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# The spectrality of symmetric additive measures

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**Abstract.** Let $\rho$ be a symmetric measure of Lebesgue type, i.e.,

$$\rho = \frac{1}{2}(\mu \times \delta_0 + \delta_0 \times \mu),$$

where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$ for $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$ and $\delta_0$ is the Dirac measure at 0. We prove that $\rho$ is a spectral measure if and only if $t \in \frac{1}{2}\mathbb{Z}$. In this case, $L^2(\rho)$ has a unique orthonormal basis of the form

$$\left\{e^{2\pi i(\lambda x - \lambda y)} : \lambda \in \Lambda_0\right\},$$

where $\Lambda_0$ is the spectrum of the Lebesgue measure supported on $[-t-1, -t] \cup [t, t+1]$. Our result answers some questions raised by Lai, Liu and Prince [JFA, 2021].

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## 1. Introduction

Let $\mu$ be a Borel probability measure on $\mathbb{R}^n$. We call $\mu$ a spectral measure if there exists a countable discrete set $\Lambda \subset \mathbb{R}^n$ such that $E(\Lambda) := \{e^{2\pi i\langle \lambda, x\rangle} : \lambda \in \Lambda\}$ forms an orthogonal basis for the Hilbert space $L^2(\mu)$, and the $\Lambda$ is called the spectrum of $\mu$. The research about spectral measures was initiated by Fuglede [7], whose famous conjecture claimed that $\Omega$ is a spectral set on $\mathbb{R}^n$ if and only if $\Omega$ is a translational tile on $\mathbb{R}^n$. Although the conjecture is disproved in dimension three or higher by Tao et al. [9, 13, 15], it is still an open problem in dimension 1 and 2. In 1998, Jorgensen and Pedersen [8] found the first singular and non-atomic spectral measure (1/4-Cantor measure), and

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the research about spectral measures was developed toward the field of fractal by this discovery. For the details and recent advances, one can refer to [2–4] and so on.

Besides spectral measures, people are also interested in looking for measures that admit exponential frames (also called Fourier frames) and exponential Riesz bases [1, 5, 6, 14]. In 2018, Lev [12] studied the addition of two measures supported respectively on two orthogonal subspaces embedded in the ambient space $\mathbb{R}^n$ and showed that these measures admit Fourier frames. Recently, Lai, Liu and Prince [10] studied the Riesz bases and orthogonal bases for these measures. In this paper, we continue the line of research into the spectrality of the addition of measures supported on two orthogonal subspaces and extend some results of Lai, Liu and Prince [10].

Recall that a Borel measure $\mu$ on $\mathbb{R}$ is continuous if $\mu(\{x\}) = 0$ for all $x \in \mathbb{R}$. Let $\mu$ and $\nu$ be two continuous Borel probability measures on $\mathbb{R}$. We embed them into the $x$ and $y$ axes in $\mathbb{R}^2$ respectively. The additive space for $\mu$ and $\nu$ is the space $L^2(\rho)$, where $\rho$ is the measure

$$\rho = \frac{1}{2}(\mu \times \delta_0 + \delta_0 \times \nu), \tag{1}$$

and $\delta_0$ is the Dirac measure at 0. We will refer to the compact support of $\mu$ and $\nu$ as the component spaces of the measure $\rho$. If $\mu = \nu$, we say that $\rho$ is symmetric. If $0 \notin \operatorname{supp}(\mu) \cap \operatorname{supp}(\nu)$, we call $\rho$ non-overlapping (Here, $\operatorname{supp}(\mu)$ denotes the compact support of $\mu$). If $\mu, \nu$ are Lebesgue measures supported on intervals of length one, we call $\rho$ the additive space of Lebesgue type, the additive space is defined as $L^2(\rho)$.

Recently, Lai, Liu and Prince [10] proved the following partial results.

