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Partial differential equations / *Équations aux dérivées partielles*

Blow-up solutions to the semilinear wave equation with overdamping term

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Abstract. This article deals with the Cauchy problem to the following damped wave equation

$$\begin{cases} u_{tt} - \Delta u + b(t) u_t = M(u), & (t,x) \in R^+ \times R^N, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in R^N, \end{cases} \tag{CP}$$

with the focusing nonlinearity $M(u) = |u|^{p-1}u,\ p > 1$. For the focusing nonlinearity $M(u) = \pm|u|^p,\ p > 1$, Ikeda and Wakasugi in [8] have showed that the solution to Problem (CP) exists globally for small data and fails to exist globally for large data. Meanwhile, they also proposed an open problem [8, Remark 1.3]. In this note, we give a positive answer to this open problem by using a method different from the test-function method. In addition, an inverse Hölder inequality associated with the solution and a differential inequality argument are used to establish a lower bound for the blow-up time.

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1. Introduction

In this paper, we consider Cauchy problem for the following wave equation

$$\begin{cases} u_{tt} - \triangle u + b(t) u_t = M(u), & (t,x) \in R^+ \times R^N, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), & x \in R^N, \end{cases} \tag{1}$$

where the coefficient of damping satisfies

$$\begin{aligned} & b^{-1}(t) \in L^1(0,\infty), \quad \text{i.e.,} \quad \int_0^\infty b^{-1}(t)\,\mathrm{d}t < \infty, \\ & b(t) > 0, t \geq 0, b(t) \in C^1[0,+\infty), \end{aligned} \tag{2}$$

and the following technical condition

$$4(1+t)b'(t) + N(p-1)b(t) \geq 0, \quad t \geq 0. \tag{3}$$

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The nonlinear term $M(u)$ satisfies the following growth condition

$$M(0)=0, \quad \text{and} \quad |M(u)-M(v)| \le C_M(1+|u|+|v|)^{p-1}|u-v|, \quad \forall\, u,v \in R, \tag{4}$$

where C_M is a positive constant and

$$1<p<\infty \quad (N=1,2), \qquad 1<p<\frac{N}{N-2} \quad (N\ge 3). \tag{5}$$

It is well known that damped wave equations may model real physical problems such as the voltage and the current on an electrical transmission line with a resistance. Generally speaking, the damping term prevents the motion of the wave and reduces its energy, while the presence of the source term drives the equation towards instability. Thus, when both of damping and source terms are present, the analysis of their interaction becomes more interesting. For example, Ikeda–Wakasugi in [8] discussed the existence and nonexistence of global solutions to (1) when the coefficient $b(t)$ satisfies (2), in which case the damping term is called as the overdamping. In the opposite case $b^{-1}(t) \not\in L^1(0,+\infty)$, the damping term is effective or noneffective, there is also a large amount of literature about global existence, asymptotic behavior and blow-up phenomena [3,11] and the references therein. In addition, for semilinear wave equations, another interesting issue is to determine the critical exponent p_c. That is, if $p > p_c$, then Problem (1) admits a global-in-time solution for small data, while, for $p \le p_c$, the solution will blow up in finite time for large data. Some related references may be found in [5,14]. However, this is without the scope of the paper because the main aim of this note is to discuss how the initial data and the coefficient $b(t)$ have an effect on the behavior of solutions. In the following, we firstly state the local existence theorem and useful lemmas without their proofs because these proofs are standard, the interested readers may refer to [8].

Theorem 1 ([8]). *Let* $(u_0,u_1) \in H^1(R^N)\times L^2(R^N)$ *and* (2)–(5) *hold. Then there exists a solution* $u \in C([0,T);H^1(R^N)) \cap C^1([0,T);L^2(R^N))$ *to* (1) *for some* T.

