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An inverse problem for a hyperbolic system in a bounded domain

Un problème inverse pour un système hyperbolique dans un domaine borné

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Abstract. In this Note we consider a two-by-two hyperbolic system defined on a bounded domain. Using Carleman inequalities, we obtain a Lipschitz stability result for the four spatially varying coefficients with measurements of only one component, given two sets of initial conditions.

Résumé. Dans cette Note, on considère un système hyperbolique de deux équations, défini dans un domaine borné. En utilisant la méthode des inégalités de Carleman, on obtient un résultat de stabilité Lipschitz pour les quatre coefficients dépendant de la variable d'espace de ce système, avec des mesures d'une seule composante de la solution et grâce à la donnée de deux ensembles de conditions initiales.

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Considérons le système (1). L'objet de cette note est d'obtenir un résultat de stabilité pour les quatre coefficients indépendants $\alpha, \beta, \gamma, \delta$ avec une observation d'une seule composante (par exemple u) sur un sous-domaine ω du domaine Ω. Nous prouvons le résultat de stabilité Lipschitz pour les quatre coefficients $\alpha, \beta, \gamma, \delta$ (cf. Théorème 4) :

La norme $H^1(\Omega)$ des coefficients $\alpha, \beta, \gamma, \delta$ est estimée par la norme $H^5(0, T, H^3(\omega))$ de u (voir (11)*).*

La méthode utilise les inégalités de Carleman (7) dans le cas hyperbolique avec observation interne sur un sous-domaine ω sans fonction de troncature en temps (cf. [15]) afin d'éviter des termes résiduels dans les équations ; on obtient ainsi un résultat de stabilité Lipschitz. Par ailleurs on utilise deux ensembles de conditions initiales afin de retrouver les quatre coefficients (cf. par exemple [2]). Enfin on élimine un terme d'observation d'une composante (par exemple v) mais cela fait apparaître des termes supplémentaires d'observations en u sur ω. On présente le cas d'un domaine borné avec une observation interne mais ce résultat peut se généraliser au cas d'un domaine borné avec une observation frontière ainsi qu'au cas d'un guide non borné de la forme $\mathbb{R}\times\omega_0$ (ω_0 étant un sous-domaine borné de $\mathbb{R}^{n-1}$) avec une observation frontière sur une partie bornée du bord (cf. pas exemple [7, 9, 11] dans le cas parabolique).

1. Introduction

Let Ω be a bounded connex domain in $\mathbb{R}^n$, $n\geq 2$ with smooth boundary. We consider the following problem

$$\begin{cases}\partial_t^2 u=\Delta u+\alpha u+\beta v+g_1 & \text{in } \Omega\times(0,T),\\ \partial_t^2 v=\Delta v+\gamma u+\delta v+g_2 & \text{in } \Omega\times(0,T),\\ u(\cdot,0)=a_1,\ v(\cdot,0)=a_2,\ \partial_t u(\cdot,0)=0,\ \partial_t v(\cdot,0)=0 & \text{in } \Omega,\\ u=0,\ v=0 & \text{in } \partial\Omega\times(0,T),\end{cases}\tag{1}$$

where $\alpha,\beta,\gamma,\delta$ are bounded coefficients defined on Ω such that

$$\alpha,\beta,\gamma,\delta\in\Lambda_1(M_0)=\left\{f\in L^\infty(\Omega),\|f\|_{L^\infty(\Omega)}\leq M_0\right\}\text{ for some } M_0>0.$$

We also suppose that

$$\alpha,\beta,\gamma,\delta\in\Lambda_2(M_0)=\left\{f\in H^1(\Omega),\|f\|_{H^1(\Omega)}\leq M_0\right\}.$$

We consider in the following solutions (u,v) of (1) in $H=(W^{5,\infty}(\Omega\times(0,T)))^2$ satisfying the a-priori bound

$$\|(u,v)\|_H\leq M$$

for some $M>0$ sufficiently large. Indeed the method of Carleman estimates requires solutions that are sufficiently regular and the Buckgheim–Klibanov method implies several time differentiations of system (1). The main purpose of this paper is to study the inverse problem of determining simultaneously the four coefficients $(\alpha,\beta,\gamma,\delta)$ from a finite number of observations of u in a sub-domain of Ω. We denote

$$G=(g_1,g_2),\quad A=(a_1,a_2),\quad B=(b_1,b_2),\quad \rho=(\alpha,\beta,\gamma,\delta),\quad \widetilde{\rho}=(\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta}).\tag{2}$$

Our result gives a Lipschitz stability result (11) for the coefficients $\alpha,\beta,\gamma,\delta$ and is the following (see Theorem 4)

$$\begin{aligned}&\|\alpha-\widetilde{\alpha}\|^2_{H^1(\Omega)}+\|\beta-\widetilde{\beta}\|^2_{H^1(\Omega)}+\|\gamma-\widetilde{\gamma}\|^2_{H^1(\Omega)}+\|\delta-\widetilde{\delta}\|^2_{H^1(\Omega)}\\ &\qquad\leq K\left(\|u_A-\widetilde{u}_A\|^2_{H^5(0,T,H^3(\omega))}+\|u_B-\widetilde{u}_B\|^2_{H^5(0,T,H^3(\omega))}\right)\end{aligned}$$

where K is a positive constant, ω is a sub-domain of Ω (see (4)) and assuming that hypotheses (9) and (10) are satisfied. We consider in the above result (u_A,v_A) (resp. $(\widetilde{u}_A,\widetilde{v}_A)$) a solution of (1) associated with (ρ,G,A) (resp. $(\widetilde{\rho},G,A)$) and (u_B,v_B) (resp. $(\widetilde{u}_B,\widetilde{v}_B)$) a solution of (1) associated with (ρ,G,B) (resp. $(\widetilde{\rho},G,B)$). The idea of choosing two different sets of initial conditions can be found for example in [2] for an hyperbolic equation in a bounded domain.

Note that our result implies a uniqueness result. There is a huge literature about inverse problems for equations or systems of Schrödinger type (see for example [1, 10, 14]) or parabolic type (see for example [4, 6, 8, 11, 23, 24]) or hyperbolic type (see for example [2, 13, 17]). When the inverse problem is about the reconstruction of coefficients, the results for equations usually concern the determination of the diffusion coefficient of the operator and/or a potentiel (see for example [1, 2, 10, 11, 13, 14]). It can also be an inverse source problem (see for example [15]). Up to our knowledge, there are few results for systems concerning the simultaneous identification of more than one coefficient in each equation (see for example [6, 7]). Also notice that there are few results where the measurements are given with only one component (see for example [4, 12]).