**Theorem A ([10, Theorem 1.3]).** *Let $\rho$ be a symmetric measure of Lebesgue type, where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$ and $-1/2 < t \le 0$.*

(1) *If $t = 0$, then $\rho$ is spectral and has a unique spectrum up to translations.*
(2) *If $t = -\frac{1}{2} + \frac{1}{2a}$, where $a > 1$ is a positive integer, then $\rho$ is not spectral.*

The authors leave many questions in [10]. In this paper, we obtain a sufficient and necessary condition for $\rho$ to be spectral under the assumption that $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$.

**Theorem 1.** *Let $\rho$ be a symmetric measure of Lebesgue type, where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$ and $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$. Then $\rho$ is a spectral measure if and only if $t \in \frac{1}{2}\mathbb{Z}$. In this case, $L^2(\rho)$ has a unique orthonormal basis of the form*

$$\left\{e^{2\pi i(\lambda x - \lambda y)} : \lambda \in \Lambda_0\right\},$$

*where $\Lambda_0$ is the spectrum of the Lebesgue measure supported on $[-t-1, -t] \cup [t, t+1]$.*

The proof depends on the analysis of the so called Orthogonality Equation. We firstly prove that the spectrum of $\rho$ is contained in a straight line if $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$ (Proposition 8). Then we give an interesting lemma about the Orthogonality Equation (Lemma 7) and extend a result about lower Beurling density (Lemma 6). At last, we prove that $\rho$ is not spectral if $t \in \mathbb{Q} \setminus (\frac{1}{2}\mathbb{Z})$ by reduction to absurdity. Furthermore, we construct the spectrum of $\rho$ under the condition that $t \in \frac{1}{2}\mathbb{Z}$.

This paper is organized as follows. Section 1 is an introduction and we state our main result. Section 2 presents some preliminary results. Section 3 is devoted to prove Theorem 1. We conclude in Section 4 with some remarks and a conjecture.

## 2. Preliminaries

In this section, we introduce some preliminary definitions and basic results which are used in our proof.

Let $\rho$ be a symmetric measure of Lebesgue type, i.e.,

$$\rho = \frac{1}{2}(\mu \times \delta_0 + \delta_0 \times \mu),$$

where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$. We study exponential functions on additive spaces. Here exponential refers to functions mapping $\mathbb{R}^2 \to \mathbb{T} \subset \mathbb{C}$ of the form

$$(x, y) \to e^{2\pi i(ax+by)},$$

where $a, b \in \mathbb{R}$. We use $e_{a,b}$ as a shorthand for this function; similarly $e_a$ is the one-dimensional function $x \to e^{2\pi i ax}$. With exponential functions, we have $e_{a,b}(x,0) = e_a(x)$ and $e_{a,b}(0, y) = e_b(y)$. Hence

$$\left\langle e_{a,b}, e_{u,v}\right\rangle_{L^2(\rho)} = \frac{1}{2}\left\langle e_a, e_u\right\rangle_{L^2(\mu)} + \frac{1}{2}\left\langle e_b, e_v\right\rangle_{L^2(\mu)}$$

Note also that for $e_{a,b}$, the $x$ projection is $e_a$, and the $y$ projection is $e_b$. If $\Lambda \subset \mathbb{R}^2$ and $(a, b) \in \Lambda$, then the $x$ projection of $(a, b)$ is $a$ and the $y$ projection is $b$. $\Lambda_x$ is the set

$$\Lambda_x = \{a \in \mathbb{R} : (a, b) \in \Lambda \text{ for some } b \in \mathbb{R}\},$$

and similarly for $\Lambda_y$. We will use these observations frequently in the paper.

Let $E(\Lambda)$ be an orthogonal set of exponential functions with exponent in $\Lambda$ for $L^2(\rho)$ and $(a, b), (c, d)$ are any two distinct points of $\Lambda$. Let $(\lambda_1, \lambda_2) = (a, b) - (c, d)$, then $\lambda_1\lambda_2 \neq 0$ and

$$e^{\pi i(\lambda_1-\lambda_2)(2t+1)}\frac{\sin(\pi\lambda_1)}{\pi\lambda_1} = -\frac{\sin(\pi\lambda_2)}{\pi\lambda_2}. \tag{2}$$

The above equation is also called Orthogonality Equation in [10]. As the right side is real, if $\lambda_1, \lambda_2 \notin \mathbb{Z}$, we obtain $(\lambda_1 - \lambda_2)(2t + 1) \in \mathbb{Z}$. This fact will be used many times in this paper.

