Hom}_{\mathscr{D}_H}(X_1, X_0) \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_2, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{D}_H}(X_0, X_n)$. Thus, $\phi \in B^n_H(\mathscr{D}_H, M)$.

Conversely, suppose that $\phi \in B^n_H(\mathscr{D}_H, M)$. Then, $\phi = b\psi$ for some $\psi \in C^{n-1}_H(\mathscr{D}_H, M)$. We now extend ψ to get an element $\psi' \in C^{n-1}_H(\mathbf{C} \otimes \mathscr{D}_H, M)$ as follows:

$$\psi'\left(m \otimes (B^0 \otimes f^0) \otimes \cdots \otimes (B^{n-1} \otimes f^{n-1})\right) = \psi(m \otimes B^0_{11} f^0 \otimes \cdots \otimes B^{n-1}_{11} f^{n-1})$$

We now set $\phi' = b\psi' \in Z^n_H(\mathbf{C} \otimes \mathscr{D}_H, M)$. We now consider the H-linear semifunctor $\rho : \mathscr{D}_H \to \mathbf{C} \otimes \mathscr{D}_H$ which fixes objects and takes any morphism f to $\mathbf{1} \otimes f$. Then, we have

$$\begin{aligned}\left(C^n_H(\rho, M)(\phi')\right)(m \otimes f^0 \otimes \cdots \otimes f^n) &= \phi'\left(m \otimes \rho(f^0) \otimes \cdots \otimes \rho(f^n)\right) = (b\psi')\left(m \otimes \rho(f^0) \otimes \cdots \otimes \rho(f^n)\right)\\ &= (b\psi)(m \otimes f^0 \otimes \cdots \otimes f^n) = \phi(m \otimes f^0 \otimes \cdots \otimes f^n)\end{aligned}$$

Since $\phi' \in Z^n_H(\mathbf{C} \otimes \mathscr{D}_H, M)$, the implication (iii) $\Rightarrow$ (ii) in Theorem 14 gives us a closed graded (H, M)-trace $\mathscr{T}^H$ of dimension n on the DGH-semicategory $(\Omega(\mathbf{C} \otimes \mathscr{D}_H), \partial_H)$ such that

$$\mathscr{T}^H_{X_0}\left(m \otimes \rho(f^0)\partial^0_H\left(\rho(f^1)\right)\dots\partial^0_H\left(\rho(f^n)\right)\right) = \phi'\left(m \otimes \rho(f^0) \otimes \cdots \otimes \rho(f^n)\right) = \phi(m \otimes f^0 \otimes \cdots \otimes f^n) \quad (49)$$

Since $(\Omega(\mathbf{C} \otimes \mathscr{D}_H))^0 = \mathbf{C} \otimes \mathscr{D}_H$ is a left H-category, we see that ϕ is the character associated to the cycle $\left(\Omega(\mathbf{C} \otimes \mathscr{D}_H), \partial_H, M, \mathscr{T}^H, \rho\right)$ over $\mathscr{D}_H$.

From the proof of Lemma 27, we know that $\mathbf{C} \otimes \mathscr{D}_H$ satisfies the assumptions in Proposition 23. Hence, $\left(\Omega(\mathbf{C} \otimes \mathscr{D}_H), \partial_H, M, \mathscr{T}^H, \rho\right)$ is a vanishing cycle over $\mathscr{D}_H$. From this, the result follows. □

For the remaining part of this section, we shall suppose that H is cocommutative. If $\mathscr{D}_H, \mathscr{D}'_H$ are left H-categories, we observe that $\mathscr{D}_H \otimes \mathscr{D}'_H$ then becomes a left H-category under the diagonal action of H.

Let M, M' be left H-comodules equipped respectively with coactions $\rho_M : M \to H \otimes M$ and $\rho_{M'} : M' \to H \otimes M'$. Since H is cocommutative, M may be treated as a right H-comodule and we can form the cotensor product $M \square_H M'$ defined by the kernel

$$M \square_H M' := \mathrm{Ker}\left(M \otimes M' \xrightarrow{\rho_M \otimes 1_{M'} - 1_M \otimes \rho_{M'}} M \otimes H \otimes M'\right)$$

in Vect_k. It follows by [9, Proposition 7.2.2] that the map $\rho_M \otimes 1_{M'}$ gives $M \square_H M'$ a left H-comodule structure. We also note that $M \otimes M'$ carries a right H-module structure via the diagonal action.

Lemma 29. *Let H be a cocommutative Hopf algebra and M, M' be right-left SAYD modules over H such that $M \square_H M'$ is a right H-submodule of $M \otimes M'$. Then, $M \square_H M'$ is also an SAYD module over H.*

Proof. For any $m \otimes m' \in M \square_H M'$, we have

$$\begin{aligned}&\left((m \otimes m')h\right)_{(-1)} \otimes \left((m \otimes m')h\right)_{(0)}\\ &\quad = (mh_1 \otimes m'h_2)_{(-1)} \otimes (mh_1 \otimes m'h_2)_{(0)} = (mh_1)_{(-1)} \otimes (mh_1)_{(0)} \otimes m'h_2\\ &\quad = S(h_{13})m_{(-1)}h_{11} \otimes m_{(0)}h_{12} \otimes m'h_2 = S(h_3)m_{(-1)}h_1 \otimes m_{(0)}h_2 \otimes m'h_4\end{aligned}$$

On the other hand, we have

$$S(h_3)(m\otimes m')_{(-1)}h_1\otimes(m\otimes m')_{(0)}h_2 = S(h_3)m_{(-1)}h_1\otimes(m_{(0)}\otimes m')h_2$$
$$= S(h_3)m_{(-1)}h_1\otimes m_{(0)}h_{21}\otimes m'h_{22} = S(h_4)m_{(-1)}h_1\otimes m_{(0)}h_2\otimes m'h_3$$

Since H is cocommutative, we see that the two expressions are the same. This proves that $M\square_H M'$ is an anti-Yetter–Drinfeld module. We now check that it is also stable. Using the cocommutativity of H and the stability of M, M', we have

$$(m\otimes m')_{(0)}(m\otimes m')_{(-1)} = m_0m_1\otimes m'm_2 = m_{00}m_{01}\otimes m'm_1 = m_0\otimes m'm_1 = m\otimes m'_0m'_{-1} = m\otimes m'$$

for any $m\otimes m'\in M\square_H M'$. □

Let $(\mathscr{S}_H,\widehat{\partial}_H)$ and $(\mathscr{S}'_H,\widehat{\partial}'_H)$ be DGH-semicategories. Then, their tensor product $\mathscr{S}_H\otimes\mathscr{S}'_H$ is the DG-semicategory defined by setting $\mathrm{Ob}(\mathscr{S}_H\otimes\mathscr{S}'_H)=\mathrm{Ob}(\mathscr{S}_H)\times\mathrm{Ob}(\mathscr{S}'_H)$ and

$$\mathrm{Hom}^n_{\mathscr{S}_H\otimes\mathscr{S}'_H}\big((X,X'),(Y,Y')\big)=\bigoplus_{i+j=n}\mathrm{Hom}^i_{\mathscr{S}_H}(X,Y)\otimes_k\mathrm{Hom}^j_{\mathscr{S}'_H}(X',Y')$$

The composition in $\mathscr{S}_H\otimes\mathscr{S}'_H$ is given by the rule:

$$(g\otimes g')\circ(f\otimes f')=(-1)^{\deg(g')\deg(f)}(gf\otimes g'f')$$

for homogeneous $f:X\to Y$, $g:Y\to Z$ in $\mathscr{S}_H$ and $f':X'\to Y'$, $g':Y'\to Z'$ in $\mathscr{S}'_H$. The differential $(\widehat{\partial}_H\otimes\widehat{\partial}'_H)^n:\mathrm{Hom}^n_{\mathscr{S}_H\otimes\mathscr{S}'_H}\big((X,X'),(Y,Y')\big)\to\mathrm{Hom}^{n+1}_{\mathscr{S}_H\otimes\mathscr{S}'_H}\big((X,X'),(Y,Y')\big)$ is determined by

$$(\widehat{\partial}_H\otimes\widehat{\partial}'_H)^n(f_i\otimes g_j)=\widehat{\partial}^i_H(f_i)\otimes g_j+(-1)^if_i\otimes\widehat{\partial}'^j_H(g_j)$$

for any $f_i\in\mathrm{Hom}^i_{\mathscr{S}_H}(X,Y)$ and $g_j\in\mathrm{Hom}^j_{\mathscr{S}'_H}(X',Y')$ such that $i+j=n$. Clearly, $(\mathscr{S}_H\otimes\mathscr{S}'_H)^0=\mathscr{S}^0_H\otimes\mathscr{S}'^0_H$.

Theorem 30. *Let H be a cocommutative Hopf algebra and M, M' be right-left SAYD modules over H such that $M\square_H M'$ is a right H-submodule of $M\otimes M'$. Let $\mathscr{D}_H$, $\mathscr{D}'_H$ be left H-categories. Then, we have a pairing*

$$HC^p_H(\mathscr{D}_H,M)\otimes HC^q_H(\mathscr{D}'_H,M')\longrightarrow HC^{p+q}_H(\mathscr{D}_H\otimes\mathscr{D}'_H,M\square_H M')$$

for $p,q\geq 0$.

Proof. Let $\phi\in Z^p_H(\mathscr{D}_H,M)$ and $\phi'\in Z^q_H(\mathscr{D}'_H,M)$. We may express ϕ and ϕ' respectively as the characters of p and q-dimensional cycles $(\mathscr{S}_H,\widehat{\partial}_H,M,\widehat{\mathscr{T}}^H,\rho)$ and $(\mathscr{S}'_H,\widehat{\partial}'_H,M',\widehat{\mathscr{T}}'^H,\rho')$ over $\mathscr{D}_H$ and $\mathscr{D}'_H$ with coefficients in M and M' respectively. We now consider the collection $\widehat{\mathscr{T}}^H\#\widehat{\mathscr{T}}'^H:=\{(\widehat{\mathscr{T}}^H\#\widehat{\mathscr{T}}'^H)_{(X,X')}:M\square_H M'\otimes\mathrm{Hom}^{p+q}_{\mathscr{S}_H\otimes\mathscr{S}'_H}\big((X,X'),(X,X')\big)\to\mathbb{C}\}_{(X,X')\in\mathrm{Ob}(\mathscr{S}_H\otimes\mathscr{S}'_H)}$ of $\mathbb{C}$-linear maps defined by

$$(\widehat{\mathscr{T}}^H\#\widehat{\mathscr{T}}'^H)_{(X,X')}(m\otimes m'\otimes f\otimes f'):=\widehat{\mathscr{T}}^H_X(m\otimes f_p)\widehat{\mathscr{T}}'^H_{X'}(m'\otimes f'_q)$$

for any $m\otimes m'\in M\square_H M'$ and $f\otimes f'=(f_i\otimes f'_j)_{i+j=p+q}\in\mathrm{Hom}^{p+q}_{\mathscr{S}_H\otimes\mathscr{S}'_H}\big((X,X'),(X,X')\big)$. We will now prove that $\widehat{\mathscr{T}}^H\#\widehat{\mathscr{T}}'^H$ is a $(p+q)$-dimensional closed graded trace on the DGH-semicategory $\mathscr{S}_H\otimes\mathscr{S}'_H$ with coefficients in $M\square_H M'$. For any $m\otimes m'\in M\square_H M'$ and $g\otimes g'=(g_i\otimes g'_j)_{i+j=p+q-1}\in\mathrm{Hom}^{p+q-1}_{\mathscr{S}_H\otimes\mathscr{S}'_H}\big((X,X'),(X,Y)\big)$, we have

$$\begin{aligned}&(\widehat{\mathscr{T}}^H\#\widehat{\mathscr{T}}'^H)_{(X,X')}\big(m\otimes m'\otimes(\widehat{\partial}_H\otimes\widehat{\partial}'_H)^{p+q-1}(g\otimes g')\big)\\&=\sum_{i+j=p+q-1}(\widehat{\mathscr{T}}^H\#\widehat{\mathscr{T}}'^H)_{(X,X')}\Big(m\otimes m'\otimes\widehat{\partial}^i_H(g_i)\otimes g'_j+(-1)^im\otimes m'\otimes g_i\otimes\widehat{\partial}'^j_H(g'_j)\Big)\\&=\widehat{\mathscr{T}}^H_X(m\otimes\widehat{\partial}^{p-1}_H(g_{p-1}))\widehat{\mathscr{T}}'^H_{X'}(m'\otimes g'_q)+(-1)^p\widehat{\mathscr{T}}^H_X(m\otimes g_p)\widehat{\mathscr{T}}'^H_{X'}(m'\otimes\widehat{\partial}'^{q-1}_H(g'_{q-1}))=0\end{aligned}$$