Therefore the major novelty of this paper is to give a stability result for four coefficients and with measurements of only one component. We give this result in a bounded domain but this can be generalized for an unbounded guide of the type $\mathbb{R}\times\omega_0$ with ω_0 a bounded domain of $\mathbb{R}^{n-1}$. This can also be generalized for systems substituting the operator $\partial_t^2 u-\Delta u$ by an operator of the type $\partial_t^2-\nabla\cdot(c\nabla u)$; we can obtain a stability result for the diffusion coefficient c but this demands a strong positivity hypothesis (see [2, 6, 11, 13]). Last we recall that the methodology based on Carleman estimates for solving inverse problems has been initiated by [5]. See also [16–22].

This Paper is organized as folows: in Section 2, we recall the weight functions (5) and the Carleman estimate (7). Then in Section 3 we state and prove our results.

2. Carleman estimate

In this section, we recall the Carleman estimate (7) which is given on $\Omega\times(-T,T)$. Therefore we will take the even extensions of the considered solutions of (1) on $(-T,0)$ (see (12)).

Denote $Q=\Omega\times(-T,T)$. We choose $\widehat{a}\in\mathbb{R}^n\setminus\overline{\Omega}$ and define $\widehat{d}(x)=|x-\widehat{a}|^2$ for $x\in\Omega$ such that

$$\widehat{d}>0 \text{ in } \Omega,\quad |\nabla\widehat{d}|>0 \text{ in } \overline{\Omega}. \tag{3}$$

Let ω be a sub-domain of Ω such that

$$\{x\in\partial\Omega,\langle x-\widehat{a},\nu(x)\rangle>0\}\subset\partial\omega. \tag{4}$$

Here $\langle\cdot,\cdot\rangle$ denotes the usual scalar product in $\mathbb{R}^n$ and $\nu(x)$ is the outward unit normal vector to $\partial\Omega$ at x. Let $k\in(0,1)$. We consider weight functions as follows, for $\lambda>0$ and $t\in(-T,T)$,

$$\psi(x,t)=\widehat{d}(x)-kt^2+M_1 \text{ where } M_1>kT^2 \text{ and } \phi(x,t)=e^{\lambda\psi(x,t)}. \tag{5}$$

Proposition 1. *There exist $T>0$ and $\epsilon>0$ such that* (3) *holds and*

$$d_1<d_0<d_2 \tag{6}$$

where

$$d_0=\inf_{\overline{\Omega}}\phi(\cdot,0),\qquad d_1=\sup_{\overline{\Omega}\times([-T,-T+2\epsilon]\cup[T-2\epsilon,T])}\phi,\qquad d_2=\sup_{\overline{\Omega}}\phi(\cdot,0).$$

Proof. First we define $\beta_0=\inf_{x\in\overline{\Omega}}\psi(x,0)=\inf_{x\in\overline{\Omega}}|x-\widehat{a}|^2+M_1$ and $\beta_1>0$ by

$$\beta_1^2=\sup_{x\in\overline{\Omega}}|x-\widehat{a}|^2-\inf_{x\in\overline{\Omega}}|x-\widehat{a}|^2.$$

Then, we consider T sufficiently large such that $\beta_2^2=kT^2-\beta_1^2>0$. With these definitions, we have $kT^2=\beta_1^2+\beta_2^2$ so we get for all $x\in\overline{\Omega}$,

$$\psi(x,\pm T)=|x-\widehat{a}|^2+M_1-\sup_{x\in\overline{\Omega}}|x-\widehat{a}|^2+\inf_{x\in\overline{\Omega}}|x-\widehat{a}|^2-\beta_2^2\le\beta_0-\beta_2^2.$$

We deduce that there exists $\epsilon>0$ such that $\epsilon<\frac{T}{2}$ and

$$\text{for all } x\in\overline{\Omega} \text{ and } t\in([-T,-T+2\epsilon]\cup[T-2\epsilon,T]),\quad \psi(x,t)<\beta_0-\frac{\beta_2^2}{2}$$

and this ends the proof of Proposition 1. □

Now we recall a global Carleman estimate (see for example [2, 3, 15]). Let $s>0$ and denote

$$I(u)=\int_Q\left(s|\nabla_{x,t}u|^2+s^3|u|^2\right)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t\quad\text{with}\quad|\nabla_{x,t}u|^2=|\partial_t u|^2+|\nabla u|^2.$$

Proposition 2 ([15, Lemma 1]). *There exist a value of $\lambda > 0$ and positive constants s_0 and C such that*

$$I(u) \leq C\|e^{s\phi} f\|^2_{L^2(Q)} + C\int_{\omega\times(-T,T)} \left(s|\nabla_{x,t} u|^2 + s^3|u|^2\right) e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t + C\int_{\Omega} \left(s|\nabla_{x,t} u(x,T)|^2 + s^3|u(x,T)|^2\right) e^{2s\phi(x,T)}\,\mathrm{d}x + C\int_{\Omega} \left(s|\nabla_{x,t} u(x,-T)|^2 + s^3|u(x,-T)|^2\right) e^{2s\phi(x,-T)}\,\mathrm{d}x \quad (7)$$

for all $s > s_0$, and all $u \in H^2(-T,T,L^2(\Omega)) \cap L^2(-T,T,H^2(\Omega)\cap H^1_0(\Omega))$ satisfying

$$\begin{cases} \partial_t^2 u - \Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\times(-T,T). \end{cases}$$

Remark 3. We could have considered a Carleman estimate with an observation on a part Γ of the boundary,

$$\Gamma = \{x \in \partial\Omega, \langle x - \widehat{a}, \nu(x)\rangle \geq 0\}. \quad (8)$$

Then the observation term $C\int_{\omega\times(-T,T)}(s|\nabla_{x,t}u|^2 + s^3|u|^2)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t$ in (7) should be replaced by $Cs\int_{\Gamma\times(-T,T)} |\partial_\nu u|^2 e^{2s\phi}\,\mathrm{d}\sigma\,\mathrm{d}t$.

In the following parts, C will be a generic positive constant which is independent of s. Let us remark that this Carleman inequality uses also λ as a second large parameter. As we will not use it, we now consider λ fixed in the sequel such that Proposition 1 holds.

3. Inverse problem

3.1. *Main result*

Consider (u_A, v_A) (resp. $(\widetilde{u}_A, \widetilde{v}_A)$) a solution of (1) associated with (ρ, G, A) defined by (2) (resp. $(\widetilde{\rho}, G, A)$). Consider also (u_B, v_B) (resp. $(\widetilde{u}_B, \widetilde{v}_B)$) a solution of (1) associated with (ρ, G, B) (resp. $(\widetilde{\rho}, G, B)$). Assume that all the coefficients $\alpha, \beta, \gamma, \delta, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}$, belong to $\Lambda_1(M_0) \cap \Lambda_2(M_0)$. The following theorem gives a stability result for the four coefficients $\alpha, \beta, \gamma, \delta$.