In the following, we give some useful results proved in [10].

**Definition 2 ([10, Definition 7]).** *Let $\Lambda$ be the set of exponents for a set of exponential functions on an additive space. The multiplicity of $\Lambda$ is the largest number of points on any vertical or horizontal line through $\Lambda$, if such a maximum exists. We say that $\Lambda$ has bounded multiplicity in this case. Similarly, the multiplicity of $u \in \Lambda_x$ (or $v \in \Lambda_y$) is the number of points on a vertical (or horizontal) line through $u$ (or $v$), if this number is finite.*

**Theorem 3 ([10, Theorem 4.1]).** *Let $E(\Lambda)$ be a frame for an additive space with measure $\rho$ and continuous component measures $\mu$ and $\nu$ defined in* (1)*. Then*

(i) *$\Lambda$ has bounded multiplicity;*
(ii) *$E(\Lambda_x)$ and $E(\Lambda_y)$ are frames for $L^2(\mu)$ and $L^2(\nu)$ respectively.*

**Theorem 4 ([10, Theorem 4.2]).** *Let $E(\Lambda)$ be an orthonormal basis for an additive space. Then*

(i) *$\Lambda$ has multiplicity one;*
(ii) *Suppose that $\mu$ and $\nu$ are Lebesgue measures supported on intervals of length one. Then $\Lambda$ cannot be a subset of $\mathbb{Z}^2$.*

For $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$, by the following lemma, we can assume that there are infinite integer points in $\Lambda$ if $E(\Lambda)$ is infinite.

**Lemma 5.** *Let $\rho$ be a symmetric measure of Lebesgue type, where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$ and $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$. Suppose that $E(\Lambda)$ is an infinite orthonormal set for $L^2(\rho)$. Then there exists $(u, v) \in \Lambda$ such that $\#((\Lambda - (u, v)) \cap \mathbb{Z}^2) = \infty$.*

**Proof.** Since $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$, we can write $2t + 1 = \frac{p}{q}$ with $\gcd(p, q) = 1$. Suppose that $\Lambda$, with $(0,0) \in \Lambda$, is an infinite orthogonal set for $\rho$. If $\#(\Lambda \setminus \mathbb{Z}^2) < \infty$, this implies that $\#(\Lambda \cap \mathbb{Z}^2) = \infty$, the lemma follows from taking $(u,v)=(0,0)$. In the following, we consider the case $\#(\Lambda\setminus\mathbb{Z}^2)=\infty$. Let $(\lambda_1,\lambda_2)\in\Lambda\setminus\mathbb{Z}^2$. Since $\langle e_{\lambda_1,\lambda_2},\ 1\rangle=0$, we have

$$e^{\pi i(\lambda_1-\lambda_2)\frac{p}{q}}\frac{\sin(\pi\lambda_1)}{\pi\lambda_1}=-\frac{\sin(\pi\lambda_2)}{\pi\lambda_2}.$$

As the right side is real and $\lambda_1,\lambda_2\notin\mathbb{Z}$, we obtain $(\lambda_1-\lambda_2)\frac{p}{q}=k_j\in\mathbb{Z}$. Let $(\lambda_1^{(j_1)},\lambda_2^{(j_1)}),(\lambda_1^{(j_2)},\lambda_2^{(j_2)})$ be any two distinct points of $\Lambda\setminus\mathbb{Z}^2$. Write

$$\left(\lambda_1^{(j_2)}-\lambda_2^{(j_2)}\right)\frac{p}{q}-\left(\lambda_1^{(j_1)}-\lambda_2^{(j_1)}\right)\frac{p}{q}=k_{j_2}-k_{j_1}\in\mathbb{Z},$$

by the Orthogonality Equation (2), we have

$$\frac{\sin\pi\left(\lambda_1^{(j_2)}-\lambda_1^{(j_1)}\right)}{\pi\left(\lambda_1^{(j_2)}-\lambda_1^{(j_1)}\right)}=(-1)^{k_{j_2}-k_{j_1}+1}\frac{\sin\pi\left(\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\right)}{\pi\left(\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\right)}.$$