This proves the condition in (24). Next for any homogeneous $f: X \to Y$, $g: Y \to X$ in $\mathscr{S}_H$ and $f': X' \to Y'$, $g': Y' \to X'$ in $\mathscr{S}'_H$, we have

$$\begin{aligned}
&(\widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H)_{(X,X')}\big(m \otimes m' \otimes (g \otimes g')(f \otimes f')\big)\\
&\quad = (-1)^{\deg(g')\deg(f)}(\widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H)_{(X,X')}(m \otimes m' \otimes gf \otimes g'f')\\
&\quad = (-1)^{\deg(g')\deg(f)}\,\widehat{\mathscr{T}}^H_X(m \otimes (gf)_p)\widehat{\mathscr{T}}'^H_{X'}(m' \otimes (g'f')_q)\\
&\quad = (-1)^{\deg(g')\deg(f)}(-1)^{\deg(g)\deg(f)}(-1)^{\deg(g')\deg(f')}\,\widehat{\mathscr{T}}^H_Y(m \otimes (fg)_p)\widehat{\mathscr{T}}'^H_{Y'}(m' \otimes (f'g')_q)\\
&\quad = (-1)^{\deg(g')\deg(f)}(-1)^{\deg(g)\deg(f)}(-1)^{\deg(g')\deg(f')}(-1)^{\deg(g)\deg(f')}\\
&\qquad\quad \times (\widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H)_{(Y,Y')}\big(m \otimes m' \otimes (f \otimes f')(g \otimes g')\big)\\
&\quad = (-1)^{\deg(g\otimes g')\deg(f\otimes f')}(\widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H)_{(Y,Y')}\big(m \otimes m' \otimes (f \otimes f')(g \otimes g')\big)
\end{aligned}$$

This proves the condition in (25). We may similarly verify the condition in (23). Thus, we get a $(p+q)$-dimensional cycle $\big(\mathscr{S}_H \otimes \mathscr{S}'_H, \widehat{\partial}_H \otimes \widehat{\partial}'_H, M\square_H M', \widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H, \rho \otimes \rho'\big)$ with coefficients in $M\square_H M'$ over the category $\mathscr{D}_H \otimes \mathscr{D}'_H$. Then, the character of this cycle, denoted by $\phi\#\phi' \in Z^{p+q}_H(\mathscr{D}_H \otimes \mathscr{D}'_H, M\square_H M')$, gives a well defined map $\gamma: Z^p_H(\mathscr{D}_H, M) \otimes Z^q_H(\mathscr{D}'_H, M') \to Z^{p+q}_H(\mathscr{D}_H \otimes \mathscr{D}'_H, M\square_H M')$.

We now verify that the map γ restricts to a pairing

$$B^p_H(\mathscr{D}_H, M) \otimes Z^q_H(\mathscr{D}'_H, M') \longrightarrow B^{p+q}_H(\mathscr{D}_H \otimes \mathscr{D}'_H, M\square_H M')$$

For this, we let $\phi \in Z^p_H(\mathscr{D}_H, M)$ be the character of a p-dimensional vanishing cycle $\big(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H, \rho\big)$ over $\mathscr{D}_H$. In particular, it follows from Definition 24 that $\mathscr{S}^0_H$ is an ordinary left H-category. From the implication (1) $\Rightarrow$ (2) in Theorem 14, it follows that we might as well take $\mathscr{S}'^0_H$ to be an ordinary left H-category. In fact, we could assume that $\mathscr{S}'_H = \Omega\mathscr{D}'_H$. Then, $\mathscr{S}^0_H \otimes \mathscr{S}'^0_H$ is an ordinary left H-category. It suffices to show that the tuple $\big(\mathscr{S}_H \otimes \mathscr{S}'_H, \widehat{\partial}_H \otimes \widehat{\partial}'_H, M\square_H M', \widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H, \rho \otimes \rho'\big)$ is a vanishing cycle.

Since $\big(\mathscr{S}_H, \widehat{\partial}_H, M, \widehat{\mathscr{T}}^H\big)$ is a vanishing cycle, we have an H-linear semifunctor $\upsilon: \mathscr{S}^0_H \to \mathscr{S}^0_H$ and an $\eta \in \mathbb{U}(\mathscr{S}^0_H \otimes M_2(\mathbb{C}))$ satisfying the conditions in Proposition 23. Extending υ, we get the H-linear semifunctor $\upsilon \otimes 1: \mathscr{S}^0_H \otimes \mathscr{S}'^0_H \to \mathscr{S}^0_H \otimes \mathscr{S}'^0_H$. Identifying, $\mathscr{S}^0_H \otimes \mathscr{S}'^0_H \otimes M_2(\mathbb{C}) \cong \mathscr{S}^0_H \otimes M_2(\mathbb{C}) \otimes \mathscr{S}'^0_H$, we obtain $\widetilde{\eta} \in \mathbb{U}(\mathscr{S}^0_H \otimes M_2(\mathbb{C}) \otimes \mathscr{S}'^0_H)$ given by

$$\begin{aligned}
\{\widetilde{\eta}(X,X') = \eta(X) \otimes 1_{X'} &\in \mathrm{Hom}_{\mathscr{S}^0_H \otimes M_2(\mathbb{C}) \otimes \mathscr{S}'^0_H}((X,X'),(X,X'))\\
&= \mathrm{Hom}_{\mathscr{S}^0_H \otimes M_2(\mathbb{C})}(X,X) \otimes \mathrm{Hom}_{\mathscr{S}'^0_H}(X',X')\}
\end{aligned}$$

It may also be easily verified that

$$\Phi_{\widetilde{\eta}}(f \otimes f' \otimes E_{11} + (\upsilon \otimes 1)(f \otimes f') \otimes E_{22}) = (\upsilon \otimes 1)(f \otimes f') \otimes E_{22}$$

Thus, we see that the category $(\mathscr{S}_H \otimes \mathscr{S}'_H)^0 = \mathscr{S}^0_H \otimes \mathscr{S}'^0_H$ satisfies the conditions in Proposition 23. Therefore, the tuple $\big(\mathscr{S}_H \otimes \mathscr{S}'_H, \widehat{\partial}_H \otimes \widehat{\partial}'_H, M\square_H M', \widehat{\mathscr{T}}^H \# \widehat{\mathscr{T}}'^H, \rho \otimes \rho'\big)$ is a vanishing cycle. This proves the result. □

6. Characters of Fredholm modules over categories

In the rest of this paper, we will study Fredholm modules and Chern characters. We fix a small $\mathbb{C}$-linear category $\mathscr{C}$. Our categorified Fredholm modules will consist of linear functors from $\mathscr{C}$ taking values in separable Hilbert spaces. Let SHilb be the category whose objects are separable Hilbert spaces and whose morphisms are bounded linear maps.

Given separable Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, let $\mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ denote the space of all bounded linear operators from $\mathscr{H}_1$ to $\mathscr{H}_2$ and $\mathscr{B}^\infty(\mathscr{H}_1, \mathscr{H}_2) \subseteq \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$ be the space of all compact operators. For any bounded operator $T \in \mathscr{B}(\mathscr{H}_1, \mathscr{H}_2)$, let $\mu_n(T)$ denote the n-th singular value of T. In other words, $\mu_n(T)$ is the n-th (arranged in decreasing order) eigenvalue of the positive operator $|T| := (T^*T)^{\frac{1}{2}}$. For $1 \leq p < \infty$, the p-th Schatten class is defined to be the space

$$\mathscr{B}^p(\mathscr{H}_1,\mathscr{H}_2) := \left\{T \in \mathscr{B}(\mathscr{H}_1,\mathscr{H}_2) \;\middle|\; \sum \mu_n(T)^p < \infty\right\}$$

Clearly, $\mathscr{B}^p(\mathscr{H}_1,\mathscr{H}_2) \subseteq \mathscr{B}^q(\mathscr{H}_1,\mathscr{H}_2)$ for $p \leq q$. For q_1, q_2, $q_3 \geq 1$ and separable Hilbert spaces $\mathscr{H}$, $\mathscr{H}'$, $\mathscr{H}''$, it follows from Hölder's inequality (see [12, p. 86]) that we have

$$T_2T_1 \in \mathscr{B}^{q_3}(\mathscr{H},\mathscr{H}''), \quad \forall\ T_1 \in \mathscr{B}^{q_1}(\mathscr{H},\mathscr{H}'),\ T_2 \in \mathscr{B}^{q_2}(\mathscr{H}',\mathscr{H}''),\ \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{q_3} \tag{50}$$

For separable Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, the space $\mathscr{B}^1(\mathscr{H}_1,\mathscr{H}_2)$ is the collection of all trace class operators from $\mathscr{H}_1$ to $\mathscr{H}_2$. For $T \in \mathscr{B}^1(\mathscr{H}_1,\mathscr{H}_2)$, we write $\mathrm{Tr}(T) := \sum \mu_n(T)$. Then, it is well known that

$$\mathrm{Tr}(T_1T_2) = \mathrm{Tr}(T_2T_1) \quad \forall\ T_1 \in \mathscr{B}^{n_1}(\mathscr{H},\mathscr{H}'),\ T_2 \in \mathscr{B}^{n_2}(\mathscr{H}',\mathscr{H}), \frac{1}{n_1} + \frac{1}{n_2} = 1 \tag{51}$$

We note that each $\mathscr{B}^p(\mathscr{H}_1,\mathscr{H}_2)$ is an "ideal" in the following sense: consider the functor

$$\mathscr{B}(\cdot,\cdot) : \mathrm{SHilb}^{op} \otimes \mathrm{SHilb} \longrightarrow \mathrm{Vect}_{\mathbb{C}} \qquad \mathscr{B}(\cdot,\cdot)(\mathscr{H}_1,\mathscr{H}_2) := \mathscr{B}(\mathscr{H}_1,\mathscr{H}_2)$$
$$\mathscr{B}(\cdot,\cdot)(\phi_1,\phi_2) : \mathscr{B}(\mathscr{H}_1,\mathscr{H}_2) \longrightarrow \mathscr{B}(\mathscr{H}_1',\mathscr{H}_2') \qquad T \longmapsto \phi_2T\phi_1$$

taking values in the category $\mathrm{Vect}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces. Then, $\mathscr{B}^p(\cdot,\cdot)$ is a subfunctor of $\mathscr{B}(\cdot,\cdot)$. In other words, for morphisms $\phi_1 : \mathscr{H}_1' \to \mathscr{H}_1$, $\phi_2 : \mathscr{H}_2 \to \mathscr{H}_2'$ and any $T \in \mathscr{B}^p(\mathscr{H}_1,\mathscr{H}_2)$, we have $\phi_2T\phi_1 \in \mathscr{B}^p(\mathscr{H}_1',\mathscr{H}_2')$.

We fix the following convention for the commutator notation: Let $\mathscr{H} : \mathscr{C} \to \mathrm{SHilb}$ be a linear functor and $\mathscr{G} := \{\mathscr{G}_X : \mathscr{H}(X) \to \mathscr{H}(X)\}_{X \in \mathrm{Ob}(\mathscr{C})}$ be a collection of bounded linear operators. Then, we set

$$[\mathscr{G},\cdot] : \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y)) \longrightarrow \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y)) \qquad [\mathscr{G},T] := \mathscr{G}_Y \circ T - T \circ \mathscr{G}_X \in \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y))$$

We now let $\mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}$ be the category whose objects are $\mathbb{Z}/2\mathbb{Z}$-graded separable Hilbert spaces and whose morphims are bounded linear maps. Let $\mathscr{H} : \mathscr{C} \to \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}$ be a linear functor and $\mathscr{G} := \{\mathscr{G}_X : \mathscr{H}(X) \to \mathscr{H}(X)\}_{X \in \mathrm{Ob}(\mathscr{C})}$ be a collection of bounded linear operators of the same degree $|\mathscr{G}|$. Then, we set

$$\begin{aligned} &[\mathscr{G},\cdot] : \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y)) \longrightarrow \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y)) \\ &[\mathscr{G},T] := \mathscr{G}_Y \circ T - (-1)^{|\mathscr{G}||T|}T \circ \mathscr{G}_X \in \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y)) \end{aligned} \tag{52}$$

for each X, $Y \in \mathscr{C}$.