Theorem 4. *Let $T > 0$ and $\widehat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 1. Assume that the following hypotheses* (9)–(10) *are satisfed*

$$|a_1 b_2 - a_2 b_1| \geq R_1 \ \text{ in } \Omega \text{ for some } R_1 > 0, \quad (9)$$

and

$$|\beta| \geq R_2 > 0 \ \text{ in } \Omega \text{ and } |\widetilde{\beta}| \geq R_2 > 0 \ \text{ in } \Omega \text{ for some } R_2 > 0. \quad (10)$$

Then the following Lipschitz stability estimates holds

$$\|\alpha - \widetilde{\alpha}\|^2_{H^1(\Omega)} + \|\beta - \widetilde{\beta}\|^2_{H^1(\Omega)} + \|\gamma - \widetilde{\gamma}\|^2_{H^1(\Omega)} + \|\delta - \widetilde{\delta}\|^2_{H^1(\Omega)} \leq K\left(\|u_A - \widetilde{u}_A\|^2_{H^5(0,T,H^3(\omega))} + \|u_B - \widetilde{u}_B\|^2_{H^5(0,T,H^3(\omega))}\right). \quad (11)$$

Here, $K > 0$ is a contant depending on R_1, R_2, M_0, M_1, M, T, ϵ and $\widehat{a}$.

Proof. Let (u_A, v_A) (resp. $(\widetilde{u}_A, \widetilde{v}_A)$) be a solution of (1) associated with (ρ, G, A) (resp. $(\widetilde{\rho}, G, A)$) and (u_B, w_B) (resp. $(\widetilde{u}_B, \widetilde{w}_B)$) be a solution of (1) associated with (ρ, G, B) (resp. $(\widetilde{\rho}, G, B)$). We decompose the proof in several steps.

First step. We do an even extension in t. Denote $(u,v)=(u_A,v_A)$, $(\widetilde{u},\widetilde{v})=(\widetilde{u}_A,\widetilde{v}_A)$ and

$$y_0=u-\widetilde{u},\ z_0=v-\widetilde{v},\ a=\alpha-\widetilde{\alpha}.\ b=\beta-\widetilde{\beta},\ c=\gamma-\widetilde{\gamma},\ d=\delta-\widetilde{\delta}. \tag{12}$$

Now we recall that we take the even extensions of all the functions on $(-T,0)$ and for simplicity, we denote the extended functions by the same notations. Denote now for $i=1,2$

$$y_i=\partial_t^i y_0,\ z_i=\partial_t^i z_0.$$

Then (y_1,z_1) and (y_2,z_2) satisfy the following systems

$$\begin{cases}\partial_t^2 y_1=\Delta y_1+\alpha y_1+\beta z_1+a\partial_t\widetilde{u}+b\partial_t\widetilde{v} & \text{in } Q,\\ \partial_t^2 z_1=\Delta z_1+\gamma y_1+\delta z_1+c\partial_t\widetilde{u}+d\partial_t\widetilde{v} & \text{in } Q,\\ y_1(\cdot,0)=z_1(\cdot,0)=0 & \text{in } \Omega,\\ \partial_t y_1(\cdot,0)=aa_1+ba_2,\ \partial_t z_1(\cdot,0)=ca_1+da_2 & \text{in } \Omega,\\ y_1=z_1=0 & \text{on } \partial\Omega\times(-T,T),\end{cases} \tag{13}$$

and

$$\begin{cases}\partial_t^2 y_2=\Delta y_2+\alpha y_2+\beta z_2+a\partial_t^2\widetilde{u}+b\partial_t^2\widetilde{v} & \text{in } Q,\\ \partial_t^2 z_2=\Delta z_2+\gamma y_2+\delta z_2+c\partial_t^2\widetilde{u}+d\partial_t^2\widetilde{v} & \text{in } Q,\\ y_2(\cdot,0)=aa_1+ba_2,\ z_2(\cdot,0)=ca_1+da_2 & \text{in } \Omega,\\ \partial_t y_2(\cdot,0)=\partial_t z_2(\cdot,0)=0 & \text{in } \Omega,\\ y_2=z_2=0 & \text{on } \partial\Omega\times(-T,T).\end{cases} \tag{14}$$

Second step. We estimate $\sum_{i=1}^2(I(y_i)+I(z_i))$ by the Carleman inequality (7).

Note that all the terms $\int_Q e^{2s\phi}(|y_i|^2+|z_i|^2)\,\mathrm{d}x\,\mathrm{d}t$ on the right-hand side of the estimate (7) will be absorbed by $I(y_i)+I(z_i)$ for s sufficiently large. So we have for s sufficiently large,

$$\begin{aligned}\sum_{i=1}^2(I(y_i)+I(z_i))&\le C\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2\big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+Cs^3\sum_{i=1}^2\int_{\omega\times(-T,T)}\big(|\nabla_{x,t}y_i|^2+|y_i|^2+|\nabla_{x,t}z_i|^2+|z_i|^2\big)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+Cs^3\sum_{i=1}^2\int_\Omega\big(|\nabla_{x,t}y_i(x,T)|^2+|y_i(x,T)|^2+|\nabla_{x,t}z_i(x,T)|^2+|z_i(x,T)|^2\big)e^{2s\phi(x,T)}\,\mathrm{d}x.\end{aligned} \tag{15}$$

Now we remove the observation term on z_i. From the first equations in (13) and (14) we have

$$\beta z_i=\partial_t^2 y_i-\Delta y_i-\alpha y_i-a\partial_t^i\widetilde{u}-b\partial_t^i\widetilde{v}\text{ in }Q. \tag{16}$$

From hypothesis (10) and deriving z_i with respect to the space variable x and to the time variable t in (16) we get

$$\begin{aligned}&\int_{\omega\times(-T,T)}\big(|\nabla_{x,t}z_i|^2+|z_i|^2\big)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le Ce^{2sd_2}\|y_i\|^2_{H^3(0,T,H^3(\omega))}+C\int_{\omega\times(-T,T)}e^{2s\phi}\big(a^2+b^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

More precisely we have in the above estimate $\|y_i\|^2_{H^3(0,T,L^2(\omega))}+\|y_i\|^2_{H^2(0,T,H^1(\omega))}+\|y_i\|^2_{H^1(0,T,H^2(\omega))}+\|y_i\|^2_{L^2(0,T,H^3(\omega))}$. So

$$\begin{aligned}&s^3\sum_{i=1}^2\int_{\omega\times(-T,T)}\big(|\nabla_{x,t}z_i|^2+|z_i|^2\big)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t\\ &\qquad\le Cs^3e^{2sd_2}\|y_0\|^2_{H^5(0,T,H^3(\omega))}+Cs^3\int_{\omega\times(-T,T)}e^{2s\phi}\big(a^2+b^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned} \tag{17}$$