Note that $\{k_j\}_{j=1}^{\infty}$ is a sequence of integers. Without loss of generality, we assume that $k_j=k_i$ (mod $p$) for any $j\neq i$. It follows that

$$\frac{\sin\pi\left(\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\right)}{\pi\left(\lambda_2^{(j_2)}-\lambda_2^{(j_1)}+\frac{(k_{j_2}-k_{j_1})q}{p}\right)}=(-1)^{k_{j_2}-k_{j_1}+1-\frac{q(k_{j_2}-k_{j_1})}{p}}\frac{\sin\pi\left(\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\right)}{\pi\left(\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\right)}.$$

Hence $\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\in\mathbb{Z}$, or $\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\notin\mathbb{Z}$ but

$$\left|\lambda_2^{(j_2)}-\lambda_2^{(j_1)}+\frac{(k_{j_2}-k_{j_1})q}{p}\right|=\left|\lambda_2^{(j_2)}-\lambda_2^{(j_1)}\right|.$$

This together with $k_{j_2}-k_{j_1}\in p\mathbb{Z}$ implies that $\lambda_2^{(j_2)}-\lambda_2^{(j_1)}=-\frac{(k_{j_2}-k_{j_1})q}{2p}\in\frac{1}{2}\mathbb{Z}$ for all $j_2>j_1$. Clearly, there exists a subsequence $\{n_k\}$ such that $\lambda_2^{(j_{n_k})}-\lambda_2^{(j_{n_1})}\in\mathbb{Z}$. In the same manner we can see that there exists a subsequence $\{n_k\}$ such that $\lambda_1^{(j_{n_k})}-\lambda_1^{(j_{n_1})}\in\mathbb{Z}$. Set $(u,v)=(\lambda_1^{(j_{n_1})},\lambda_2^{(j_{n_1})})$, the lemma is proved. □

Recall the Beurling density of countable sets. Let $\Lambda$ be a countable set in $\mathbb{R}^n$. For $r>0$, the lower Beurling density corresponding to $r$ (or $r$ -Beurling density) of $\Lambda$ is defined by the formula

$$D_r^-:=\liminf_{h\to\infty}\inf_{x\in\mathbb{R}^n}\frac{\#(\Lambda\cap B(x;h))}{h^r},$$

where $B(x;h)=\{y\in\mathbb{R}^n:\mathrm{dist}(x,y)<h\}$. If $r=1$, we write simply $D^-$. Then we give the following lemma.

**Lemma 6.** *Let $L,L'\subset\mathbb{Z}$ and $L\cap L'=\emptyset$. Suppose that $\alpha\neq 0$ and $\beta$ is a constant. Then the lower Beurling density of $\Lambda:=\alpha L\cup(-\alpha L'+\beta)$ is smaller than $\frac{1}{|\alpha|}$.*

**Proof.** Without loss of generality, we assume that $\alpha,\beta>0$. It follows that

$$\begin{aligned}D^-(\Lambda)&=\underline{\lim}_{h\to\infty}\inf_{x\in\mathbb{R}}\frac{\#(\Lambda\cap(x-\frac{1}{2}h,x+\frac{1}{2}h))}{h}\\&\le\underline{\lim}_{h\to\infty}\frac{\#(\Lambda\cap(-\frac{1}{2}h,\frac{1}{2}h))}{h}\\&\le\underline{\lim}_{h\to\infty}\frac{h+\beta}{\alpha h}=\frac{1}{\alpha}.\end{aligned}$$

□

We end this section with a simple but useful lemma.