Further, if $\operatorname{supp}(u_0,u_1) \subset \{x\in R^N : |x|\le L\} := B_L (L>1)$, *then*

$$\operatorname{supp} u(t,\cdot) \subset \left\{x\in R^N : |x| \le L+t\right\} := B_{L+t}.$$

For the sake of simplicity, we define

$$E(t)=\frac{1}{2}\|u_t\|^2_{L^2(R^N)}+\frac{1}{2}\|\nabla u(t)\|^2_{L^2(R^N)}-\int_{R^N}\int_0^u M(s)\mathrm{d}s\mathrm{d}x.$$

Then

Lemma 2 ([8]). *Under all the assumptions of Theorem 1, the energy functional* $E(t)$ *is non-increasing with respect to time variable. Namely*

$$E'(t)=-b(t)\|u_t\|^2_{L^2(R^N)}, \text{ for } t>0,$$

which implies that

$$E(t)-E(0)=-\int_0^t b(s)\|u_s\|^2_{L^2(R^N)}\mathrm{d}s. \tag{6}$$

Remark 3. Actually, it is easy to see (6) shows that the set

$$\left\{(u,u_t)\in H^1(R^N)\times L^2(R^N) : E(t)\le 0,\ t\ge 0\right\}$$

is invariant under the semi-flow of Problem (1) which will come into play to finish our proof.

Ikeda–Wakasugi in [8] applied the test-function method to give some sufficient conditions of blow-up solutions for the focusing nonlinearity $M(u)=\pm|u|^p$. However, for another case $M(u)=|u|^{p-1}u$, Ikeda–Wakasugi proposed an open problem:

> "**Remark 1.3.** For another focusing nonlinearity $M(z)=|z|^{p-1}z$, we expect similar results as Theorem 1.4 and 1.5 hold, while they still remain open."

In fact, when we check their proofs step by step, it seems that some methods used in [8] can not be applied to our problem because the nonlinearity $M(u)$ may change the sign. In this note, our strategy for getting around this difficulty is to construct an explicit sub-solution and apply comparison principle of second order differential inequalities [11, 15] to end our proof. More precisely, our first main result is as follows:

Theorem 4. *Suppose that all the conditions of Theorem 1,* (3) *and* (14) *are fulfilled and* $M(u) = |u|^p u$*, then the solution of Problem* (1) *blows up at some finite time* T^* *in the sense that*

$$\lim_{t\to T^{*-}} \|u\|_{L^2(R^N)} = +\infty,$$

if the following condition holds

$$E(0) \le 0, \int_{R^N} u_0(x)u_1(x)\mathrm{d}x > 0.$$

Example 5. The hypotheses of Theorem 4 hold for the following function $b(t)$ on the interval $[0,\infty)$

(i) $b(t) = t^2 \sin t + \dfrac{2t^3}{6+(p-1)N} + \dfrac{(6+pN-N)^2}{27}, \ t \ge 0;$

(ii) $b(t) = b_0(1+t)^{-\beta}, \ \beta < -1, \ b_0 > 0;$

(iii) $b(t) = \dfrac{t(1+t)^k}{\ln(1+t)}, \ k > 1.$

Remark 6. Actually, if $b(t)$ satisfies (ii), Nishihara in [12] also obtained similar results. Therefore, we find that the nonnegativity of $b'(t)$ plays an important role in his corresponding proof. Naturally, a question arises: are similar results valid if the derivative of $b(t)$ changes the sign? To give a positive or negative answer to this problem, one has to invent some methods.

Another problem is whether there exists a function $b(t)$ satisfying (2) and (3) as well as the uncertainty of the sign of $b'(t)$ or not. In fact, the following remark answers this problem.

Remark 7. If $b(t)$ is chosen as (i) of Example 5, then it is not difficult to verify that (2) and (3) are true, but the sign of

$$b'(t) = 2t\sin t + t^2 \cos t + \frac{6t^2}{6+Np-N}$$

is uncertain. In this note, we give more general conditions of $b(t)$ which guarantee that the solution blows up in finite time.