Now we estimate

$$\int_\Omega \big(|\nabla_{x,t} y_i(x,T)|^2 + |y_i(x,T)|^2 + |\nabla_{x,t} z_i(x,T)|^2 + |z_i(x,T)|^2\big) e^{2s\phi(x,T)}\, \mathrm{d}x\, \mathrm{d}t$$

using a classical energy estimate. Multiplying the first equation of (13) or (14) by $\partial_t y_i$ and the second one by $\partial_t z_i$, we get for all $0 \le t \le T$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_\Omega \big(|\nabla_{x,t} y_i(x,t)|^2 + |\nabla_{x,t} z_i(x,t)|^2\big)\, \mathrm{d}x\right)&\\ \le C\int_\Omega \big(|y_i(x,t)|^2 + |z_i(x,t)|^2 + |\partial_t y_i(x,t)|^2 + |\partial_t z_i(x,t)|^2\big)\, \mathrm{d}x &+ C\int_\Omega \big(a^2+b^2+c^2+d^2\big)\, \mathrm{d}x.\end{aligned}$$

By Gronswall' Lemma and Poincaré inequality, we have

$$\begin{aligned}\int_\Omega \big(|\nabla_{x,t} y_i(x,T)|^2 + |y_i(x,T)|^2 + |\nabla_{x,t} z_i(x,T)|^2 + |z_i(x,T)|^2\big)\, \mathrm{d}x&\\ \le C\int_\Omega \big(|\nabla_{x,t} y_i(x,0)|^2 + |y_i(x,0)|^2 + |\nabla_{x,t} z_i(x,0)|^2 + |z_i(x,0)|^2\big)\, \mathrm{d}x&\\ + C\int_Q \big(a^2+b^2+c^2+d^2\big)\, \mathrm{d}x\, \mathrm{d}t.&\end{aligned}$$

Since $e^{2s\phi(\cdot,T)} \le e^{2sd_1}$ we get

$$\begin{aligned}\int_\Omega \big(|\nabla_{x,t} y_i(x,T)|^2 + |y_i(x,T)|^2 + |\nabla_{x,t} z_i(x,T)|^2 + |z_i(x,T)|^2\big) e^{2s\phi(x,T)}\, \mathrm{d}x&\\ \le Ce^{2sd_1}\int_\Omega \big(|\nabla_{x,t} y_i(x,0)|^2 + |y_i(x,0)|^2 + |\nabla_{x,t} z_i(x,0)|^2 + |z_i(x,0)|^2\big)\, \mathrm{d}x&\\ + Ce^{2sd_1}\int_Q \big(a^2+b^2+c^2+d^2\big)\, \mathrm{d}x\, \mathrm{d}t.&\end{aligned}$$

From (13) and (14) we obtain

$$\begin{aligned}s^3\sum_{i=1}^{2}\int_\Omega \big(|\nabla_{x,t} y_i(x,T)|^2 + |y_i(x,T)|^2 + |\nabla_{x,t} z_i(x,T)|^2 + |z_i(x,T)|^2\big) e^{2s\phi(x,T)}\, \mathrm{d}x&\\ \le Cs^3e^{2sd_1}\int_\Omega \big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\, \mathrm{d}x.& \quad (18)\end{aligned}$$

So from (15), (17), (18) we have

$$\begin{aligned}\sum_{i=1}^{2}(I(y_i)+I(z_i)) \le Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\, \mathrm{d}x\, \mathrm{d}t&\\ + Cs^3e^{2sd_1}\int_\Omega \big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\, \mathrm{d}x + Cs^3e^{2sd_2}F_0(\omega)& \quad (19)\end{aligned}$$

with $F_0(\omega) = \|y_0\|_{H^5(0,T,H^3(\omega))}$.

Third step. Now we estimate $\int_{\Omega_L} e^{2s\phi(x,0)}(|y_2(x,0)|^2 + |z_2(x,0)|^2 + |\nabla y_2(x,0)|^2 + |\nabla z_2(x,0)|^2)\, \mathrm{d}x$. Consider η be a C^∞ cut-off function satisfying $0 \le \eta \le 1$ and

$$\eta(t) = \begin{cases} 0 & \text{if } t \in [-T,-T+\epsilon] \cup [T-\epsilon, T],\\ 1 & \text{if } t \in [-T+2\epsilon, T-2\epsilon],\end{cases} \quad (20)$$

with ϵ defined in Proposition 1. Multiplying now the first equation of (14) by $2\eta\partial_t y_2$, we get

$$\begin{aligned}2\int_{\Omega\times(-T,0)} \eta e^{2s\phi}\partial_t y_2\big(\partial_t^2 y_2 - \Delta y_2\big)\, \mathrm{d}x\, \mathrm{d}t&\\ = 2\int_{\Omega\times(-T,0)} \eta e^{2s\phi}\big(\alpha y_2 + \beta z_2 + a\partial_t^2\widetilde{u} + b\partial_t^2\widetilde{v}\big)\partial_t y_2\, \mathrm{d}x\, \mathrm{d}t.& \quad (21)\end{aligned}$$

The right-hand side of (21) is less than $C\int_Q e^{2s\phi}(a^2+b^2+|y_2|^2+|z_2|^2+|\partial_t y_2|^2)\, \mathrm{d}x\, \mathrm{d}t$.

For the left-hand side of (21), since $\eta(-T)=0$, $\partial_t y_2 = 0$ on $\partial\Omega\times(-T,T)$, we get

$$2\int_{\Omega\times(-T,0)} \eta e^{2s\phi}\partial_t y_2\big(\partial_t^2 y_2 - \Delta y_2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_\Omega \big[\eta e^{2s\phi}\big(|\partial_t y_2|^2 + |\nabla y_2|^2\big)\big]_{t=-T}^{t=0}\,\mathrm{d}x - \int_{\Omega\times(-T,0)} \partial_t(\eta e^{2s\phi})\big(|\partial_t y_2|^2+|\nabla y_2|^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+\int_{\Omega\times(-T,0)} 4s\eta e^{2s\phi}\partial_t y_2 \nabla y_2\cdot\nabla\phi\,\mathrm{d}x\,\mathrm{d}t.$$

We deduce that

$$\int_\Omega e^{2s\phi(x,0)}\big(|\partial_t y_2(x,0)|^2 + \nabla y_2(x,0)|^2\big)\,\mathrm{d}x$$
$$\le Cs\int_Q e^{2s\phi}\big(|y_2|^2+|z_2|^2+|\nabla_{x,t}y_2|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\int_Q e^{2s\phi}\big(a^2+b^2\big)\,\mathrm{d}x\,\mathrm{d}t.$$

So

$$\int_\Omega e^{2s\phi(x,0)}|\nabla y_2(x,0)|^2\,\mathrm{d}x \le Cs\int_Q e^{2s\phi}\big(|y_2|^2+|z_2|^2+|\nabla_{x,t}y_2|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\int_Q e^{2s\phi}\big(a^2+b^2\big)\,\mathrm{d}x\,\mathrm{d}t.$$

Similarly for z_2 so we have

$$\int_\Omega e^{2s\phi(x,0)}\big(|\nabla y_2(x,0)|^2 + |\nabla z_2(x,0)|^2\big)\,\mathrm{d}x$$
$$\le Cs\int_Q e^{2s\phi}\big(|y_2|^2+|z_2|^2+|\nabla_{x,t}y_2|^2+|\nabla_{x,t}z_2|^2\big)\,\mathrm{d}x\,\mathrm{d}t + C\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$\le C(I(y_2)+I(z_2)) + C\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2\big)\,\mathrm{d}x\,\mathrm{d}t.$$