**Lemma 7.** *Let $0<|\alpha|<1$. If*

$$f(x_0):=\frac{\sin\alpha x_0}{\sin x_0}=\pm\alpha,$$

*then there exists a positive constant $\delta_\alpha$ such that $|x_0|>(1+\delta_\alpha)\pi$.*

**Proof.** Since $f(x)=\frac{\sin\alpha x}{\sin x}$ is an even function, we only need to consider the case $x>0$. A simple calculation gives

$$f'(x)=\frac{\alpha\cos\alpha x\sin x-\cos x\sin\alpha x}{\sin^2 x}.$$

Set $g(x)=\alpha\cos\alpha x\sin x-\cos x\sin\alpha x$, then for $x\in(0,\pi)$,

$$g'(x)=(1-\alpha^2)\sin\alpha x\sin x\begin{cases}<0 & (-1<\alpha<0),\\ >0 & (0<\alpha<1).\end{cases}$$

If $-1<\alpha<0$, then $g(x)$ is decreasing in $(0,\pi)$, i.e., $f'<0$. Then $f(x)<\lim_{x\to 0}f(x)=\alpha$ in $(0,\pi)$. Note that $f(x)$ is continuous in $(\pi,2\pi)$ and

$$\lim_{x\to\pi^+}f(x)=\lim_{x\to\pi^+}\frac{\sin\alpha x}{\sin\alpha\pi}\frac{\sin\alpha\pi}{\sin x}=\lim_{x\to\pi^+}\frac{\sin\alpha\pi}{\sin x}=+\infty.$$

We know that there exists $\delta_\alpha$ such that $f(x)>|\alpha|$ when $x\in(\pi,(1+\delta_\alpha)\pi)$. If there exists $x_0$ such that $f(x_0)\in\{\pm\alpha\}$, then $x_0\geq(1+\delta_\alpha)\pi$.

If $0<\alpha<1$, then $g(x)$ is increasing in $(0,\pi)$, i.e., $f'>0$. Then $f(x)>\lim_{x\to 0}f(x)=\alpha$ in $(0,\pi)$. Note that $f(x)$ is continuous in $(\pi,2\pi)$ and

$$\lim_{x\to\pi^+}f(x)=\lim_{x\to\pi^+}\frac{\sin\alpha x}{\sin\alpha\pi}\frac{\sin\alpha\pi}{\sin x}=\lim_{x\to\pi^+}\frac{\sin\alpha\pi}{\sin x}=-\infty.$$

We know that there exists $\delta_\alpha$ such that $|f(x)|>|\alpha|$ when $x\in(\pi,(1+\delta_\alpha)\pi)$. If there exists $x_0$ such that $f(x_0)\in\{\pm\alpha\}$, then $x_0\geq(1+\delta_\alpha)\pi$. □

## 3. Proof of Theorem 1

In this section, we will give the proof of Theorem 1. Firstly, unlike the case $t=-\frac{1}{2}$ which was proved by Lai, Liu and Prince in [10, Proposition 7.1], we prove that the points of spectrum of $\rho$ are contained in a straight line if $t\in\mathbb{Q}\setminus\{-\frac{1}{2}\}$.

**Proposition 8.** *If the symmetric measure of Lebesgue type $\rho$ is a spectral measure with the spectrum $\Lambda$ and $t\in\mathbb{Q}\setminus\{-\frac{1}{2}\}$, then the points of $\Lambda$ are contained in a straight line.*

**Proof.** Since $t\in\mathbb{Q}\setminus\{-\frac{1}{2}\}$, we can write $2t+1=\frac{p}{q}\in\mathbb{Q}\setminus\{0\}$ with $\gcd(p,q)=1$. By Lemma 5, we can assume that $\Lambda$, with $(0,0)\in\Lambda$, is the spectrum of $\rho$ and $\#(\Lambda\cap\mathbb{Z}^2)=\infty$. By Theorem 4 (ii), we know $\Lambda\setminus\mathbb{Z}^2\neq\emptyset$. For any $(\lambda_1,\lambda_2)\in\Lambda\setminus\mathbb{Z}^2$ and $(k_1,k_2)\in\Lambda\cap\mathbb{Z}^2$, the Orthogonality Equation (2) shows that