Definition 31. *Let $\mathscr{C}$ be a small $\mathbb{C}$-category and let $p \in [1,\infty)$. We consider a pair $(\mathscr{H},\mathscr{F})$ as follows.*

(i) *A linear functor $\mathscr{H} : \mathscr{C} \to \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}$ such that $\mathscr{H}(f) : \mathscr{H}(X) \to \mathscr{H}(Y)$ is a linear operator of degree 0 for each $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$.*

(ii) *A collection $\mathscr{F} := \{\mathscr{F}_X : \mathscr{H}(X) \to \mathscr{H}(X)\}_{X \in \mathrm{Ob}(\mathscr{C})}$ of bounded linear operators of degree 1 such that $\mathscr{F}_X^2 = 1_{\mathscr{H}(X)}$ for each $X \in \mathrm{Ob}(\mathscr{C})$.*

The pair $(\mathscr{H},\mathscr{F})$ is said to be a p-summable even Fredholm module over the category $\mathscr{C}$ if every $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ satisfies

$$[\mathscr{F},f] := \left(\mathscr{F}_Y \circ \mathscr{H}(f) - \mathscr{H}(f) \circ \mathscr{F}_X\right) \in \mathscr{B}^p\left(\mathscr{H}(X),\mathscr{H}(Y)\right) \tag{53}$$

Taking $H = \mathbb{C} = M$ in Definition 12, we note that a closed graded trace of dimension n on a DG-semicategory $(\mathscr{S},\widehat{\partial})$ is a collection of $\mathbb{C}$-linear maps $\widehat{T} := \{\widehat{T}_X : \mathrm{Hom}^n_{\mathscr{S}}(X,X) \to \mathbb{C}\}_{X \in \mathrm{Ob}(\mathscr{S})}$ satisfying the following two conditions

$$\widehat{T}_X\left(\widehat{\partial}^{n-1}(f)\right) = 0 \qquad \widehat{T}_X(gg') = (-1)^{ij}\ \widehat{T}_Y(g'g) \tag{54}$$

for all $f \in \mathrm{Hom}^{n-1}_{\mathscr{S}}(X,X)$, $g \in \mathrm{Hom}^i_{\mathscr{S}}(Y,X)$, $g' \in \mathrm{Hom}^j_{\mathscr{S}}(X,Y)$ and $i + j = n$. Accordingly, we will consider cycles $(\mathscr{S},\widehat{\partial},\mathbb{C},\widehat{T},\rho)$ over $\mathscr{C}$ by setting $H = \mathbb{C} = M$ in Definition 13. In the rest of this paper, since we always have $H = \mathbb{C} = M$, we will suppress the SAYD module $\mathbb{C}$ and write a cycle over $\mathscr{C}$ simply as $(\mathscr{S}, \widehat{\partial}, \widehat{T}, \rho)$.

Let $(\mathscr{H}, \mathscr{F})$ be a pair that satisfies conditions (i) and (ii) in Definition 31. We define a graded semicategory $\Omega'\mathscr{C} = \Omega_{(\mathscr{H},\mathscr{F})}\mathscr{C}$ as follows: we put $Ob(\Omega'\mathscr{C}) := Ob(\mathscr{C})$ and for any X, $Y \in \mathscr{C}$, $j \geq 0$, we set $Hom^j_{\Omega'\mathscr{C}}(X, Y)$ to be the linear span in $\mathscr{B}(\mathscr{H}(X), \mathscr{H}(Y))$ of the operators

$$\mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j] \tag{55}$$

where $\widetilde{f}^0 \otimes f^1 \otimes \cdots \otimes f^j$ is a homogeneous element of degree j in $Hom_{\Omega\mathscr{C}}(X, Y)$. Here, we write $\mathscr{H}(\widetilde{f}^0) = \mathscr{H}(f^0) + \mu \cdot 1$, where $\widetilde{f}^0 = f^0 + \mu$. Using the fact that

$$[\mathscr{F}, f]\mathscr{H}(f') = [\mathscr{F}, f \circ f'] - \mathscr{H}(f)[\mathscr{F}, f']$$

for composable morphisms f, f' in $\mathscr{C}$, we observe that $\Omega'\mathscr{C}$ is closed under composition. We set

$$\partial' := [\mathscr{F}, \cdot] : \mathscr{B}(\mathscr{H}(X), \mathscr{H}(Y)) \longrightarrow \mathscr{B}(\mathscr{H}(X), \mathscr{H}(Y))$$
$$\partial' T = [\mathscr{F}, T] = \mathscr{F}_Y \circ T - (-1)^{|T|} T \circ \mathscr{F}_X$$

We now have the following Lemma.

Lemma 32. *Let $(\mathscr{H}, \mathscr{F})$ be a pair that satisfies conditions (1) and (2) in Definition 31. Then,*

(i) *$(\Omega'\mathscr{C}, \partial')$ is a DG-semicategory and $\Omega'^0\mathscr{C}$ is an ordinary category.*

(ii) *There is a canonical semifunctor $\rho' = \rho_{\mathscr{H}} : \mathscr{C} \to \Omega'^0\mathscr{C}$ which is identity on objects and takes any $f \in Hom_{\mathscr{C}}(X, Y)$ to $\mathscr{H}(f) \in \mathscr{B}(\mathscr{H}(X), \mathscr{H}(Y))$. This extends to a unique DG-semifunctor $\widehat{\rho}' = \widehat{\rho}_{\mathscr{H}} : (\Omega\mathscr{C}, \partial) \to (\Omega'\mathscr{C}, \partial')$ such that the restriction of $\widehat{\rho}'$ to $\mathscr{C}$ is identical to ρ'.*

(iii) *Suppose that $(\mathscr{H}, \mathscr{F})$ is a p-summable Fredholm module. Choose $n \geq p - 1$. Then, for X, $Y \in Ob(\mathscr{C})$ and $1 \leq k \leq n+1$, we have $Hom^k_{\Omega'\mathscr{C}}(X, Y) \subseteq \mathscr{B}^{(n+1)/k}(\mathscr{H}(X), \mathscr{H}(Y))$.*

Proof.

(i). Since each $\mathscr{F}_X$ is a degree 1 operator and $\mathscr{F}_Y[\mathscr{F}, f] = -[\mathscr{F}, f]\mathscr{F}_X$ for any $f \in Hom_{\mathscr{C}}(X, Y)$, we have $\partial'\big(Hom^j_{\Omega'\mathscr{C}}(X, Y)\big) \subseteq Hom^{j+1}_{\Omega'\mathscr{C}}(X, Y)$. We now check that $\partial'^2 = 0$. For any homogeneous element $\mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j]$ of degree j, we have

$$\begin{aligned}
\partial'^2\left(\mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j]\right) &= \partial'\left(\mathscr{F}_Y \circ \left(\mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j]\right)\right) \\
&\quad - (-1)^j \partial'\left(\left(\mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j]\right) \circ \mathscr{F}_X\right) \\
&= \mathscr{F}_Y^2 \circ \mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j] \\
&\quad - (-1)^{j+1} \mathscr{F}_Y \circ \mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j] \circ \mathscr{F}_X \\
&\quad - (-1)^j \big(\mathscr{F}_Y \circ \mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j] \circ \mathscr{F}_X \\
&\quad - (-1)^{j+1} \mathscr{H}(\widetilde{f}^0)[\mathscr{F}, f^1][\mathscr{F}, f^2]\ldots[\mathscr{F}, f^j] \circ \mathscr{F}_X^2\big) \\
&= 0
\end{aligned}$$

The fact that ∂' is compatible with composition follows by direct computation. It is also easy to see that $\Omega'^0\mathscr{C}$ is an ordinary category.

(ii). This is immediate using the universal property in Proposition 9.

(iii). This is a consequence of Hölder's inequality used as in (50) and the condition (53) in Definition 31. □

For any $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space $\mathscr{H} = \mathscr{H}_0 \oplus \mathscr{H}_1$, the grading operator $\epsilon_{\mathscr{H}}$ on $\mathscr{H}$ is determined by setting $\epsilon_{\mathscr{H}}(x) := (-1)^{\deg(x)} x$ for any homogeneous element $x \in \mathscr{H}$. When $\mathscr{H}$ is clear from context, we will often denote the grading operator $\epsilon_{\mathscr{H}}$ simply by ϵ. For any $T \in \mathscr{B}(\mathscr{H}(X),\mathscr{H}(Y))$ such that $[\mathscr{F},T] \in \mathscr{B}^1(\mathscr{H}(X),\mathscr{H}(Y))$, we define

$$\mathrm{Tr}_s(T) := \frac{1}{2}\,\mathrm{Tr}(\epsilon\mathscr{F}_Y[\mathscr{F},T]) = \frac{1}{2}\,\mathrm{Tr}\left(\epsilon\mathscr{F}_Y\partial'(T)\right) = \frac{1}{2}\,\mathrm{Tr}\left(\epsilon\mathscr{F}_Y(\mathscr{F}_Y\circ T-(-1)^{|T|}\;T\circ\mathscr{F}_X)\right)$$

Proposition 33. *Let $(\mathscr{H},\mathscr{F})$ be a p-summable Fredholm module over $\mathscr{C}$. Take $2m \geq p-1$. Then, the collection*

$$\widehat{\mathrm{Tr}}_s = \left\{\mathrm{Tr}_s : \mathrm{Hom}^{2m}_{\Omega'\mathscr{C}}(X,X) \longrightarrow \mathbb{C}\right\}_{X\in \mathrm{Ob}(\mathscr{C})} \tag{56}$$

defines a closed graded trace of dimension $2m$ on $(\Omega'\mathscr{C},\partial')$.

Proof. From the proof of Lemma 32 (i), it is clear that for any $T \in \mathrm{Hom}^{2m}_{\Omega'\mathscr{C}}(X,X)$, we have $[\mathscr{F},T] \in \mathrm{Hom}^{2m+1}_{\Omega'\mathscr{C}}(X,X)$. Applying Lemma 32 (iii), it follows that $[\mathscr{F},T] \in \mathscr{B}^1(\mathscr{H}(X),\mathscr{H}(X))$. Accordingly, each of the maps $\mathrm{Tr}_s : \mathrm{Hom}^{2m}_{\Omega'\mathscr{C}}(X,X) \to \mathbb{C}$ is well-defined.

For $T' \in \mathrm{Hom}^{2m-1}_{\Omega'\mathscr{C}}(X,X)$, we notice that

$$\mathrm{Tr}_s(\partial' T') = \frac{1}{2}\,\mathrm{Tr}\left(\epsilon\mathscr{F}_X(\partial'^2 T')\right) = 0$$

We now consider $T_1 \in \mathrm{Hom}^{i}_{\Omega'\mathscr{C}}(X,Y)$, $T_2 \in \mathrm{Hom}^{j}_{\Omega'\mathscr{C}}(Y,X)$ such that $i+j=2m$. We notice that

$$\epsilon\mathscr{F}_Y\partial'(T_1) = \partial'(T_1)\epsilon\mathscr{F}_X \qquad \epsilon\mathscr{F}_X\partial'(T_2) = \partial'(T_2)\epsilon\mathscr{F}_Y \tag{57}$$

We note that $i \equiv j \pmod 2$. Using (57) and (51), we now have

$$\begin{aligned}
2\cdot\mathrm{Tr}_s(T_1T_2) = \mathrm{Tr}\left(\epsilon\mathscr{F}_Y\partial'(T_1T_2)\right) &= \mathrm{Tr}\left(\epsilon\mathscr{F}_Y\partial'(T_1)T_2\right) + (-1)^i\,Tr\left(\epsilon\mathscr{F}_Y T_1\partial'(T_2)\right)\\
&= \mathrm{Tr}\left(\partial'(T_1)\epsilon\mathscr{F}_X T_2\right) + (-1)^i\,Tr\left(\partial'(T_2)\epsilon\mathscr{F}_Y T_1\right)\\
&= \mathrm{Tr}\left(\epsilon\mathscr{F}_X T_2\partial'(T_1)\right) + (-1)^i\,Tr\left(\epsilon\mathscr{F}_X\partial'(T_2)T_1\right)\\
&= \mathrm{Tr}\left(\epsilon\mathscr{F}_X T_2\partial'(T_1)\right) + (-1)^j\,Tr\left(\epsilon\mathscr{F}_X\partial'(T_2)T_1\right)\\
&= (-1)^{ij}2\cdot\mathrm{Tr}_s(T_2T_1)
\end{aligned}$$

□

Theorem 34. *Let $(\mathscr{H},\mathscr{F})$ be a p-summable Fredholm module over $\mathscr{C}$. Take $2m \geq p-1$. Then, the tuple $(\Omega'\mathscr{C},\partial',\widehat{\mathrm{Tr}}_s,\rho')$ defines a $2m$-dimensional cycle over $\mathscr{C}$. Then, $\phi^{2m} \in CN^{2m}(\mathscr{C}) = C^{2m}_{\mathbb{C}}(\mathscr{C},\mathbb{C}) = \mathrm{Hom}(CN_{2m}(\mathscr{C}),\mathbb{C})$ defined by*

$$\phi^{2m}(f^0\otimes f^1\otimes\cdots\otimes f^{2m}) := \mathrm{Tr}_s\left(\mathscr{H}(f^0)[\mathscr{F},f^1][\mathscr{F},f^2]\dots[\mathscr{F},f^{2m}]\right)$$

for any $f^0\otimes f^1\otimes\cdots\otimes f^{2m} \in \mathrm{Hom}_{\mathscr{C}}(X_1,X)\otimes\mathrm{Hom}_{\mathscr{C}}(X_2,X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{C}}(X,X_{2m})$ is a cyclic cocycle over $\mathscr{C}$.