So from (19) we get

$$\int_\Omega e^{2s\phi(x,0)}\big(|\nabla y_2(x,0)|^2 + |\nabla z_2(x,0)|^2\big)\,\mathrm{d}x \le Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ Cs^3e^{2sd_1}\int_\Omega\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\,\mathrm{d}x + Cs^3e^{2sd_2}F_0(\omega). \quad (22)$$

We can similarly argue to have

$$\int_\Omega e^{2s\phi(x,0)}\big(|y_2(x,0)|^2 + |z_2(x,0)|^2\big)\,\mathrm{d}x$$
$$= \int_\Omega e^{2s\phi(x,0)}\big(|\partial_t y_1(x,0)|^2 + |\partial_t z_1(x,0)|^2\big)\,\mathrm{d}x$$
$$\le Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ Cs^3e^{2sd_1}\int_\Omega\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\,\mathrm{d}x + Cs^3e^{2sd_2}F_0(\omega). \quad (23)$$

From (22) and (23) we obtain

$$\int_\Omega e^{2s\phi(x,0)}\big(|y_2(x,0)|^2 + |z_2(x,0)|^2 + |\nabla y_2(x,0)|^2 + |\nabla z_2(x,0)|^2\big)\,\mathrm{d}x$$
$$\le Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t$$
$$+ Cs^3e^{2sd_1}\int_\Omega\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\,\mathrm{d}x + Cs^3e^{2sd_2}F_0(\omega). \quad (24)$$

Fourth step. We estimate $\int_\Omega e^{2s\phi(x,0)}(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2)\,\mathrm{d}x$.

We choose here the two sets of initial conditions A and B such that (9) holds. Consider (u_A, v_A) (resp. $(\widetilde{u_A}, \widetilde{v_A})$) a solution of (1) associated with (ρ, G, A) (resp. $(\widetilde{\rho}, G, A)$ and (u_B, v_B) (resp. $(\widetilde{u_B}, \widetilde{v_B})$)

a solution of (1) associated with (ρ, G, B) (resp. $(\widetilde{\rho}, G, B)$). From the above second step (see (19)), we denote now

$$F_{0A}(\omega) = \|u_A - \widetilde{u_A}\|_{H^5(0,T,H^3(\omega))} \quad \text{and} \quad F_{0B}(\omega) = \|u_B - \widetilde{u_B}\|_{H^5(0,T,H^3(\omega))}.$$

From now on, each function f defined in the precedent steps is denoted either f_A or f_B when it is related either by the conditions A or B. We have from (14)

$$y_{2A}(\cdot,0) = aa_1 + ba_2 \quad \text{and} \quad y_{2B}(\cdot,0) = ab_1 + bb_2 \ \text{ in } \Omega.$$

Multiplying the first equation by b_2 and the second one by a_2, we eliminate the coefficient b and we get

$$a(a_1b_2 - a_2b_1) = b_2 y_{2A}(\cdot,0) - a_2 y_{2B}(\cdot,0) \ \text{ in } \Omega. \tag{25}$$

Using the hypothesis (9) we get from (25)

$$\int_\Omega e^{2s\phi(x,0)} a^2 \,\mathrm{d}x \le C \int_\Omega e^{2s\phi(x,0)} \left(|y_{2A}(x,0)|^2 + |y_{2B}(x,0)|^2\right) \mathrm{d}x \tag{26}$$

and from (24) we have

$$\begin{aligned}\int_\Omega e^{2s\phi(x,0)} a^2 \,\mathrm{d}x &\le Cs^3 \int_Q e^{2s\phi}\left(a^2 + b^2 + c^2 + d^2 + |\nabla a|^2 + |\nabla b|^2\right) \mathrm{d}x\,\mathrm{d}t \\ &\quad + Cs^3 e^{2sd_1} \int_\Omega \left(a^2 + b^2 + c^2 + d^2 + |\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2 + |\nabla d|^2\right) \mathrm{d}x + Cs^3 e^{2sd_2} F_1(\omega).\end{aligned} \tag{27}$$

with $F_1(\omega) = F_{0A}(\omega) + F_{0B}(\omega)$. Similarly we have $b(a_2b_1 - a_1b_2) = b_1 y_{2A}(\cdot,0) - a_1 y_{2B}(\cdot,0)$, so (27) is still valid with a replaced by b on the left-hand side of the estimate. We proceed by the same way to obtain c and d using this time (14) for z_{2A} and z_{2B} and the hypothesis (9). Therefore

$$\begin{aligned}\int_\Omega e^{2s\phi(x,0)} \left(a^2 + b^2 + c^2 + d^2\right) \mathrm{d}x &\le Cs^3 \int_Q e^{2s\phi}\left(a^2 + b^2 + c^2 + d^2 + |\nabla a|^2 + |\nabla b|^2\right) \mathrm{d}x\,\mathrm{d}t \\ &\quad + Cs^3 e^{2sd_1} \int_\Omega \left(a^2 + b^2 + c^2 + d^2 + |\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2 + |\nabla d|^2\right) \mathrm{d}x + Cs^3 e^{2sd_2} F_1(\omega).\end{aligned} \tag{28}$$

Deriving now (25) with respect to x_i for any integer $i = 1, \dots, n$, still using the hypothesis (9) we get

$$\int_\Omega e^{2s\phi(x,0)} |\nabla a|^2 \,\mathrm{d}x \le C \int_\Omega e^{2s\phi(x,0)} \left(|y_{2A}(x,0)|^2 + |y_{2B}(x,0)|^2 + |\nabla y_{2A}(x,0)|^2 + |\nabla y_{2B}(x,0)|^2 + a^2\right) \mathrm{d}x.$$

Similarly for b, c, d so

$$\begin{aligned}&\int_\Omega e^{2s\phi(x,0)} \left(|\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2 + |\nabla d|^2\right) \mathrm{d}x \\ &\quad \le C \int_\Omega e^{2s\phi(x,0)} \big(|y_{2A}(x,0)|^2 + |y_{2B}(x,0)|^2 + |\nabla y_{2A}(x,0)|^2 + |\nabla y_{2B}(x,0)|^2 + |z_{2A}(x,0)|^2 \\ &\qquad + |z_{2B}(x,0)|^2 + |\nabla z_{2A}(x,0)|^2 + |\nabla z_{2A}(x,0)|^2 + a^2 + b^2 + c^2 + d^2\big) \mathrm{d}x.\end{aligned}$$