The second theorem is devoted to giving the lower bound for blow-up time T^*.

Theorem 8. *If all the conditions of Theorem 4 and the condition*

$$1 < p < \begin{cases} 1+\frac{4}{N}, & N = 3; \\ 1+\frac{2}{N-2}, & N \ge 4, \end{cases}$$

hold, then the blow-up time T^* *satisfies*

$$T^* \ge \int_{\psi(0)}^{\infty} \frac{1}{C_1\xi^\sigma + \xi}\mathrm{d}\xi, \quad \sigma = \frac{4(N+2p-Np)}{N+4-Np} > 1,$$

where $\psi(0)$*,* C_1 *are defined in* (19).

2. Proofs of our main results

In this section, we borrow comparison principle of second-order differential inequality and energy estimate method to complete our proof.

Proof of Theorem 1. We will divide this proof into some steps.

Step 1. Establish second-order differential inequality. Define $\psi(t) = \frac{1}{2}\int_{R^N}|u(x,t)|^2\mathrm{d}x$. Then $u \in C(0,T;L^2)$, $u_t \in C^1(0,T;L^2)$ allows us to make a direct computation

$$\psi'(t) = \int_{R^N} u(x,t)u_t(x,t)\mathrm{d}x, \tag{7}$$

$$\begin{aligned}\psi''(t) &= \int_{R^N} u(x,t)u_{tt}(x,t)\mathrm{d}x + \int_{R^N}|u_t|^2\mathrm{d}x\\ &= \int_{R^N}|u_t|^2\mathrm{d}x - \int_{R^N}|\nabla u|^2\mathrm{d}x - b(t)\int_{R^N}u_t u\mathrm{d}x + \int_{R^N}|u|^{p+1}\mathrm{d}x\\ &= 2\int_{R^N}|u_t|^2\mathrm{d}x - 2E(t) + \frac{p-1}{p+1}\int_{R^N}|u|^{p+1}\mathrm{d}x - b(t)\int_{R^N}u_t u\mathrm{d}x.\end{aligned}$$

This becomes

$$\psi''(t) + b(t)\psi'(t) = 2\int_{R^N}|u_t|^2\mathrm{d}x - 2E(t) + \frac{p-1}{p+1}\int_{R^N}|u|^{p+1}\mathrm{d}x. \tag{8}$$

In addition, with the help of Hölder inequality and Theorem 1, we have

$$\int_{B_{t+L}}|u|^{p+1}\mathrm{d}x \ge L^{-\frac{(p-1)N}{2}}(1+t)^{-\frac{(p-1)N}{2}}\psi^{\frac{p+1}{2}}(t). \tag{9}$$

Moreover, together with (9) and the fact $E(t) \le E(0) \le 0$, (8) may be rewritten as the following

$$b^{-1}(t)\psi''(t) + \psi'(t) \ge C_1 h(t)\psi^{\frac{p+1}{2}}(t), \tag{10}$$

where

$$C_1 = \frac{p-1}{p+1}L^{-\frac{(p-1)N}{2}}, \quad h(t) = (1+t)^{-\frac{(p-1)N}{2}}b^{-1}(t). \tag{11}$$

Step 2. Construct blow-up sub-solution. Due to the nonlinearity of differential inequality, in general, an explicit solution to the differential inequality (10) is hardly solved. To this end, we construct an explicit sub-solution to compare with the solution to (10). Set

$$\varphi(t) = \left(\psi^{-\frac{p-1}{4}}(0) - \varepsilon\int_0^t h(\tau)\mathrm{d}\tau\right)^{-\frac{4}{p-1}}, \quad 0 < t \le T^*,$$

where the positive constants ε, T^* will be determined later to make the function $\varphi(t)$ be well-defined. Indeed, a simple computation shows that