Proof. It follows directly from Lemma 32 and Proposition 33 that $(\Omega'\mathscr{C},\partial',\widehat{\mathrm{Tr}}_s,\rho')$ is a $2m$-dimensional cycle over $\mathscr{C}$. The rest follows by applying Theorem 14 with $H=\mathbb{C}=M$. □

We will refer to ϕ^{2m} as the $2m$-dimensional character associated with the p-summable even Fredholm module $(\mathscr{H},\mathscr{F})$ over the category $\mathscr{C}$.

Remark 35. The appearance of only even cyclic cocycles in Theorem 34 is due to the following fact from [12, Lemma 2 (a)]: if $T \in \mathscr{B}(\mathscr{H}(X),\mathscr{H}(X))$ is homogeneous of odd degree, then $\mathrm{Tr}_s(T)=0$.

7. Periodicity of Chern character for Fredholm modules

We continue with $\mathscr{C}$ being a small $\mathbb{C}$-category. Taking $H=\mathbb{C}=M$, we denote the cyclic cohomology groups of $\mathscr{C}$ by $H^\bullet_\lambda(\mathscr{C}) := HC^\bullet_{\mathbb{C}}(\mathscr{C},\mathbb{C})$. The cyclic complex corresponding to the cocyclic module $\{CN^n(\mathscr{C}) = \mathrm{Hom}_{\mathbb{C}}(CN_n(\mathscr{C}),\mathbb{C})\}_{n\geq 0}$ as in (14) will be denoted by $C^\bullet_\lambda(\mathscr{C})$. The cocycles of this complex will be denoted by $Z^\bullet_\lambda(\mathscr{C}) := Z^\bullet_{\mathbb{C}}(\mathscr{C},\mathbb{C})$ and the coboundaries by $B^\bullet_\lambda(\mathscr{C}) := B^\bullet_{\mathbb{C}}(\mathscr{C},\mathbb{C})$.

Let $(\mathscr{H},\mathscr{F})$ be a p-summable Fredholm module over $\mathscr{C}$. We take $2m \geq p-1$. Let ϕ^{2m} be the $2m$-dimensional character associated to the Fredholm module $(\mathscr{H},\mathscr{F})$. We denote by $\mathrm{ch}^{2m}(\mathscr{H},\mathscr{F}) \in H^{2m}_\lambda(\mathscr{C})$ the cohomology class of ϕ^{2m}. Since $\mathscr{B}^p(\mathscr{H}(X),\mathscr{H}(Y)) \subseteq \mathscr{B}^q(\mathscr{H}(X),\mathscr{H}(Y))$ for any $p \leq q$, the Fredholm module $(\mathscr{H},\mathscr{F})$ is also $(p+2)$-summable. Using Theorem 34, we then have the $(2m+2)$-dimensional character ϕ^{2m+2} associated to $(\mathscr{H},\mathscr{F})$. We will show that the cyclic cocycles ϕ^{2m} and ϕ^{2m+2} are related to each other via the periodicity operator.

If $\mathscr{C}$ and $\mathscr{C}'$ are small $\mathbb{C}$-categories, from the proof of Theorem 30 it follows that there is a pairing on cyclic cocycles

$$Z^r_\lambda(\mathscr{C}) \otimes Z^s_\lambda(\mathscr{C}') \longrightarrow Z^{r+s}_\lambda(\mathscr{C}\otimes\mathscr{C}') \qquad \phi\otimes\phi' \longmapsto \phi\#\phi' \tag{58}$$

which descends to a pairing on cyclic cohomologies:

$$H^r_\lambda(\mathscr{C}) \otimes H^s_\lambda(\mathscr{C}') \longrightarrow H^{r+s}_\lambda(\mathscr{C}\otimes\mathscr{C}') \tag{59}$$

given by

$$(\widehat{T}^\phi\#\widehat{T}^{\phi'})_{(X,X')}(f\otimes f') := \widehat{T}^\phi_X(f_r)\widehat{T}^{\phi'}_{X'}(f'_s) \tag{60}$$

for any $f\otimes f' = \sum_{i+j=r+s}(f_i\otimes f'_j) \in \mathrm{Hom}^{r+s}_{\mathscr{S}\otimes\mathscr{S}'}\big((X,X'),(X,X')\big)$. Here ϕ and ϕ' are expressed respectively as the characters of r and s-dimensional cycles $(\mathscr{S},\widehat{\partial},\widehat{T}^\phi,\rho)$ and $(\mathscr{S}',\widehat{\partial}',\widehat{T}^{\phi'},\rho')$ over $\mathscr{C}$ and $\mathscr{C}'$. In particular, $\phi\#\phi'$ is the character of the $(r+s)$-dimensional cycle $\big(\mathscr{S}\otimes\mathscr{S}',\widehat{\partial}\otimes\widehat{\partial}',\widehat{T}^\phi\#\widehat{T}^{\phi'},\rho\otimes\rho'\big)$ over $\mathscr{C}\otimes\mathscr{C}'$. For a morphism f in $\mathscr{C}$, we will often suppress the functor ρ and write the morphism $\rho(f)$ in $\mathscr{S}^0$ simply as f. Similarly, when there is no danger of confusion, we will often write the morphism $\mathscr{H}(f)$ simply as f.

Now setting $\mathscr{C}' = \mathbb{C}$ (the category with one object) and considering the cyclic cocycle $\psi \in H^2_\lambda(\mathbb{C})$ determined by $\psi(1,1,1) = 1$, we obtain the periodicity operator:

$$S : Z^r_\lambda(\mathscr{C}) \longrightarrow Z^{r+2}_\lambda(\mathscr{C}) \qquad S(\phi) := \phi\#\psi$$

for any $r \geq 0$ and $\phi \in Z^r_\lambda(\mathscr{C})$.

Lemma 36. *Let $\phi \in Z^r_\lambda(\mathscr{C})$. For any $f^0\otimes f^1\otimes\cdots\otimes f^{r+2} \in CN_{r+2}(\mathscr{C})$, we have*

$$\begin{aligned}(S(\phi))(f^0\otimes f^1\otimes\cdots\otimes f^{r+2}) = {} & \widehat{T}^\phi_X(f^0f^1f^2\widehat{\partial}f^3\ldots\widehat{\partial}f^{r+2}) + \widehat{T}^\phi_X(f^0\widehat{\partial}f^1(f^2f^3)\ldots\widehat{\partial}f^{r+2}) + \cdots \\ & + \widehat{T}^\phi_X(f^0\widehat{\partial}f^1\ldots\widehat{\partial}f^{i-1}(f^if^{i+1})\widehat{\partial}f^{i+2}\ldots\widehat{\partial}f^{r+2}) + \cdots \\ & + \widehat{T}^\phi_X(f^0\widehat{\partial}f^1\ldots\widehat{\partial}f^r(f^{r+1}f^{r+2}))\end{aligned}$$

Proof. We consider the 2-dimensional trace $\widehat{T}^\psi$ on the DG-semicategory $(\Omega\mathbb{C},\partial)$ such that $\psi \in Z^2_\lambda(\mathbb{C})$ is the character of the corresponding cycle over $\mathbb{C}$. We first observe that we have the following equalities in $\Omega\mathbb{C}$:

$$\partial 1 = (\partial 1)1 + 1(\partial 1), \qquad 1(\partial 1)1 = 0, \qquad 1(\partial 1)^2 = (\partial 1)^2 1$$

We illustrate the proof for $r = 2$. The general case will follow similarly. By definition, we have

$$\begin{aligned}
&(S(\phi))(f^0\otimes f^1\otimes f^2\otimes f^3\otimes f^4)\\
&= (\phi\#\psi)(f^0\otimes f^1\otimes f^2\otimes f^3\otimes f^4)\\
&= (\widehat{T}^{\phi}\#\widehat{T}^{\psi})\left((f^0\otimes 1)(\widehat{\partial}\otimes\partial)(f^1\otimes 1)(\widehat{\partial}\otimes\partial)(f^2\otimes 1)(\widehat{\partial}\otimes\partial)(f^3\otimes 1)(\widehat{\partial}\otimes\partial)(f^4\otimes 1)\right)\\
&= (\widehat{T}^{\phi}\#\widehat{T}^{\psi})\left((f^0\otimes 1)(\widehat{\partial}f^1\otimes 1+f^1\otimes\partial 1)(\widehat{\partial}f^2\otimes 1+f^2\otimes\partial 1)(\widehat{\partial}f^3\otimes 1+f^3\otimes\partial 1)(\widehat{\partial}f^4\otimes 1+f^4\otimes\partial 1)\right)\\
&= (\widehat{T}^{\phi}\#\widehat{T}^{\psi})\Big(f^0\widehat{\partial}f^1\widehat{\partial}f^2\widehat{\partial}f^3\widehat{\partial}f^4\otimes 1+f^0\widehat{\partial}f^1\widehat{\partial}f^2\widehat{\partial}f^3f^4\otimes 1\partial 1+f^0\widehat{\partial}f^1\widehat{\partial}f^2f^3f^4\otimes 1(\partial 1)^2\\
&\qquad +f^0\widehat{\partial}f^1f^2f^3\widehat{\partial}f^4\otimes 1(\partial 1)^2 1+f^0\widehat{\partial}f^1f^2f^3f^4\otimes 1(\partial 1)^3+f^0f^1f^2\widehat{\partial}f^3\widehat{\partial}f^4\otimes 1(\partial 1)^2\\
&\qquad +f^0f^1f^2\widehat{\partial}f^3f^4\otimes 1(\partial 1)^3-f^0f^1f^2f^3\widehat{\partial}f^4\otimes 1(\partial 1)^3 1+f^0f^1f^2f^3f^4\otimes 1(\partial 1)^4\Big)\\
&= \widehat{T}^{\phi}\left(f^0\widehat{\partial}f^1\widehat{\partial}f^2f^3f^4\right)\widehat{T}^{\psi}\left(1(\partial 1)^2\right)+\widehat{T}^{\phi}\left(f^0\widehat{\partial}f^1f^2f^3\widehat{\partial}f^4\right)\widehat{T}^{\psi}\left(1(\partial 1)^2 1\right)\\
&\quad +\widehat{T}^{\phi}\left(f^0f^1f^2\widehat{\partial}f^3\widehat{\partial}f^4\right)\widehat{T}^{\psi}\left(1(\partial 1)^2\right)\\
&= \widehat{T}^{\phi}\left(f^0\widehat{\partial}f^1\widehat{\partial}f^2f^3f^4\right)+\widehat{T}^{\phi}\left(f^0\widehat{\partial}f^1f^2f^3\widehat{\partial}f^4\right)+\widehat{T}^{\phi}\left(f^0f^1f^2\widehat{\partial}f^3\widehat{\partial}f^4\right)
\end{aligned}$$

The last equality follows by using the fact that $\widehat{T}^{\psi}\left(1(\partial 1)^2\right)=\psi(1,1,1)=1$. □

Proposition 37. *Let ϕ be the character of an r-dimensional cycle $(\mathscr{S},\widehat{\partial},\widehat{T}^{\phi},\rho)$ over $\mathscr{C}$. Then, $S(\phi)$ is a coboundary. In particular, we have $S(\phi)=b\psi$, where $\psi\in CN^{r+1}(\mathscr{C})$ is given by*

$$\psi(f^0\otimes f^1\otimes\cdots\otimes f^{r+1})=\sum_{j=1}^{r+1}(-1)^{j-1}\ \widehat{T}^{\phi}\left(f^0\widehat{\partial}f^1\ldots\widehat{\partial}f^{j-1}f^j\widehat{\partial}f^{j+1}\ldots\widehat{\partial}f^{r+1}\right)$$

Proof. Again, we illustrate the case of $r=2$. The general computation is similar.