$$\frac{\sin(\pi\lambda_1)}{\pi\lambda_1}=(-1)^{l+1}\frac{\sin(\pi\lambda_2)}{\pi\lambda_2}\tag{3}$$

and

$$\frac{\sin\pi(k_1-\lambda_1)}{\pi(k_1-\lambda_1)}=(-1)^{k_{1,2}-l+1}\frac{\sin\pi(k_2-\lambda_2)}{\pi(k_2-\lambda_2)},\tag{4}$$

where $l=\frac{(\lambda_1-\lambda_2)p}{q}\in\mathbb{Z}$, $k_{1,2}=\frac{(k_1-k_2)p}{q}\in\mathbb{Z}$. This implies that $q|(k_1-k_2)$. Since $k_i\in\mathbb{Z}$ and $\sin\pi(k_i-\lambda_i)=(-1)^{k_i+1}\sin\pi\lambda_i$, by dividing (3) by (4), we have

$$(-1)^{k_{1,2}+k_1-k_2}\frac{k_1-\lambda_1}{k_2-\lambda_2}=\frac{\lambda_1}{\lambda_2},$$

i.e.,

$$\begin{cases}\frac{\lambda_1}{\lambda_2}=\frac{k_1}{k_2}, & k_{1,2}+k_1-k_2\in 2\mathbb{Z};\\ 2=\frac{k_1}{\lambda_1}+\frac{k_2}{\lambda_2}, & k_{1,2}+k_1-k_2\in 2\mathbb{Z}+1.\end{cases}$$

We can suppose that there exists $(k_1,k_2)\in\Lambda\cap\mathbb{Z}^2$ such that $k_{1,2}+k_1-k_2\in 2\mathbb{Z}$. If not, we can choose $\Lambda-(k_1,k_2)$ to be the new spectrum that we need.

If $k_{1,2}+k_1-k_2\in 2\mathbb{Z}$ for all $(k_1,k_2)\in\Lambda\cap\mathbb{Z}^2$, then the proposition follows from taking $\alpha=\frac{\lambda_1}{\lambda_2}$, and $\Lambda\subset\{(\alpha\lambda,\lambda):\lambda\in\mathbb{R}\}$. Furthermore, $\alpha\neq 0,1$.

If $k_{1,2}+k_1-k_2\in 2\mathbb{Z}+1$ for some $(k_1,k_2)\in\Lambda$, we firstly claim that $\#(\Lambda\setminus\mathbb{Z}^2)=1$. In fact, if $(\lambda_1,\lambda_2),(\lambda_1',\lambda_2')\in\Lambda\setminus\mathbb{Z}^2$ and $k_{1,2}+k_1-k_2\in 2\mathbb{Z}+1$, then

$$\begin{cases} 2=\frac{k_1}{\lambda_1}+\frac{k_2}{\lambda_2}, \\ 2=\frac{k_1}{\lambda_1'}+\frac{k_2}{\lambda_2'}. \end{cases} \tag{5}$$

Since there exists $(k_1',k_2')$ satisfying $k_{1,2}'+k_1'-k_2'\in 2\mathbb{Z}$, we obtain $\lambda_1/\lambda_2=\lambda_1'/\lambda_2'=k_1'/k_2'$. Take it into (5), hence $\lambda_1=\lambda_1',\lambda_2=\lambda_2'$. Then the claim follows. We now turn to prove that the case of $k_{1,2}+k_1-k_2\in 2\mathbb{Z}+1$ is impossible. Let $(\lambda_1,\lambda_2)\in\Lambda\setminus\mathbb{Z}^2$ and $k_{1,2}+k_1-k_2\in 2\mathbb{Z}+1$, $k_{1,2}'+k_1'-k_2'\in 2\mathbb{Z}$, we obtain that

$$\Lambda\subset\left\{(\lambda_1,\lambda_2),\ \left(-\frac{\lambda_1}{\lambda_2}k_2+2\lambda_1,k_2\right),\ \left(\frac{\lambda_1}{\lambda_2}k_2',k_2'\right)\right\}.$$