From (24) and (28) we get

$$\begin{aligned}&\int_\Omega e^{2s\phi(x,0)} \left(|\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2 + |\nabla d|^2\right) \mathrm{d}x \\ &\le Cs^3 \int_Q e^{2s\phi}\left(a^2 + b^2 + c^2 + d^2 + |\nabla a|^2 + |\nabla b|^2\right) \mathrm{d}x\,\mathrm{d}t \\ &\quad + Cs^3 e^{2sd_1} \int_\Omega \left(a^2 + b^2 + c^2 + d^2 + |\nabla a|^2 + |\nabla b|^2 + |\nabla c|^2 + |\nabla d|^2\right) \mathrm{d}x + Cs^3 e^{2sd_2} F_1(\omega).\end{aligned} \tag{29}$$

Thus from (24) and (29) we obtain

$$\begin{aligned}&\int_\Omega e^{2s\phi(x,0)}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\,\mathrm{d}x\\&\le Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t\\&\qquad+Cs^3e^{2sd_1}\int_\Omega\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\,\mathrm{d}x+Cs^3e^{2sd_2}F_1(\omega).\end{aligned}\tag{30}$$

Now we proceed as in [2, 13, 15] to prove that the term $s^3\int_Q e^{2s\phi}(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2)\,\mathrm{d}x\,\mathrm{d}t$ on the right-hand side of (30) can be absorbed by the left-hand side of the estimate for s sufficiently large.

Indeed,

$$\begin{aligned}&s^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t\\&\qquad=\int_\Omega e^{2s\phi(x,0)}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\left(\int_{-T}^{T}s^3e^{2s(\phi(x,t)-\phi(x,0))}\,\mathrm{d}t\right)\mathrm{d}x.\end{aligned}$$

But $\phi(x,t)-\phi(x,0)=-e^{\lambda(d(x)+M_1)}(1-e^{-\lambda kt^2})$ and there exists a positive constant C such that $\phi(x,t)-\phi(x,0)\le -C(1-e^{-\lambda kt^2})$. Therefore $\int_{-T}^{T}s^3e^{2s(\phi(x,t)-\phi(x,0))}\,\mathrm{d}t\le\int_{-T}^{T}s^3e^{-2sC(1-e^{-\lambda kt^2})}\,\mathrm{d}t$ uniformly in x.

Moreover by the Lebesgue convergence theorem, we have

$$\int_{-T}^{T}s^3e^{-2sC(1-e^{-\lambda kt^2})}\,\mathrm{d}t\to 0\text{ as }s\to\infty.$$

Thus we can neglect $s^3\int_Q e^{2s\phi}(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2)\,\mathrm{d}x\,\mathrm{d}t$ on the right-hand side of (30) for s sufficiently large. Furthermore since $e^{2sd_0}\le e^{2s\phi(\cdot,0)}$ we deduce that

$$e^{2sd_0}\big(1-Cs^3e^{2s(d_1-d_0)}\big)\int_\Omega\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2+|\nabla c|^2+|\nabla d|^2\big)\,\mathrm{d}x\le Cs^3e^{2sd_2}F_1(\omega).$$

Since $d_1<d_0$ we can choose s sufficiently large such that $1-Cs^3e^{2s(d_1-d_0)}\ge\frac{1}{2}$ so we get

$$\|\alpha-\widetilde{\alpha}\|^2_{H^1(\Omega)}+\|\beta-\widetilde{\beta}\|^2_{H^1(\Omega)}+\|\gamma-\widetilde{\gamma}\|^2_{H^1(\Omega)}+\|\delta-\widetilde{\delta}\|^2_{H^1(\Omega)}\le Cs^3e^{2s(d_2-d_0)}F_1(\omega).$$

So we conclude for Theorem 4. □

Remark 5. Notice that in the Carleman inequality $I(y_i)$ for y_i, the observation term is $\|y_i\|^2_{H^1(0,T,H^1(\omega))}$ (for $i=1,2$) and so $\|y_0\|^2_{H^3(0,T,H^1(\omega))}$. Since we removed the observation terms on z_i, this added additional terms in y_i and so in y_0 in our final estimate (11).

Remark 6. We can prove a similar result when we use a Carleman estimate with an observation term on the sub-boundary Γ defined by (8). The left-hand side of the Lipschitz stability result is unchanged as the right-hand side has an additional term $\|\partial_\nu\partial_t^2(u_A-\widetilde{u}_A)\|^2_{L^2(\Gamma\times(0,T))}+\|\partial_\nu\partial_t^2(u_B-\widetilde{u}_B)\|^2_{L^2(\Gamma\times(0,T))}$.

3.2. *Other result*

Our main theorem (Theorem 4) gives a Lipschitz stability result for the four spatiallly coefficients $\alpha,\beta,\gamma,\delta$ in $H^1(\Omega)$. When these coefficients belong to $L^2(\Omega)$ we can prove a similar result (see Theorem 7) but we get a Hölder and not Lipschitz stability result. Assume that all the coefficients $\alpha,\beta,\gamma,\delta,\widetilde{\alpha},\widetilde{\beta},\widetilde{\gamma},\widetilde{\delta}$, belong to $\Lambda_1(M_0)$. The following theorem (Theorem 7 (i)) gives a stability result for the four coefficients $\alpha,\beta,\gamma,\delta$ in $L^2(\Omega)$ when $\alpha=\widetilde{\alpha}$ and $\beta=\widetilde{\beta}$ in ω. That means that these two coefficients α and β are supposed known in ω. We relax this last hypothesis in Theorem 7 (ii) where an estimate of these four coefficients is given for $\alpha,\beta\in\Lambda_2(M_0)\cap\Lambda_1(M_0)$ and $\gamma,\delta\in\Lambda_1(M_0)$.

Consider (u_A, v_A) (resp. $(\widetilde{u}_A, \widetilde{v}_A)$) a solution of (1) associated with (ρ, G, A) defined by (2) (resp. $(\widetilde{\rho}, G, A)$). Consider also (u_B, v_B) (resp. $(\widetilde{u}_B, \widetilde{v}_B)$) a solution of (1) associated with (ρ, G, B) (resp. $(\widetilde{\rho}, G, B)$).