$$\begin{aligned}b^{-1}(t)\varphi''(t) + \varphi'(t) &\le \frac{4\varepsilon^2(p+3)}{(p-1)^2}\frac{h^2(t)}{b(t)}\varphi^{\frac{p+1}{2}}(t) + \frac{4\varepsilon}{p-1}\frac{h'(t)}{b(t)}\varphi^{\frac{p+3}{4}}(t) + \frac{4\varepsilon h(t)}{p-1}\varphi^{\frac{p+3}{4}}(t)\\ &= \left[\frac{4\varepsilon^2(p+3)}{(p-1)^2}\frac{h^2(t)}{b(t)} + \frac{4\varepsilon h(t)}{p-1}\varphi^{\frac{1-p}{4}}(t)\right]\varphi^{\frac{p+1}{2}}(t) + \frac{4\varepsilon}{p-1}\frac{h'(t)}{b(t)}\varphi^{\frac{p+3}{4}}(t)\\ &\le \left[\frac{4\varepsilon^2(p+3)}{(p-1)^2 b^2(0)} + \frac{4\varepsilon}{p-1}\varphi^{\frac{1-p}{4}}(0)\right]h(t)\varphi^{\frac{p+1}{2}}(t), \quad \text{for } 0 < t < T^*.\end{aligned} \tag{12}$$

Here we have used the condition (3) and the following facts

$$\begin{aligned}&\frac{h'(t)}{b(t)} = \frac{-h(t)}{b(t)}\left[\frac{b'(t)}{b(t)} + \frac{(p-1)N}{2(1+t)}\right] \le 0,\\ &\frac{h(t)}{b(t)} \le \frac{1}{b^2(0)}, \quad \varphi(t) \ge \varphi(0) \ge \psi(0), \quad \text{for } t > 0.\end{aligned}$$

Step 3. Establish the constants ε, T^*. First, to choose

$$\varepsilon=\varepsilon_0=\min\left\{1,\frac{2^{\frac{p-7}{4}}(p-1)b(0)\int_{R^N}u_0u_1\mathrm{d}x}{(\int_{R^N}|u_0|^2\mathrm{d}x)^{\frac{p+3}{4}}},\frac{2^{\frac{p-7}{4}}(p-1)^3L^{-\frac{(p-1)N}{2}}}{(p+3)(p+1)+(p^2-1)(\int_{R^N}|u_0|^2\mathrm{d}x)^{\frac{1-p}{4}}}\right\},$$

and then to apply (10) and (12) yield

$$\begin{cases} b^{-1}(t)\varphi''(t)+\varphi'(t)\le C_1h(t)\varphi^{\frac{p+1}{2}}(t), & \text{for } 0<t<T^*;\\ b^{-1}(t)\psi''(t)+\psi'(t)\ge C_1h(t)\psi^{\frac{p+1}{2}}(t), & \text{for } 0<t<T^*;\\ \varphi(0)=\psi(0)=\frac{1}{2}\int_{R^N}|u_0(x)|^2\mathrm{d}x, \quad \varphi'(0)<\psi'(0)=\int_{R^N}u_0(x)u_1(x)\mathrm{d}x. \end{cases} \tag{13}$$

Next, we estimate the value of T^* such that the function $\varphi(t)$ is well defined. Set

$$F(t)=\psi^{-\frac{p-1}{4}}(0)-\varepsilon\int_0^t h(\tau)\mathrm{d}\tau.$$

On the one hand, it is not hard to verify that $F(t)$ satisfies

$$F(0)=\psi(0)>0,\ F'(t)=-\varepsilon h(t)\le 0.$$

On the other hand, if we assume that

$$\left[\int_0^\infty(1+t)^{-\frac{(p-1)N}{2}}b^{-1}(t)\mathrm{d}t\right]^{-1}<\psi^{\frac{p-1}{4}}(0)\varepsilon_0, \tag{14}$$

then $\lim_{t\to+\infty}F(t)$ is negative. So, with the aid of the intermediate value theorem, we know that there exists a unique constant $0<T^*<+\infty$ satisfying $F(T^*)=0$, which indicates that the function $\varphi(t)$ works well.