$$\begin{aligned}
&(b\psi)(f^0\otimes f^1\otimes f^2\otimes f^3\otimes f^4)\\
&=\psi(f^0f^1\otimes f^2\otimes f^3\otimes f^4)-\psi(f^0\otimes f^1f^2\otimes f^3\otimes f^4)+\psi(f^0\otimes f^1\otimes f^2f^3\otimes f^4)\\
&\quad -\psi(f^0\otimes f^1\otimes f^2\otimes f^3f^4)+\psi(f^4f^0\otimes f^1\otimes f^2\otimes f^3)\\
&=\widehat{T}^{\phi}(f^0f^1f^2\widehat{\partial}f^3\widehat{\partial}f^4)-\widehat{T}^{\phi}(f^0f^1\widehat{\partial}f^2f^3\widehat{\partial}f^4)+\widehat{T}^{\phi}(f^0f^1\widehat{\partial}f^2\widehat{\partial}f^3f^4)-\widehat{T}^{\phi}(f^0f^1f^2\widehat{\partial}f^3\widehat{\partial}f^4)\\
&\quad +\widehat{T}^{\phi}(f^0\widehat{\partial}(f^1f^2)f^3\widehat{\partial}f^4-\widehat{T}^{\phi}(f^0\widehat{\partial}(f^1f^2)\widehat{\partial}f^3f^4)+\widehat{T}^{\phi}(f^0f^1\widehat{\partial}(f^2f^3)\widehat{\partial}f^4)-\widehat{T}^{\phi}(f^0\widehat{\partial}f^1f^2f^3\widehat{\partial}f^4)\\
&\quad +\widehat{T}^{\phi}(f^0\widehat{\partial}f^1\widehat{\partial}(f^2f^3)f^4)-\widehat{T}^{\phi}(f^0f^1\widehat{\partial}f^2\widehat{\partial}(f^3f^4))+\widehat{T}^{\phi}(f^0\widehat{\partial}f^1f^2\widehat{\partial}(f^3f^4))-\widehat{T}^{\phi}(f^0\widehat{\partial}f^1\widehat{\partial}f^2f^3f^4)\\
&\quad +\widehat{T}^{\phi}(f^4f^0f^1\widehat{\partial}f^2\widehat{\partial}f^3)-\widehat{T}^{\phi}(f^4f^0\widehat{\partial}f^1f^2\widehat{\partial}f^3)+\widehat{T}^{\phi}(f^4f^0\widehat{\partial}f^1\widehat{\partial}f^2f^3)\\
&=\widehat{T}^{\phi}(f^0f^1f^2\widehat{\partial}f^3\widehat{\partial}f^4)+\widehat{T}^{\phi}(f^0\widehat{\partial}f^1(f^2f^3)\widehat{\partial}f^4)+\widehat{T}^{\phi}(f^0\widehat{\partial}f^1\widehat{\partial}f^2f^3f^4)\\
&=(S(\phi))(f^0\otimes f^1\otimes f^2\otimes f^3\otimes f^4)
\end{aligned}$$

□

Theorem 38. *Let $\mathscr{C}$ be a small $\mathbb{C}$-category and let $(\mathscr{H},\mathscr{F})$ be a p-summable even Fredholm module over $\mathscr{C}$. Take $2m\geq p-1$. Then,*

$$S(\phi^{2m})=-(m+1)\phi^{2m+2}\qquad \textit{in } H_{\lambda}^{2m+2}(\mathscr{C})$$

Proof. We will show that $S(\phi^{2m})+(m+1)\phi^{2m+2}=b\psi$ for some $\psi\in Z_{\lambda}^{2m+1}(\mathscr{C})$. By Theorem 34, we know that ϕ^{2m} is the character of the $2m$-dimensional cycle $(\Omega'\mathscr{C},\partial',\widehat{\mathrm{Tr}}_s,\rho')$ over the category $\mathscr{C}$. Applying Lemma 36 and using the fact that $\mathrm{Tr}_s(T)=0$ for any homogeneous T of odd degree, we have

$$\begin{aligned}
&(S(\phi^{2m}))(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})\\
&\qquad\qquad =\sum_{j=0}^{2m+1}\mathrm{Tr}_s\left(f^0[\mathscr{F},f^1]\ldots[\mathscr{F},f^{j-1}](f^jf^{j+1})[\mathscr{F},f^{j+2}]\ldots[\mathscr{F},f^{2m+2}]\right)
\end{aligned}$$

Further,

$$\phi^{2m+2}(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})=\mathrm{Tr}_s\left(f^0[\mathscr{F},f^1]\ldots\ldots[\mathscr{F},f^{2m+2}]\right)$$

so that

$$\begin{aligned}\left(S(\phi^{2m})+(m+1)\phi^{2m+2}\right)&(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})\\&=\sum_{j=0}^{2m+1}\mathrm{Tr}_s\left(f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}](f^jf^{j+1})[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]\right)\\&\qquad+(m+1)\,\mathrm{Tr}_s\left(f^0[\mathscr{F},f^1]\dots\dots[\mathscr{F},f^{2m+2}]\right)\end{aligned}\tag{61}$$

We now consider $\psi=\sum_{j=0}^{2m+1}(-1)^{j-1}\psi^j$, where

$$\psi^j(f^0\otimes f^1\otimes\cdots\otimes f^{2m+1})=\mathrm{Tr}\left(\epsilon\mathscr{F}f^j[\mathscr{F},f^{j+1}]\dots[\mathscr{F},f^{2m+1}][\mathscr{F},f^0][\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]\right)\tag{62}$$

Since $2m\geq p-1$ and $(\mathscr{H},\mathscr{F})$ is a p-summable even Fredholm module over $\mathscr{C}$, it follows that the operator $\epsilon\mathscr{F}f^j[\mathscr{F},f^{j+1}]\dots[\mathscr{F},f^{2m+1}][\mathscr{F},f^0][\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]$ is trace class.

We observe that $\tau\psi^j=\psi^{j-1}$ for $1\leq j\leq 2m+1$ and $\tau\psi^0=\psi^{2m+1}$. It follows that $(1-\lambda)(\psi)=0$. Hence, $\psi\in C_\lambda^{2m+1}(\mathscr{C})=\mathrm{Ker}(1-\lambda)$. Using (62), we have

$$\begin{aligned}(b\psi^j)&(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})\\&=\sum_{i=0}^{2m+1}(-1)^i\,\psi^j(f^0\otimes\cdots\otimes f^if^{i+1}\otimes\cdots\otimes f^{2m+2})+\psi^j(f^{2m+2}f^0\otimes f^1\otimes\cdots\otimes f^{2m+1})\\&=\mathrm{Tr}\left(\epsilon\mathscr{F}f^{j+1}[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^j]\right)\\&\quad+(-1)^{j-1}\,\mathrm{Tr}\left(\epsilon\mathscr{F}f^{j+1}[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}][\mathscr{F},f^0][\mathscr{F},f^1]\dots f^j\right)\\&\quad+\mathrm{Tr}\left(\epsilon\mathscr{F}f^j[\mathscr{F},f^{j+1}]\dots[\mathscr{F},f^{2m+2}]f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]\right)\end{aligned}$$

We now set $\beta^j=[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]$. Then, we have

$$\begin{aligned}[\mathscr{F},\beta^j]=\mathscr{F}\beta^j-(-1)^{2m}\beta^j\mathscr{F}&=\mathscr{F}[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]\\&\quad-[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]\mathscr{F}\\&=(-1)^{j-1}[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}][\mathscr{F},f^0][\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]\end{aligned}$$

With $\alpha^j=f^j\mathscr{F}f^{j+1}$, we get

$$\begin{aligned}(-1)^{j-1}\,\mathrm{Tr}&\left(\epsilon\mathscr{F}f^{j+1}[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}][\mathscr{F},f^0][\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}]f^j\right)\\&=\mathrm{Tr}\left(\epsilon\mathscr{F}f^{j+1}[\mathscr{F},\beta^j]f^j\right)=\mathrm{Tr}_s\left(\alpha^j[\mathscr{F},\beta^j]\right)=\mathrm{Tr}_s([\mathscr{F},\alpha^j]\beta^j)\end{aligned}\tag{63}$$

where we have used the fact that Tr_s is a closed graded trace and $\mathrm{Tr}_s(T)=\mathrm{Tr}(\epsilon T)$ for any operator that is trace class (see [12, Lemma 2]). Thus, we have

$$(b\psi^j)(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})=-\mathrm{Tr}_s\left([\mathscr{F},f^j]\mathscr{F}f^{j+1}\beta^j\right)+\mathrm{Tr}_s([\mathscr{F},\alpha^j]\beta^j)+\mathrm{Tr}_s\left(\mathscr{F}f^j[\mathscr{F},f^{j+1}]\beta^j\right)$$

Since

$$\mathscr{F}[\mathscr{F},f^jf^{j+1}]=\mathscr{F}[\mathscr{F},f^j]f^{j+1}+\mathscr{F}f^j[\mathscr{F},f^{j+1}]=-[\mathscr{F},f^j]\mathscr{F}f^{j+1}+\mathscr{F}f^j[\mathscr{F},f^{j+1}],$$

we obtain

$$(b\psi^j)(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})=\mathrm{Tr}_s\left(\left(\mathscr{F}[\mathscr{F},f^jf^{j+1}]+[\mathscr{F},\alpha^j]\right)\beta^j\right)$$

As

$$\mathscr{F}[\mathscr{F},f^jf^{j+1}]+[\mathscr{F},\alpha^j]=\mathscr{F}[\mathscr{F},f^jf^{j+1}]+\mathscr{F}\alpha^j+\alpha^j\mathscr{F}=[\mathscr{F},f^j][\mathscr{F},f^{j+1}]+2f^jf^{j+1}$$

we get

$$\begin{aligned}(b\psi)(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2}) &= \sum_{j=0}^{2m+1}(-1)^{j-1}(b\psi^j)(f^0\otimes f^1\otimes\cdots\otimes f^{2m+2})\\ &= \sum_{j=0}^{2m+1}(-1)^{j-1}\Big(2\,\mathrm{Tr}_s\big(f^jf^{j+1}\beta^j\big)+\mathrm{Tr}_s\big([\mathscr{F},f^j][\mathscr{F},f^{j+1}]\beta^j\big)\Big)\\ &= \sum_{j=0}^{2m+1}2\,\mathrm{Tr}_s\Big(f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}](f^jf^{j+1})[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]\Big)\\ &\quad+\sum_{j=0}^{2m+1}\mathrm{Tr}_s\big(f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{2m+2}]\big)\\ &= \sum_{j=0}^{2m+1}2\,\mathrm{Tr}_s\Big(f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{j-1}](f^jf^{j+1})[\mathscr{F},f^{j+2}]\dots[\mathscr{F},f^{2m+2}]\Big)\\ &\quad+(2m+2)\,\mathrm{Tr}\big(f^0[\mathscr{F},f^1]\dots[\mathscr{F},f^{2m+2}]\big)\end{aligned}$$

The result now follows by (61). □

8. Homotopy invariance of the Chern character

Let SHilb_2 be the full subcategory of $\mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}$ whose objects are of the form $\mathscr{D}=\mathscr{D}_0\oplus\mathscr{D}_1$ with $\mathscr{D}_0=\mathscr{D}_1=\mathscr{H}$ for some separable Hilbert space $\mathscr{H}$, and whose morphisms are bounded linear maps. Given separable Hilbert spaces $\mathscr{H}$, $\mathscr{H}'$, a sequence of operators $\{T_n\in\mathscr{B}(\mathscr{H},\mathscr{H}')\}_{n\geq 1}$ converges to $T\in\mathscr{B}(\mathscr{H},\mathscr{H}')$ in the strong topology if $T_nx\to Tx$ in the norm on $\mathscr{H}'$ for each $x\in\mathscr{H}$ (see, for instance, [41, Section 4.1]). A function $\phi:[0,1]\to\mathscr{B}(\mathscr{H},\mathscr{H}')$ is strongly C^1 if it is differentiable and its derivative is continuous with respect to the strong topology on $\mathscr{B}(\mathscr{H},\mathscr{H}')$.

In this section, if $\mathscr{D}=\mathscr{H}\oplus\mathscr{H}\in\mathrm{SHilb}_2$, we will denote by $F(\mathscr{D})$ the morphism in $\mathrm{SHilb}_2(\mathscr{D},\mathscr{D})=\mathscr{B}(\mathscr{H}\oplus\mathscr{H},\mathscr{H}\oplus\mathscr{H})$ given by the matrix $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$ swapping the two copies of $\mathscr{H}$.

Lemma 39. *Let $\mathscr{C}$ be a small $\mathbb{C}$-category and $\{\mathscr{H}_t:\mathscr{C}\to\mathrm{SHilb}_2\}_{t\in[0,1]}$ be a family of linear functors such that for each $X\in\mathrm{Ob}(\mathscr{C})$, we have $\mathscr{H}_t(X)=\mathscr{H}_{t'}(X)$ for all t, $t'\in[0,1]$. We put $\mathscr{H}(X):=\mathscr{H}_t(X)$ for all $t\in[0,1]$. For each $f:X\to Y$ in $\mathscr{C}$, we assume that the function*

$$p_f:[0,1]\longrightarrow\mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}(\mathscr{H}_t(X),\mathscr{H}_t(Y))\qquad t\longmapsto\mathscr{H}_t(f)$$

is strongly C^1. Then if $\delta_t(f):=p'_f(t)$, we have

$$\delta_t(fg)=\mathscr{H}_t(f)\circ\delta_t(g)+\delta_t(f)\circ\mathscr{H}_t(g)$$

for composable morphisms f, g in $\mathscr{C}$.