Theorem 7. *Let $T > 0$ and $\widehat{a} \in \mathbb{R}^n \setminus \Omega$ satisfying the conditions of Proposition 1. Assume that hypotheses* (9)–(10) *are satisfed.*

(i) *Assume that $\alpha, \beta, \gamma, \delta, \widetilde{\alpha}, \widetilde{\beta}, \widetilde{\gamma}, \widetilde{\delta}, \in \Lambda_1(M_0)$. We also suppose that $\alpha = \widetilde{\alpha}$ and $\beta = \widetilde{\beta}$ in ω. Then the following Hölder stability estimates holds*

$$\|\alpha - \widetilde{\alpha}\|^2_{L^2(\Omega)} + \|\beta - \widetilde{\beta}\|^2_{L^2(\Omega)} + \|\gamma - \widetilde{\gamma}\|^2_{L^2(\Omega)} + \|\delta - \widetilde{\delta}\|^2_{L^2(\Omega)} \leq K\left(\|u_A - \widetilde{u}_A\|^2_{H^4(0,T,H^3(\omega))} + \|u_B - \widetilde{u}_B\|^2_{H^4(0,T,H^3(\omega))}\right)^{\kappa}. \quad (31)$$

(ii) *Assume that $\alpha, \beta, \widetilde{\alpha}, \widetilde{\beta} \in \Lambda_1(M_0) \cap \Lambda_2(M_0)$ and $\gamma, \delta, \widetilde{\gamma}, \widetilde{\delta} \in \Lambda_1(M_0)$. Then the following Hölder stability estimate holds*

$$\|\alpha - \widetilde{\alpha}\|^2_{H^1(\Omega)} + \|\beta - \widetilde{\beta}\|^2_{H^1(\Omega)} + \|\gamma - \widetilde{\gamma}\|^2_{L^2(\Omega)} + \|\delta - \widetilde{\delta}\|^2_{L^2(\Omega)} \leq K\left(\|u_A - \widetilde{u}_A\|^2_{H^5(0,T,H^3(\omega))} + \|u_B - \widetilde{u}_B\|^2_{H^5(0,T,H^3(\omega))}\right)^{\kappa}. \quad (32)$$

Here, $K > 0$ and $\kappa \in (0,1)$ are two constants depending on R_1, R_2, M_0, M_1, M, T and $\widehat{a}$.

Proof. As for Theorem 4 we decompose the proof in several steps. We will widely follow the ideas described in the proof of Theorem 4. We consider the case (ii) where α and β are in $H^1(\Omega)$ but not necessarily γ and δ.

First step. In this step we still make an even extension in t. We keep the notations of Theorem 4: $(u, v) = (u_A, v_A)$, $(\widetilde{u}, \widetilde{v}) = (\widetilde{u}_A, \widetilde{v}_A)$ and the definitions of a, b, c, d (see (12)) and we take the even extensions of all the functions on $(-T, 0)$. But for $i = 0, 1, 2$ we denote here

$$y_0 = \eta(u - \widetilde{u}),\ z_0 = \eta(v - \widetilde{v}), y_i = \partial_t^i y_0,\ z_i = \partial_t^i z_0 \quad (33)$$

with η the truncature function defined by (20). Then these new (y_1, z_1) and (y_2, z_2) satisfy the following systems

$$\begin{cases} \partial_t^2 y_1 = \Delta y_1 + \alpha y_1 + \beta z_1 + a\partial_t(\eta\widetilde{u}) + b\partial_t(\eta\widetilde{v}) + \partial_t R_1 & \text{in } Q, \\ \partial_t^2 z_1 = \Delta z_1 + \gamma y_1 + \delta z_1 + c\partial_t(\eta\widetilde{u}) + +d\partial_t(\eta\widetilde{v}) & \text{in } Q, \\ y_1(\cdot,0) = z_1(\cdot,0) = 0 & \text{in } \Omega, \\ \partial_t y_1(\cdot,0) = aa_1 + ba_2,\ \partial_t z_1(\cdot,0) = ca_1 + da_2 & \text{in } \Omega, \\ y_1 = z_1 = 0 & \text{on } \partial\partial\Omega \times (-T,T), \end{cases} \quad (34)$$

and

$$\begin{cases} \partial_t^2 y_2 = \Delta y_2 + \alpha y_2 + \beta z_2 + a\partial_t^2(\eta\widetilde{u}) + b\partial_t^2(\eta\widetilde{v}) + \partial_t^2 R_1 & \text{in } Q, \\ \partial_t^2 z_2 = \Delta z_2 + \gamma y_2 + \delta z_2 + c\partial_t^2(\eta\widetilde{u}) + d\partial_t^2(\eta\widetilde{v}) + \partial_t^2 R_2 & \text{in } Q, \\ y_2(\cdot,0) = aa_1 + ba_2,\ z_2(\cdot,0) = ca_1 + da_2 & \text{in } \Omega, \\ \partial_t y_2(\cdot,0) = aa_3 + ba_4,\ \partial_t z_2(\cdot,0) = ca_3 + da_4 & \text{in } \Omega, \\ y_2 = z_2 = 0 & \text{on } \partial\Omega \times (-T,T), \end{cases} \quad (35)$$

with

$$R_1 = (\partial_t^2\eta)(u - \widetilde{u}) + 2\partial_t\eta\partial_t(u - \widetilde{u}),\ R_2 = (\partial_t^2\eta)(v - \widetilde{v}) + 2\partial_t\eta\partial_t(v - \widetilde{v}).$$

Second step. We estimate $\sum_{i=1}^{2}(I(y_i)+I(z_i))$ by the Carleman inequality (7).

Note that all the terms with derivatives of η will be bounded above by Ce^{2sd_1} with C a positive constant and due to the truncature function η, we have $y_i(\cdot,\pm T)=0$, $\partial_t y_i(\cdot,\pm T)=0$ and $\nabla y_i(\cdot,\pm T)=0$ in Ω. So we have for s sufficiently large,

$$\sum_{i=1}^{2}(I(y_i)+I(z_i)) \le C\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2\big)\,\mathrm{d}x\,\mathrm{d}t + Ce^{2sd_1} + Cs^3\sum_{i=1}^{2}\int_{\omega\times(-T,T)}\big(|\nabla_{x,t}y_i|^2+|y_i|^2+|\nabla_{x,t}z_i|^2+|z_i|^2\big)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t. \quad (36)$$

Now we remove the observation term on z_i. From the first equations in (34) and (35) we have

$$\beta z_i = \partial_t^2 y_i - \Delta y_i - \alpha y_i - a\partial_t^i(\eta\widetilde{u}) - b\partial_t^i(\eta\widetilde{v}) - \partial_t^i R_1 \text{ in } Q. \quad (37)$$

From hypothesis (10) and (37) we get

$$\sum_{i=1}^{2}\int_{\omega\times(-T,T)}\big(|\nabla_{x,t}z_i|^2+|z_i|^2\big)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t \le Ce^{2sd_2}\|y_0\|^2_{H^5(0,T,H^3(\omega))} + \int_{\omega\times(-T,T)} e^{2s\phi}\big(a^2+b^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t + Ce^{2sd_1}. \quad (38)$$

(Note that if we suppose that $\alpha=\widetilde{\alpha}$ and $\beta=\widetilde{\beta}$ in ω as in the case i), then (38) becomes $\sum_{i=1}^{2}\int_{\omega\times(-T,T)}(|\nabla_{x,t}z_i|^2+|z_i|^2)e^{2s\phi}\,\mathrm{d}x\,\mathrm{d}t \le Ce^{2sd_2}\|y_0\|^2_{H^5(0,T,H^3(\omega))}+Ce^{2sd_1}$.)

So from (33) and (38) we have

$$\sum_{i=1}^{2}(I(y_i)+I(z_i)) \le Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t + Cs^3e^{2sd_1} + Cs^3e^{2sd_2}F_0(\omega) \quad (39)$$

with $F_0(\omega)=\|y_0\|_{H^5(0,T,H^3(\omega))}$ (same definition as before).