Applying Theorem 4, then the projection of $\Lambda$ on the $x$-axis is

$$\Lambda_x\subset\{\lambda_1\}\cup(-\alpha L+2\lambda_1)\cup\alpha L'$$

for some integer set $L\cap L'=\emptyset$ and $\alpha=\frac{\lambda_1}{\lambda_2}$. By Lemma 6, the lower Beurling density of $\Lambda_x$ is less than $\frac{1}{|\alpha|}$. Similarly,

$$\Lambda_y\subset\{\lambda_2\}\cup\left(-\frac{1}{\alpha}M+2\lambda_2\right)\cup\frac{1}{\alpha}M'$$

for some integer set $M\cap M'=\emptyset$. Therefore, the lower Beurling density of $\Lambda_y$ is less than $|\alpha|$. By Theorem 3 and Landau's theorem [11], one has $\alpha=\pm1$. But $\alpha=1$ and $\alpha=-1$ are both impossible. If $\alpha=1$, then $\lambda_1=\lambda_2$. This is in contradiction with (3). Assume $\alpha=-1$. Note that $k_2-k_1=-2\lambda_1\in\mathbb{Z}$ for some $(k_1,k_2)\in\Lambda\cap\mathbb{Z}^2$. This implies that $\lambda_1=\frac{lq}{2p}=\frac{l'q}{2}$ for some integer $l'=\frac{l}{p}$. Since $\lambda_1\notin\mathbb{Z}$ and by (3), we obtain that $q,l',l\notin 2\mathbb{Z}$. Since

$$k_{1,2}+k_1-k_2=(k_1-k_2)\frac{p+q}{q}\in 2\mathbb{Z}+1,$$

and $q|(k_1-k_2)$, we know $p$ is even. Then $l=pl'\in 2\mathbb{Z}$, a contradiction.

Hence, the proof is complete, i.e.,

$$\Lambda\subset\{(\alpha\lambda,\lambda):\lambda\in\mathbb{R}\},$$

where $\alpha=\frac{\lambda_1}{\lambda_2}$ for any $(\lambda_1,\lambda_2)\in\Lambda\setminus\mathbb{Z}^2$. Moreover, $\alpha\neq 0,1$. □

With the help of above preparations, we can prove our main result.

**Proof of Theorem 1.** Suppose that $\rho$ is a spectral measure with the spectrum $\Lambda$, and $(0,0)\in\Lambda$. According to Proposition 8, there exists $\alpha\neq 0,1$ such that $\lambda_1/\lambda_2\equiv\alpha$ for $(\lambda_1,\lambda_2)\in\Lambda$. If $t\in\mathbb{Q}\setminus\{-\frac{1}{2}\}$, we can write $2t+1=\frac{p}{q}\in\mathbb{Q}\setminus\{0\}$ with $\gcd(p,q)=1$. Without loss of generality, we only consider the case $p,q>0$.

We divide the following proof into four steps.

**Step 1. We prove that** $\alpha=-1$**.** We firstly claim that

$$\lambda_1=\frac{\alpha kq}{p(\alpha-1)},\quad \lambda_2=\frac{kq}{p(\alpha-1)},\quad k\in\mathbb{Z}. \tag{6}$$

In fact, by Theorem 4 (ii), we know $\Lambda\setminus\mathbb{Z}^2\neq\emptyset$. For any $(\lambda_1,\lambda_2)\in\Lambda\setminus\mathbb{Z}^2$, since $(0,0)\in\Lambda$, by the Orthogonality Equation (2), we have $(\lambda_1-\lambda_2)\frac{p}{q}\in\mathbb{Z}$. If $(\lambda_1',\lambda_2')\in\mathbb{Z}^2$, take $(\lambda_1-\lambda_1',\lambda_2-\lambda_2')$ into the Orthogonality Equation (2), we have

$$[\lambda_1-\lambda_1'-(\lambda_2-\lambda_2')]\frac{p}{q}=(\lambda_1-\lambda_2)\frac{p}{q}-(\lambda_1'-\lambda_2')\frac{p}{q}\in\mathbb{Z},\ \text{i.e.,}\ (\lambda_1'-\lambda_2')\frac{p}{q}\in\mathbb{Z}.$$