Proof. We have

$$\begin{aligned}&\delta_t(fg)-\mathscr{H}_t(f)\circ\delta_t(g)-\delta_t(f)\circ\mathscr{H}_t(g)\\ &=p'_{fg}(t)-\mathscr{H}_t(f)\circ p'_g(t)-p'_f(t)\circ\mathscr{H}_t(g)\\ &=\lim_{s\to 0}\frac{1}{s}\big(p_{fg}(t+s)-p_{fg}(t)-\mathscr{H}_t(f)\circ p_g(t+s)+\mathscr{H}_t(f)\circ p_g(t)-p_f(t+s)\circ\mathscr{H}_t(g)+p_f(t)\circ\mathscr{H}_t(g)\big)\\ &=\lim_{s\to 0}\frac{1}{s}\big(\mathscr{H}_{t+s}(fg)-\mathscr{H}_t(fg)-\mathscr{H}_t(f)\mathscr{H}_{t+s}(g)+\mathscr{H}_t(f)\mathscr{H}_t(g)-\mathscr{H}_{t+s}(f)\mathscr{H}_t(g)+\mathscr{H}_t(f)\mathscr{H}_t(g)\big)\\ &=\lim_{s\to 0}\frac{1}{s}\big(\mathscr{H}_{t+s}(f)-\mathscr{H}_t(f)\big)\big(\mathscr{H}_{t+s}(g)-\mathscr{H}_t(g)\big)\\ &=\lim_{s\to 0}\frac{1}{s}\big(p_f(t+s)-p_f(t)\big)\big(p_g(t+s)-p_g(t)\big)\\ &=p'_f(t)\lim_{s\to 0}\big(p_g(t+s)-p_g(t)\big)=0\end{aligned}$$

□

For each $n \in \mathbb{Z}_{\geq 0}$, we now define an operator $A : CN^n(\mathscr{C}) \to CN^n(\mathscr{C})$ given by

$$A := 1 + \lambda + \lambda^2 + \cdots + \lambda^n$$

where λ is the (signed) cyclic operator. We observe that if $\psi \in C^n_\lambda(\mathscr{C}) = \mathrm{Ker}(1-\lambda)$, then $A\psi = (n+1)\psi$. From the relation

$$(1-\lambda)(1 + 2\lambda + 3\lambda^2 + \cdots + (n+1)\lambda^n) = A - (n+1)\cdot 1$$

it is immediate that $\mathrm{Ker}(A) \subseteq \mathrm{Im}(1-\lambda)$. Let $B_0 : CN^{n+1}(\mathscr{C}) \to CN^n(\mathscr{C})$ be the map defined as follows:

$$(B_0\phi)(f^0 \otimes \cdots \otimes f^n) := \phi(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^n) - (-1)^{n+1}\phi(f^0 \otimes \cdots \otimes f^n \otimes 1_{X_0})$$

for any $f^0 \otimes f^1 \otimes \cdots \otimes f^n \in \mathrm{Hom}_{\mathscr{C}}(X_1, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_2, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_n)$. We now set

$$B := AB_0 : CN^{n+1}(\mathscr{C}) \longrightarrow CN^n(\mathscr{C})$$

Lemma 40. *We have*

(i) $bA = Ab'$.
(ii) $bB + Bb = 0$.

Proof.

(i). This follows from the general fact that the dual $CN^\bullet(\mathscr{C})$ of the cyclic nerve of $\mathscr{C}$ is a cocyclic module (see, for instance, [34, Section 2.5]).

(ii). For any $f^0 \otimes f^1 \otimes \cdots \otimes f^n \in \mathrm{Hom}_{\mathscr{C}}(X_1, X_0) \otimes \mathrm{Hom}_{\mathscr{C}}(X_2, X_1) \otimes \cdots \otimes \mathrm{Hom}_{\mathscr{C}}(X_0, X_n)$ and $\phi \in CN^n\mathscr{C}$, we have

$$\begin{aligned}(B_0 b\phi)(f^0 \otimes \cdots \otimes f^n) &= (b\phi)(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^n) - (-1)^{n+1}(b\phi)(f^0 \otimes \cdots \otimes f^n \otimes 1_{X_0}) \\ &= \phi(f^0 \otimes \cdots \otimes f^n) + \sum_{i=0}^{n-1}(-1)^{i+1}\phi(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^i f^{i+1} \otimes \cdots \otimes f^n) \\ &\quad + (-1)^{n+1}\phi(f^n \otimes f^0 \otimes \cdots \otimes f^{n-1}) \\ &\quad - (-1)^{n+1}\left(\sum_{i=0}^{n-1}(-1)^i \phi(f^0 \otimes \cdots \otimes f^i f^{i+1} \otimes \cdots \otimes f^n \otimes 1_{X_0})\right)\end{aligned}$$

On the other hand,

$$\begin{aligned}(b' B_0\phi)(f^0 \otimes \cdots \otimes f^n) = &\sum_{i=0}^{n-1}(-1)^i \phi(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^i f^{i+1} \otimes \cdots \otimes f^n) \\ &- (-1)^n \sum_{i=0}^{n-1}(-1)^i \phi(f^0 \otimes \cdots \otimes f^i f^{i+1} \otimes \cdots \otimes f^n \otimes 1_{X_0})\end{aligned}$$

Thus, we obtain

$$(B_0 b + b' B_0)(\phi)(f^0 \otimes \cdots \otimes f^n) = \phi(f^0 \otimes \cdots \otimes f^n) + (-1)^{n+1}\phi(f^n \otimes f^0 \otimes \cdots \otimes f^{n-1})$$

Therefore,

$$(B_0 b + b' B_0)(\phi) = \phi - \lambda\phi \tag{64}$$

Now, by applying the operator A to both sides of (64), we have

$$AB_0 b + Ab' B_0 = 0$$

The result now follows from part (i). □

Proposition 41. *The image of the map $B : CN^{n+1}(\mathscr{C}) \to CN^n(\mathscr{C})$ is $C^n_\lambda(\mathscr{C})$.*

Proof. Let $\phi \in C^n_\lambda(\mathscr{C})$ and let $R := \bigoplus_{X,Y\in \mathrm{Ob}(\mathscr{C})} \mathrm{Hom}(X,Y)$. Then R is an algebra with mutiplication given by composition wherever possible and 0 otherwise. We choose a linear map $\eta : R \to \mathbb{C}$ such that

$$\eta(f) = 0 \qquad \text{for } f \in \mathrm{Hom}_{\mathscr{C}}(X,Y),\ X \neq Y$$
$$\eta(1_X) = 1 \qquad \forall\ X \in \mathrm{Ob}(\mathscr{C})$$

We now define $\psi \in CN^{n+1}(\mathscr{C})$ by setting

$$\begin{aligned}\psi(f^0\otimes\cdots\otimes f^{n+1}) &:= \eta(f^0)\phi(f^1\otimes\cdots\otimes f^{n+1})\\ &\quad + (-1)^n\left(\phi\left(f^0\otimes f^1\otimes\cdots\otimes f^n\right)\eta(f^{n+1}) - \eta(f^0)\phi\left(1_{X_1}\otimes f^1\otimes\cdots\otimes f^n\right)\eta(f^{n+1})\right)\end{aligned}$$

for any $f^0\otimes f^1\otimes\cdots\otimes f^{n+1} \in \mathrm{Hom}_{\mathscr{C}}(X_1,X_0)\otimes\mathrm{Hom}_{\mathscr{C}}(X_2,X_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{C}}(X_0,X_{n+1})$. We observe that if the tuple $(f^1,\dots,f^{n+1})$ is not cyclically composable, i.e., $X_0 \neq X_1$, then the first term vanishes as $\eta(f^0) = 0$. Similarly, if the tuple $(f^0,\dots,f^n)$ is not cyclically composable, i.e., $X_{n+1} \neq X_0$, then the second term vanishes. For the last term, $\eta(f^0)$ and $\eta(f^{n+1})$ will be non zero only if $X_1 = X_0$ and $X_0 = X_{n+1}$ which means that $X_{n+1} = X_1$ and the tuple $(1_{X_1}, f^1,\dots,f^n)$ is cyclically composable.

Then, for any $g^0\otimes g^1\otimes\cdots\otimes g^n \in \mathrm{Hom}_{\mathscr{C}}(Y_1,Y_0)\otimes\mathrm{Hom}_{\mathscr{C}}(Y_2,Y_1)\otimes\cdots\otimes\mathrm{Hom}_{\mathscr{C}}(Y_0,Y_n)$, we have

$$\begin{aligned}&\psi(1_{Y_0}\otimes g^0\otimes\cdots\otimes g^n)\\ &\quad = \eta(1_{Y_0})\phi(g^0\otimes\cdots\otimes g^n)\\ &\qquad + (-1)^n\Big(\phi\left(1_{Y_0}\otimes g^0\otimes\cdots\otimes g^{n-1}\right)\eta(g^n) - \phi\left(\eta(1_{Y_0})1_{Y_0}\otimes g^0\otimes\cdots\otimes g^{n-1}\right)\eta(g^n)\Big)\\ &\quad = \phi(g^0\otimes\cdots\otimes g^n)\end{aligned}$$

Also

$$\begin{aligned}\psi(g^0\otimes\cdots\otimes g^n\otimes 1_{Y_0}) &= \eta(g^0)\phi(g^1\otimes\cdots\otimes g^n\otimes 1_{Y_0})\\ &\quad + (-1)^n\Big(\phi\left(g^0\otimes\cdots\otimes g^n\right)\eta(1_{Y_0}) - \phi\left(\eta(g^0)1_{Y_1}\otimes g^1\otimes\cdots\otimes g^n\right)\eta(1_{Y_0})\Big)\\ &= (-1)^n\phi\left(g^0\otimes\cdots\otimes g^n\right)\end{aligned}$$

where the second equality follows from the fact that $\phi \in C^n_\lambda(\mathscr{C})$ and that $\eta(g^0) = 0$ whenever $Y_1 \neq Y_0$. Thus,

$$\begin{aligned}(B_0\psi)(g^0\otimes\cdots\otimes g^n) &= \psi(1_{Y_0}\otimes g^0\otimes\cdots\otimes g^n) - (-1)^{n+1}\psi(g^0\otimes\cdots\otimes g^n\otimes 1_{Y_0})\\ &= 2\phi(g^0\otimes\cdots\otimes g^n)\end{aligned}$$

Since $\phi \in \mathrm{Ker}(1-\lambda)$, we now have $B\psi = 2A\phi = 2(n+1)\phi$. Thus, $\phi \in \mathrm{Im}(B)$. Conversely, let $\phi \in \mathrm{Im}(B)$. Then, $\phi = B\psi$ for some $\psi \in CN^{n+1}(\mathscr{C})$. Using the fact that $(1-\lambda)A = 0$, we have

$$(1-\lambda)(\phi) = (1-\lambda)(B\psi) = ((1-\lambda)AB_0)\psi = 0$$

This proves the result. □

Proposition 42. *Let $\psi \in CN^n(\mathscr{C})$ be such that $b\psi \in C^{n+1}_\lambda(\mathscr{C})$. Then,*

(i) $B\psi \in Z^{n-1}_\lambda(\mathscr{C})$ *i.e.,* $b(B\psi) = 0$ *and* $(1-\lambda)(B\psi) = 0$.
(ii) $S(B\psi) = n(n+1)b\psi$ *in* $H^{n+1}_\lambda(\mathscr{C})$.

Proof.

(i). We know that $(1-\lambda)(B\psi) = (1-\lambda)(AB_0)(\psi) = 0$. Further, for any $\phi \in \mathrm{Ker}(1-\lambda)$, we have $B_0\phi = 0$. Therefore, it follows that $bB\psi = -Bb\psi = -AB_0b\psi = 0$.