Step 4. Blow-up in finite time. Now, we prove that $\lim_{t\to T^{*-}}\psi(t)=+\infty$. In fact, according to comparison principle of Lemma 3.1 in [11], we have $\psi(t)\ge\varphi(t)$, for $0<t<T^*$. This completes the proof of Theorem 4. □

At last, we complete the proof of Theorem 8.

Proof of Theorem 8. To show this, we need to build up first order differential inequality associated with the functional $\psi(t)$. In fact, to follow the proof of Inequality (7) and use the well-known inequality $ab\le\frac{a^2}{2}+\frac{b^2}{2}$ and the definition of $E(t)$ as well as (6) yield

$$\psi'(t)\le\int_{R^N}|u_t|^2\mathrm{d}x+\int_{R^N}|u|^2\mathrm{d}x\le\int_{R^N}|u|^{p+1}\mathrm{d}x+\int_{R^N}|u|^2\mathrm{d}x. \tag{15}$$

Subsequently, we need to use $\psi(t)$ to control the term $\int_{R^N}|u|^{p+1}\mathrm{d}x$. Therefore, due to the failure of the embedding inequality $\|u\|_{L^{p+1}(R^N)}\le C\|\nabla u\|_{L^2(R^N)}(C>0)$ for p satisfying (5), it is not possible to apply some methods used in [13] to obtain similar controlled relations between $\psi(t)$ and the source. To show this, we will make use of interpolation inequality and (6) to establish an inverse Hölder inequality. Actually, the well-known Gagliardo–Nirenberg inequality in [1, Corollary 9.14] or [4, Theorem 7.10] shows that there exists a positive constant C depending on $|\Omega|, p, N$ such that

$$\int_{R^N}|u|^{p+1}\mathrm{d}x\leqslant C\left(\int_{R^N}|u|^2\mathrm{d}x\right)^{\frac{(1-\theta)(p+1)}{2}}\left(\int_{R^N}|\nabla u|^2\mathrm{d}x\right)^{\frac{\theta(p+1)}{2}}, \tag{16}$$

where

$$\frac{1}{p+1}=\frac{1-\theta}{2}+\frac{\theta}{2^*},\quad 2^*=\frac{2N}{N-2}(N\geqslant 3).$$

Keeping $0 < \frac{\theta(p+1)}{2} < 1$ in mind and collecting Young inequality and (16), we have

$$\int_{R^N} |u|^{p+1}\mathrm{d}x \le C\varepsilon^{\frac{2}{\theta p-\theta-2}} \left(\int_{R^N} |u|^2\mathrm{d}x\right)^{\frac{(1-\theta)(p+1)}{(2-\theta p-\theta)}} + C\varepsilon^{\frac{2}{\theta p+\theta}} \int_{R^N} |\nabla u|^2\mathrm{d}x.$$

Further, choosing $C\varepsilon^{\frac{2}{\theta p+\theta}} = 1$ and using energy identity (6), we obtain the following inverse Hölder inequality

$$\int_{R^N} |u|^{p+1}\mathrm{d}x \le \left(\int_{R^N} |u|^2\mathrm{d}x\right)^{\frac{4(N+2p-Np)}{N+4-Np}}. \tag{17}$$

Finally, together with (15) and (17), we have

$$T^* \ge \int_{\psi(0)}^{\infty} \frac{1}{C_1\xi^\sigma + \xi}\mathrm{d}\xi, \tag{18}$$

where

$$\sigma = \frac{4(N+2p-Np)}{N+4-Np} > 1, \quad C_1 = \frac{p-1}{p+1}C^{\frac{4}{4+N-Np}}, \quad \psi(0) = \frac{1}{2}\int_{R^N} |u_0|^2\mathrm{d}x. \tag{19}$$

□

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