Notice that contrary to Theorem 4 we do not have terms in ∇c and ∇d in the above estimate because, due to the truncature function η, we do not have to estimate the terms for $t=\pm T$ in (7).

Third step. We estimate $\int_{\Omega_L} e^{2s\phi(x,0)}(|y_2(x,0)|^2+|z_2(x,0)|^2+|\nabla y_2(x,0)|^2)\,\mathrm{d}x$ as in Theorem 4. Since we have no terms in ∇c and ∇d in (39) we no longer need to estimate $\int_\Omega e^{2s\phi(x,0)}|\nabla z_2(x,0)|^2\,\mathrm{d}x$. We get

$$\int_\Omega e^{2s\phi(x,0)}\big(|y_2(x,0)|^2+|z_2(x,0)|^2+|\nabla y_2(x,0)|^2\big)\,\mathrm{d}x \le Cs^3e^{2sd_1}+Cs^3e^{2sd_2}F_0(\omega)+Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t. \quad (40)$$

Fourth step. We estimate $\int_\Omega e^{2s\phi(x,0)}(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2)\,\mathrm{d}x$ as in Theorem 4.

Thus we get

$$\int_\Omega e^{2s\phi(x,0)}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x \le Cs^3e^{2sd_1}+Cs^3e^{2sd_2}F_1(\omega)+Cs^3\int_Q e^{2s\phi}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x\,\mathrm{d}t \quad (41)$$

with $F_1(\omega)=F_{0A}(\omega)+F_{0B}(\omega)$. Notice that as in Theorem 4 the term $s^3\int_Q e^{2s\phi}(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2)\,\mathrm{d}x\,\mathrm{d}t$ can be absorbed by the left-hand side of the estimate for s sufficiently large ($s\ge s_2$), so we get

$$\int_\Omega e^{2s\phi(x,0)}\big(a^2+b^2+c^2+d^2+|\nabla a|^2+|\nabla b|^2\big)\,\mathrm{d}x \le Cs^3e^{2sd_1}+Cs^3e^{2sd_2}F_1(\omega).$$

Since $e^{2sd_0} \le e^{2s\phi(\cdot,0)}$ we deduce that

$$\|\alpha-\widetilde{\alpha}\|^2_{H^1(\Omega)}+\|\beta-\widetilde{\beta}\|^2_{H^1(\Omega)}+\|\gamma-\widetilde{\gamma}\|^2_{L^2(\Omega)}+\|\delta-\widetilde{\delta}\|^2_{L^2(\Omega)} \le Cs^3\left(e^{2s(d_2-d_0)}F_1(\omega)+e^{2s(d_1-d_0)}\right). \quad (42)$$

As $d_1-d_0<0$ and $d_2-d_0>0$, we can optimize the above inequality with respect to s (see for example [7,9,11]). Indeed, note that if $F_1(\omega)=0$, since (42) holds for any $s\ge s_2$ and $d_1-d_0<0$ we get (32). Now if $F_1(\omega)\neq 0$ is sufficiently small ($F_1(\omega)<\frac{d_0-d_1}{d_2-d_0}$), we optimize (42) with respect to s. Indeed denote

$$f(s)=e^{2s(d_2-d_0)}F_1(\omega)+e^{2s(d_1-d_0)}.$$

Moreover the function f has a minimum in

$$s_3=\frac{1}{2(d_2-d_1)}\ln\left(\frac{d_0-d_1}{(d_2-d_0)F_1(\omega)}\right) \text{ and } f(s_3)=K'F_1(\omega)^{\kappa}$$

with $\kappa=\frac{d_0-d_1}{d_2-d_1}$ and $K'=\left(\frac{d_0-d_1}{d_2-d_0}\right)^{\frac{d_2-d_0}{d_2-d_1}}+\left(\frac{d_0-d_1}{d_2-d_0}\right)^{\frac{d_1-d_0}{d_2-d_0}}$. Finally the minimum s_3 is sufficiently large ($s_3\ge s_2$) if the following condition $F_1(\omega)\le\frac{d_0-d_1}{(d_2-d_0)e^{2s_2(d_2-d_1)}}$ is satisfied.

So we conclude for Theorem 7. □

Remark 8. If we consider the case (i) in Theorem 7 where the coefficients α and β are supposed known in ω, then there is no term in ∇a nor ∇b in (38) and therefore in the estimate (39) of $\sum_{i=1}^2(I(y_i)+I(z_i))$. Thus we need not to differentiate (25) with respect to the space variable x_i and so to evaluate $\int_\Omega e^{2s\phi(x,0)}|\nabla y_2(x,0)|^2\,\mathrm{d}x$. Consequently there is no term in ∇a nor ∇b on the right-hand sides of all the estimates and we just get an estimate of the L^2-norms of a and b, besides the L^2-norms of c and d. Moreover in the fourth step when we estimate the coefficients a,b,c,d, we just have to estimate $\int_\Omega e^{2s\phi(x,0)}(|y_2(x,0)|^2+|z_2(x,0)|^2)\,\mathrm{d}x=\int_\Omega e^{2s\phi(x,0)}(|\partial_t y_1(x,0)|^2+|\partial_t z_1(x,0)|^2)\,\mathrm{d}x$. Therefore in the case (i) of Theorem 7, we just need to evaluate $I(y_1)+I(z_1)$ and not $\sum_{i=1}^2(I(y_i)+I(z_i))$. This explains why the observation terms on the right-hand side of (31) are given in $H^4(0,T,H^3(\omega))$-norms instead of $H^5(0,T,H^3(\omega))$-norms.

Remark 9. Last, notice that in the third step of each theorem we could have estimated $\int_\Omega e^{2s\phi(x,0)}(|y_2(x,0)|^2+|z_2(x,0)|^2)\,\mathrm{d}x$ in another way using the following lemma (see [13, Lemma 4.2])

$$\int_\Omega e^{2s\phi(x,0)}|f(x,0)|^2\,\mathrm{d}x\le Cs\int_Q e^{2s\phi}|f|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{C}{s}\int_Q e^{2s\phi}|\partial_t f|^2\,\mathrm{d}x\,\mathrm{d}t$$

for all s sufficiently large and $f\in H^1(-T,T;L^2(\Omega))$. Indeed

$$\begin{aligned}\int_\Omega & e^{2s\phi(x,0)}(|y_2(x,0)|^2+|z_2(x,0)|^2)\,\mathrm{d}x\\ &\le Cs\int_Q e^{2s\phi}(|y_2|^2+|z_2|^2)\,\mathrm{d}x\,\mathrm{d}t+\frac{C}{s}\int_Q e^{2s\phi}(|\partial_t y_2|^2+|\partial_t z_2|^2)\,\mathrm{d}x\,\mathrm{d}t\le\frac{C}{s^2}(I(y_2)+I(z_2)).\end{aligned}$$

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