(ii). We have to show that $SB\psi - n(n+1)b\psi = b\zeta$ for some $\zeta \in C_\lambda^n(\mathscr{C})$. We set $\phi = B\psi$. Then, ϕ is the character of an $(n-1)$-dimensional cycle $(\mathscr{S}, \widehat{\partial}, \widehat{T}, \rho)$ over $\mathscr{C}$. By Proposition 37, we have $S\phi = b\psi'$, where $\psi' \in CN^n(\mathscr{C})$ is given by

$$\psi'(f^0 \otimes \cdots \otimes f^n) = \sum_{j=1}^{n} (-1)^{j-1}\ \widehat{T}\left(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^n\right)$$

Suppose we have $\psi'' \in CN^n(\mathscr{C})$ such that $\psi'' - \psi \in B^n(\mathscr{C})$ and $\zeta = \psi' - n(n+1)\psi'' \in C_\lambda^n(\mathscr{C})$. This would give

$$b\zeta = b\psi' - n(n+1)b\psi'' = SB\psi - n(n+1)b\psi$$

We set $\theta := B_0\psi$, $\theta' := \frac{1}{n}\phi$ and $\theta'' := \theta - \theta' \in CN^{n-1}(\mathscr{C})$. Since $B\psi \in Z_\lambda^{n-1}(\mathscr{C})$, we have

$$A\theta'' = AB_0\psi - \frac{1}{n}A\phi = B\psi - \frac{1}{n}AB\psi = B\psi - \frac{1}{n}nB\psi = 0$$

Since $\mathrm{Ker}(A) \subseteq \mathrm{Im}(1-\lambda)$, we have $\theta'' = (1-\lambda)(\psi_1)$ for some $\psi_1 \in CN^{n-1}(\mathscr{C})$. We take $\psi'' = \psi - b\psi_1$. We now show that $(1-\lambda)(\zeta) = 0$, i.e., $(1-\lambda)(\psi') = n(n+1)(1-\lambda)(\psi'')$ where $\zeta = \psi' - n(n+1)\psi''$. We see that

$$(\tau_n\psi')(f^0 \otimes \cdots \otimes f^n) = \psi'(f^n \otimes f^0 \otimes \cdots \otimes f^{n-1}) = \sum_{j=0}^{n-1} (-1)^j\ \widehat{T}\left(\widehat{\partial}f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1} f^n\right)$$

where we have used the fact that $\widehat{T}$ is a graded trace. For $1 \leq j \leq n-1$, we now set

$$\omega_j := f^0(\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1}) f^j (\widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1}) f^n$$

Then,

$$\begin{aligned}\widehat{\partial}\omega_j = (\widehat{\partial}f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1}) f^j (\widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1}) f^n + (-1)^{j-1} f^0 (\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1}\widehat{\partial}f^j\widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1}) f^n \\ + (-1)^n f^0(\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1}) f^j (\widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1}\widehat{\partial}f^n)\end{aligned}$$

Thus,

$$\begin{aligned}0 = \widehat{T}(\widehat{\partial}\omega_j) = \widehat{T}&\left(\widehat{\partial}f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1} f^n\right) \\ &+ (-1)^{j-1}\widehat{T}\left(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1}\widehat{\partial}f^j\widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1} f^n\right) \\ &+ (-1)^n \widehat{T}\left(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1}\widehat{\partial}f^n\right)\end{aligned}$$

Therefore,

$$\begin{aligned}(1-\lambda)(\psi')(f^0 \otimes \cdots \otimes f^n) &= -\sum_{j=1}^{n} (-1)^j\ \widehat{T}\left(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^n\right) \\ &\quad - (-1)^n \sum_{j=0}^{n-1} (-1)^j\ \widehat{T}\left(\widehat{\partial}f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1} f^n\right) \\ &= -\Big((-1)^n \widehat{T}\left(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{n-1} f^n\right) + (-1)^n \widehat{T}\left(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{n-1} f^n\right) \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \Big(\widehat{T}\big(f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^n\big) \\ &\quad + (-1)^n \widehat{T}\big(\widehat{\partial}f^0\widehat{\partial}f^1 \ldots \widehat{\partial}f^{j-1} f^j \widehat{\partial}f^{j+1} \ldots \widehat{\partial}f^{n-1} f^n\big)\Big)\Big) \\ &= (-1)^{n+1}(n+1)\widehat{T}(f^n f^0 \widehat{\partial}f^1 \ldots \widehat{\partial}f^{n-1}) \\ &= (-1)^{n+1}(n+1)\phi(f^n f^0 \otimes f^1 \otimes \cdots \otimes f^{n-1})\end{aligned}$$

Hence,

$$(1-\lambda)(\psi')(f^0 \otimes \cdots \otimes f^n) = (-1)^{n+1}\ (n+1)\phi(f^n f^0 \otimes f^1 \otimes \cdots \otimes f^{n-1}) \tag{65}$$

On the other hand, using the definition of ψ'' and the fact that $(1-\lambda)b = b'(1-\lambda)$, we have

$$(1-\lambda)(\psi'') = (1-\lambda)(\psi) - (1-\lambda)(b\psi_1) = (1-\lambda)(\psi) - b'(1-\lambda)(\psi_1) = (1-\lambda)(\psi) - b'\theta''$$

Since $b\psi \in C_\lambda^{n+1}(\mathscr{C})$, we have from (64) that $(1-\lambda)(\psi) = (B_0 b + b' B_0)(\psi) = b' B_0 \psi = b'\theta = b'\theta' + b'\theta''$. Hence,

$$(1-\lambda)(\psi'') = b'\theta' = \frac{1}{n}\, b'\phi$$

Since $\phi = B\psi \in Z_\lambda^{n-1}(\mathscr{C})$, $b\phi = 0$ and therefore

$$(1-\lambda)(\psi'')(f^0 \otimes \cdots \otimes f^n) = \frac{1}{n}\,(b'\phi)(f^0 \otimes \cdots \otimes f^n) = \frac{1}{n}\,(-1)^{n-1}\phi(f^n f^0 \otimes f^1 \otimes \cdots \otimes f^{n-1}) \tag{66}$$

The result now follows by comparing (65) and (66). □

Proposition 43. *Let $\mathscr{C}$ be a small $\mathbb{C}$-category and $\{\mathscr{H}_t : \mathscr{C} \to \mathrm{SHilb}_2\}_{t\in[0,1]}$ be a family of linear functors such that for each $X \in \mathrm{Ob}(\mathscr{C})$, we have $\mathscr{H}_t(X) = \mathscr{H}_{t'}(X)$ for all t, $t' \in [0,1]$ and $\mathscr{H}_t(f)$ is of degree zero for each $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$ and $t \in [0,1]$. We put $\mathscr{H}(X) := \mathscr{H}_t(X)$ for all $t \in [0,1]$.*

Let $\mathscr{F}$ be the family of operators

$$\mathscr{F} = \left\{ (F(\mathscr{H}(X)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}_{X \in \mathrm{Ob}(\mathscr{C})} \tag{67}$$

Let $p = 2m$ be an even integer. We assume that

(i) *for each $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, the association $t \mapsto [\mathscr{F}, \mathscr{H}_t(f)]$ is a continuous map*

$$\zeta_f : [0,1] \longrightarrow \mathscr{B}^p(\mathscr{H}(X), \mathscr{H}(Y)) \qquad t \longmapsto [\mathscr{F}, \mathscr{H}_t(f)]$$

(ii) *for each $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, the association*

$$p_f : [0,1] \longrightarrow \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}(\mathscr{H}_t(X), \mathscr{H}_t(Y)) \qquad t \longmapsto \mathscr{H}_t(f)$$

is piecewise strongly C^1.

Let $(\mathscr{H}_t, \mathscr{F})$ be the corresponding p-summable Fredholm modules over $\mathscr{C}$. Then, the class in $H_\lambda^{p+2}(\mathscr{C})$ of the $(p+2)$-dimensional character of the Fredholm module $(\mathscr{H}_t, \mathscr{F})$ is independent of t.

Proof. For any $t \in [0,1]$, let ϕ_t be the p-dimensional character of the Fredholm module $(\mathscr{H}_t, \mathscr{F})$. We will show that $S(\phi_{t_1}) = S(\phi_{t_2})$ for any $t_1, t_2 \in [0,1]$.

By assumption, we know that there exists a finite set $R = \{0 = r_0 \leq r_1 < \cdots < r_k \leq r_{k+1} = 1\} \subseteq [0,1]$ such that $p_f : [0,1] \to \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}(\mathscr{H}_t(X), \mathscr{H}_t(Y))$ is continuously differentiable in each $[r_i, r_{i+1}]$. By abuse of notation, we set for each $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$:

$$\delta_t(f) := p'_f(t) \in \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}(\mathscr{H}_t(X), \mathscr{H}_t(Y)) \tag{68}$$

Here, it is understood that if $t = r_i$ for some $1 \leq i \leq k$, we use the right hand derivative when r_i is treated as a point of $[r_i, r_{i+1}]$ and the left hand derivative when r_i is treated as a point of $[r_{i-1}, r_i]$.

Using Lemma 39, we know that

$$\delta_t(fg) = \mathscr{H}_t(f) \circ \delta_t(g) + \delta_t(f) \circ \mathscr{H}_t(g) \tag{69}$$

for any $t \in [0,1]$ and for any pair of composable morphisms f and g in $\mathscr{C}$.

For any $t \in [0,1]$ and $1 \leq j \leq p+1$, we set

$$\begin{aligned}
&\psi_t^j(f^0 \otimes \cdots \otimes f^{p+1}) \\
&\qquad := \mathrm{Tr}\Big(\epsilon \mathscr{H}_t(f^0)[\mathscr{F}, \mathscr{H}_t(f^1)] \ldots [\mathscr{F}, \mathscr{H}_t(f^{j-1})]\delta_t(f^j)[\mathscr{F}, \mathscr{H}_t(f^{j+1})] \ldots [\mathscr{F}, \mathscr{H}_t(f^{p+1})]\Big)
\end{aligned}$$

Using the expression in (69) and the fact that $\epsilon\mathscr{H}(f) = \mathscr{H}(f)\epsilon$ for any morphism $f \in \mathscr{C}$, it may be easily verified that $b\psi_t^j = 0$. For example, when $j = 1$, we have (suppressing the functor $\mathscr{H}$)

$$\begin{aligned}
&(b\psi_t^1)(f^0 \otimes \cdots \otimes f^{p+2})\\
&= \sum_{i=0}^{p+1} \psi_t^1(f^0 \otimes \dots f^i f^{i+1} \otimes \cdots \otimes f^{p+2}) + \psi_t^j(f^{p+2} f^0 \otimes f^1 \otimes \cdots \otimes f^{p+2})\\
&= \mathrm{Tr}\left(\epsilon f^0 f^1 \delta_t(f^2)[\mathscr{F}, f^3]\dots[\mathscr{F}, f^{p+2}]\right) - \mathrm{Tr}\left(\epsilon f^0 \delta_t(f^1 f^2)[\mathscr{F}, f^3]\dots[\mathscr{F}, f^{p+2}]\right)\\
&\quad + \mathrm{Tr}\left(\epsilon f^0 \delta_t(f^1)[\mathscr{F}, f^2 f^3]\dots[\mathscr{F}, f^{p+2}]\right) - \mathrm{Tr}\left(\epsilon f^0 \delta_t(f^1)[\mathscr{F}, f^2][\mathscr{F}, f^3 f^4]\dots[\mathscr{F}, f^{p+2}]\right) + \dots\\
&\quad \cdots - \mathrm{Tr}\left(\epsilon f^0 \delta_t(f^1)[\mathscr{F}, f^2]\dots[\mathscr{F}, f^{p+1} f^{p+2}]\right) + \mathrm{Tr}\left(\epsilon f^{p+2} f^0 \delta_t(f^1)[\mathscr{F}, f^2][\mathscr{F}, f^3 f^4]\dots[\mathscr{F}, f^{p+1}]\right)\\
&= 0
\end{aligned}$$

We then define

$$\psi_t := \sum_{j=0}^{p+1} (-1)^{j-1} \psi_t^j$$

We have $b\psi_t = 0$.

For fixed f, it follows from the compactness of $[0,1]$ and the assumptions (1) and (2) that the families $\{\mathscr{H}_t(f)\}_{t\in[0,1]}$, $\{p_f(t)\}_{t\in[0,1]}$ and $\{\delta_t(f)\}_{t\in[0,1]}$ are uniformly bounded. For the sake of simplicity, we assume that there is only a single point $r \in R$ such that $t_1 \le r \le t_2$. Then, we form $\psi \in CN^{p+1}(\mathscr{C})$ by setting

$$\psi(f^0 \otimes \cdots \otimes f^{p+1}) := \int_{t_1}^{r} \psi_t(f^0 \otimes \cdots \otimes f^{p+1})\mathrm{d}t + \int_{r}^{t_2} \psi_t(f^0 \otimes \cdots \otimes f^{p+1})\mathrm{d}t$$

We now have

$$\begin{aligned}
&\psi(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^p)\\
&= \int_{t_1}^{r} \psi_t(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^p)\mathrm{d}t + \int_{r}^{t_2} \psi_t(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^p)\mathrm{d}t\\
&= \int_{t_1}^{r} \left(\sum_{j=0}^{p} (-1)^j \mathrm{Tr}\Big(\epsilon[\mathscr{F}, \mathscr{H}_t(f^0)]\dots[\mathscr{F}, \mathscr{H}_t(f^{j-1})]\delta_t(f^j)[\mathscr{F}, \mathscr{H}_t(f^{j+1})]\dots[\mathscr{F}, \mathscr{H}_t(f^p)]\Big)\right)\mathrm{d}t\\
&\quad + \int_{r}^{t_2} \left(\sum_{j=0}^{p} (-1)^j \mathrm{Tr}\Big(\epsilon[\mathscr{F}, \mathscr{H}_t(f^0)]\dots[\mathscr{F}, \mathscr{H}_t(f^{j-1})]\delta_t(f^j)[\mathscr{F}, \mathscr{H}_t(f^{j+1})]\dots[\mathscr{F}, \mathscr{H}_t(f^p)]\Big)\right)\mathrm{d}t
\end{aligned}$$

Let $\phi : [0,1] \to Z_\lambda^p(\mathscr{C})$ be the map given by $t \mapsto \phi_t$. We now claim that

$$\psi(1_{X_0} \otimes f^0 \otimes \cdots \otimes f^p) = \int_{t_1}^{r} \phi'(t)(f^0 \otimes \cdots \otimes f^p)\mathrm{d}t + \int_{r}^{t_2} \phi'(t)(f^0 \otimes \cdots \otimes f^p)\mathrm{d}t$$

Indeed, we have

$$\begin{aligned}
\phi'(t)(f^0 \otimes \cdots \otimes f^p) &= \lim_{s\to 0} \frac{1}{s}(\phi_{t+s} - \phi_t)(f^0 \otimes \cdots \otimes f^p)\\
&= \lim_{s\to 0}\Bigg(\mathrm{Tr}\Big(\epsilon \frac{1}{s}\left(\mathscr{H}_{t+s}(f^0) - \mathscr{H}_t(f^0)\right)[\mathscr{F}, \mathscr{H}_{t+s}(f^1)]\dots[\mathscr{F}, \mathscr{H}_{t+s}(f^p)]\Big)\\
&\quad + \mathrm{Tr}\Big(\epsilon\mathscr{H}_t(f^0)\Big[\mathscr{F}, \frac{1}{s}\left(\mathscr{H}_{t+s}(f^1) - \mathscr{H}_t(f^1)\right)\Big][\mathscr{F}, \mathscr{H}_{t+s}(f^2)]\dots[\mathscr{F}, \mathscr{H}_{t+s}(f^p)]\Big)\\
&\quad + \cdots + \mathrm{Tr}\Big(\epsilon\mathscr{H}_t(f^0)[\mathscr{F}, \mathscr{H}_t(f^1)]\dots\Big[\mathscr{F}, \frac{1}{s}\left(\mathscr{H}_{t+s}(f^p) - \mathscr{H}_t(f^p)\right)\Big]\Big)\Bigg)
\end{aligned}$$

By (i), we know that the association $t \mapsto [\mathscr{F}, \mathscr{H}_t(f)]$ is a continuous map for each morphism $f \in \mathscr{C}$. Therefore, we have

$$\begin{aligned}
&\lim_{s\to 0}\left(\mathrm{Tr}\left(\epsilon\mathscr{H}_t(f^0)[\mathscr{F},\mathscr{H}_t(f^1)]\dots[\mathscr{F},\mathscr{H}_t(f^{j-1})]\left[\mathscr{F},\frac{1}{s}\left(\mathscr{H}_{t+s}(f^j)-\mathscr{H}_t(f^j)\right)\right]\dots[\mathscr{F},\mathscr{H}_{t+s}(f^p)]\right)\right)\\
&=\lim_{s\to 0}(-1)^j\left(\mathrm{Tr}\left(\epsilon[\mathscr{F},\mathscr{H}_t(f^0)]\dots[\mathscr{F},\mathscr{H}_t(f^{j-1})]\right.\right.\\
&\qquad\left.\left.\times\frac{1}{s}\left(\mathscr{H}_{t+s}(f^j)-\mathscr{H}_t(f^j)\right)[\mathscr{F},\mathscr{H}_{t+s}(f^{j+1})]\dots[\mathscr{F},\mathscr{H}_{t+s}(f^p)]\right)\right)\\
&=(-1)^j\,\mathrm{Tr}\big(\epsilon[\mathscr{F},\mathscr{H}_t(f^0)][\mathscr{F},\mathscr{H}_t(f^1)]\dots[\mathscr{F},\mathscr{H}_t(f^{j-1})]\delta_t(f^j)[\mathscr{F},\mathscr{H}_t(f^{j+1})]\dots[\mathscr{F},\mathscr{H}_t(f^p)]\big)
\end{aligned}$$

From this, we obtain

$$\begin{aligned}
&\int_{t_1}^{r}\phi'(t)(f^0\otimes\cdots\otimes f^p)\mathrm{d}t+\int_{r}^{t_2}\phi'(t)(f^0\otimes\cdots\otimes f^p)\mathrm{d}t\\
&\quad=\int_{t_1}^{r}\sum_{j=0}^{p}(-1)^j\,\mathrm{Tr}\big(\epsilon[\mathscr{F},\mathscr{H}_t(f^0)][\mathscr{F},\mathscr{H}_t(f^1)]\dots\\
&\qquad\qquad[\mathscr{F},\mathscr{H}_t(f^{j-1})]\delta_t(f^j)[\mathscr{F},\mathscr{H}_t(f^{j+1})]\dots[\mathscr{F},\mathscr{H}_t(f^p)]\big)\mathrm{d}t\\
&\qquad+\int_{r}^{t_2}\sum_{j=0}^{p}(-1)^j\,\mathrm{Tr}\big(\epsilon[\mathscr{F},\mathscr{H}_t(f^0)][\mathscr{F},\mathscr{H}_t(f^1)]\dots\\
&\qquad\qquad[\mathscr{F},\mathscr{H}_t(f^{j-1})]\delta_t(f^j)[\mathscr{F},\mathscr{H}_t(f^{j+1})]\dots[\mathscr{F},\mathscr{H}_t(f^p)]\big)\mathrm{d}t\\
&\quad=\psi(1_{X_0}\otimes f^0\otimes\cdots\otimes f^p)
\end{aligned}$$

Hence

$$\begin{aligned}
\psi(1_{X_0}\otimes f^0\otimes\cdots\otimes f^p)&=\phi_{t_2}(f^0\otimes\cdots\otimes f^p)-\phi_r(f^0\otimes\cdots\otimes f^p)+\phi_r(f^0\otimes\cdots\otimes f^p)-\phi_{t_1}(f^0\otimes\cdots\otimes f^p)\\
&=\phi_{t_2}(f^0\otimes\cdots\otimes f^p)-\phi_{t_1}(f^0\otimes\cdots\otimes f^p)
\end{aligned}$$

Since $\psi(f^0\otimes\cdots\otimes f^p\otimes 1_{X_0})=0$, we now have

$$\begin{aligned}
(B_0\psi)(f^0\otimes\cdots\otimes f^p)&=\psi(1_{X_0}\otimes f^0\otimes\cdots\otimes f^p)-\psi(f^0\otimes\cdots\otimes f^p\otimes 1_{X_0})\\
&=(\phi_{t_2}-\phi_{t_1})(f^0\otimes\cdots\otimes f^p)
\end{aligned}$$

Since $b\psi=0$, using Proposition 42 and the fact that $\phi_{t_2}-\phi_{t_1}\in\mathrm{Ker}(1-\lambda)$, we have

$$0=S(B\psi)=S(AB_0\psi)=(p+1)S(\phi_{t_2}-\phi_{t_1})$$

This proves the result. □

Theorem 44. *Let $\mathscr{C}$ be a small $\mathbb{C}$-category and $\{\rho_t:\mathscr{C}\to\mathrm{SHilb}_2\}_{t\in[0,1]}$ be a family of linear functors such that for each $X\in\mathrm{Ob}(\mathscr{C})$, we have $\rho_t(X)=\rho_{t'}(X)$ for all t, $t'\in[0,1]$. We put $\rho(X):=\rho_t(X)$ for all $t\in[0,1]$. Further, for each $t\in[0,1]$, let*

$$\mathscr{F}_t:=\left\{\mathscr{F}_t(X):=\begin{pmatrix}0&\mathscr{Q}_t(X)\\\mathscr{P}_t(X)&0\end{pmatrix}:\rho(X)\longrightarrow\rho(X)\right\}_{X\in\mathrm{Ob}(\mathscr{C})}\tag{70}$$

with $\mathscr{P}_t(X)=\mathscr{Q}_t^{-1}(X)$ be such that $(\rho_t,\mathscr{F}_t)$ is a p-summable Fredholm module over the category $\mathscr{C}$. We set $\rho(X)=\rho'(X)\oplus\rho'(X)\in\mathrm{SHilb}_2$. We further assume that for some even integer p and for any $f\in\mathrm{Hom}_{\mathscr{C}}(X,Y)$, we have

(i) *$t\mapsto\rho_t^+(f)-\mathscr{Q}_t\rho_t^-(f)\mathscr{P}_t$ is a continuous map from $[0,1]$ to $\mathscr{B}^p(\rho'(X),\rho'(Y))$, where $\rho_t^\pm$ are the two components of the morphism ρ_t of degree zero.*

(ii) *$t\mapsto\rho_t^+(f)$ and $t\mapsto\mathscr{Q}_t\rho_t^-(f)\mathscr{P}_t$ are piecewise strongly C^1 maps from $[0,1]$ to $\mathrm{SHilb}(\rho'(X),\rho'(Y))$.*

Then, the $(p+2)$-dimensional character $\mathrm{ch}^{p+2}(\rho_t,\mathscr{F}_t)\in H_\lambda^{p+2}(\mathscr{C})$ is independent of $t\in[0,1]$.

Proof. For each $t \in [0,1]$, we set $\mathcal{T}_t := \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{Q}_t \end{pmatrix}$. Then, $\mathcal{T}_t^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \mathcal{P}_t \end{pmatrix}$ and $\mathcal{F}'_t := \mathcal{T}_t \mathcal{F}_t \mathcal{T}_t^{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

For each $t \in [0,1]$, we also define a linear functor $\mathcal{H}_t : \mathscr{C} \to \mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}$ given by

$$\mathcal{H}_t(X) := \rho(X) \qquad \mathcal{H}_t(f) := \mathcal{T}_t \rho_t(f) \mathcal{T}_t^{-1}$$

Then, we have

$$[\mathcal{F}'_t, \mathcal{H}_t(f)] = \begin{pmatrix} 0 & \mathcal{Q}_t \rho_t^{-}(f)\mathcal{P}_t - \rho_t^{+}(f) \\ \rho_t^{+}(f) - \mathcal{Q}_t \rho_t^{-}(f)\mathcal{P}_t & 0 \end{pmatrix}$$

Therefore, using assumption (i), we see that the map $t \mapsto [\mathcal{F}', \mathcal{H}_t(f)]$ from $[0,1]$ to $\mathcal{B}^p(\mathcal{H}_t(X), \mathcal{H}_t(Y))$ is continuous for each $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$. Further,

$$\mathcal{H}_t(f) = \mathcal{T}_t \rho_t(f) \mathcal{T}_t^{-1} = \begin{pmatrix} \rho_t^{+}(f) & 0 \\ 0 & \mathcal{Q}_t \rho_t^{-}(f)\mathcal{P}_t \end{pmatrix}$$

Therefore, by applying assumption (ii), we see that the map $t \mapsto \mathcal{H}_t(f)$ is piecewise strongly C^1. Since trace is invariant under similarity, the result now follows using Proposition 43. □

Theorem 45. *Let $\mathscr{C}$ be a small $\mathbb{C}$-category and $\{\rho_t : \mathscr{C} \to \mathrm{SHilb}_2\}_{t \in [0,1]}$ be a family of linear functors such that for each $X \in \mathrm{Ob}(\mathscr{C})$, we have $\rho_t(X) = \rho_{t'}(X)$ for all $t, t' \in [0,1]$. We put $\rho(X) := \rho_t(X)$ for all $t \in [0,1]$. Further, for each $t \in [0,1]$ and $X \in \mathrm{Ob}(\mathscr{C})$, let*

$$\mathcal{F}_t(X) := \begin{pmatrix} 0 & \mathcal{Q}_t(X) \\ \mathcal{P}_t(X) & 0 \end{pmatrix} : \rho(X) \longrightarrow \rho(X)$$

with $\mathcal{Q}_t^{-1} = \mathcal{P}_t$ be such that $(\rho_t, \mathcal{F}_t)$ is a p-summable Fredholm module over the category $\mathscr{C}$. We further assume that for some even integer p, we have

(i) *For any $f \in \mathrm{Hom}_{\mathscr{C}}(X,Y)$, $t \mapsto \rho_t(f)$ is a strongly C^1-map from $[0,1]$ to $\mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}(\rho(X), \rho(Y))$.*

(ii) *For any $X \in \mathscr{C}$, $t \mapsto \mathcal{F}_t(X)$ is a strongly C^1-map from $[0,1]$ to $\mathrm{SHilb}_{\mathbb{Z}/2\mathbb{Z}}(\rho(X), \rho(X))$.*

Then, the $(p+2)$-dimensional character $\mathrm{ch}^{p+2}(\rho_t, \mathcal{F}_t) \in H_\lambda^{p+2}(\mathscr{C})$ is independent of $t \in [0,1]$.

Proof. By definition, $\rho_t(f) = \begin{pmatrix} \rho^{+}(f) & 0 \\ 0 & \rho^{-}(f) \end{pmatrix}$ and $\mathcal{F}_t(X) = \begin{pmatrix} 0 & \mathcal{Q}_t(X) \\ \mathcal{P}_t(X) & 0 \end{pmatrix}$. As such, it is clear that a system satisfying the assumptions (1) and (2) above also satisfies the assumptions in Theorem 44. This proves the result. □

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