Hence, for any $(\lambda_1,\lambda_2)\in\Lambda$, we always have $(\lambda_1-\lambda_2)\frac{p}{q}\in\mathbb{Z}$. Write $\lambda_1=\lambda_2+\frac{kq}{p}$, $k\in\mathbb{Z}$, combining with $\lambda_1/\lambda_2\equiv\alpha$, (6) is obtained.

Let $(\lambda_1^{(j_1)},\lambda_2^{(j_1)}),(\lambda_1^{(j_2)},\lambda_2^{(j_2)})\in\Lambda$, where $\lambda_1^{(j_i)}=\lambda_2^{(j_i)}+\frac{k_{j_i}q}{p}$, $k_{j_i}\in\mathbb{Z}, i=1,2$. Combining with $\lambda_1^{(j_i)}/\lambda_2^{(j_i)}\equiv\alpha$, we set

$$\gamma=(\lambda_1^{(j_2)},\lambda_2^{(j_2)})-(\lambda_1^{(j_1)},\lambda_2^{(j_1)})=\left(\frac{q(k_{j_2}-k_{j_1})\alpha}{p(\alpha-1)},\frac{q(k_{j_2}-k_{j_1})}{p(\alpha-1)}\right):=(\gamma_x,\gamma_y).$$

Taking $\gamma$ into the Orthogonality Equation (2), we have

$$\sin(\pi\gamma_x)=(-1)^{k_{j_2}-k_{j_1}+1}\alpha\sin(\pi\gamma_y).$$

Since $\alpha\neq 0,1$, we know that both $\gamma_x,\gamma_y$ are integers or neither is an integer.

If $\gamma\in\mathbb{Z}^2$, one has $p|(k_{j_2}-k_{j_1})$. Then for $|\alpha|>0$,

$$|\gamma_x|=|\alpha\gamma_y|\geq\frac{|\alpha|q}{|\alpha-1|},\ \ |\gamma_y|\geq\frac{q}{|\alpha-1|}. \tag{7}$$

If $\gamma\notin\mathbb{Z}^2$, we obtain

$$\frac{\sin\pi\gamma_x}{\sin\pi\gamma_y}=(-1)^{k_{j_2}-k_{j_1}+1}\alpha. \tag{8}$$

From now on, we prove that $\alpha=-1$ by excluding three cases.

**Case I.** $q>1$ **and** $0<|\alpha|<1$**.** By (8) and Lemma 7, there exists a positive constant $\delta_\alpha$ such that

$$|\gamma_y|\geq 1+\delta_\alpha.$$

Combining with (7), we have

$$|\gamma_y|\geq\min\left\{1+\delta_\alpha,\frac{q}{|\alpha-1|}\right\}.$$

By Lemma 6, the lower Beurling density of $\Lambda_y$ satisfies

$$D^-(\Lambda_y)\leq\frac{1}{\min\left\{1+\delta_\alpha,\frac{q}{|\alpha-1|}\right\}}\leq\max\left\{\frac{1}{1+\delta_\alpha},\frac{|\alpha-1|}{q}\right\}<1.$$

**Case II.** $q>1$ **and** $|\alpha|>1$**.** Write (8) as

$$\frac{\sin(\pi\gamma_x\alpha^{-1})}{\sin(\pi\gamma_x)}=(-1)^{k_{j_2}-k_{j_1}+1}\alpha^{-1}.$$

Since $0<|\alpha^{-1}|<1$, by Lemma 7, there exists a positive constant $\delta_{\alpha^{-1}}$ such that

$$|\gamma_x|\geq 1+\delta_{\alpha^{-1}}.$$

Combining with (7), we have

$$|\gamma_x|\geq\min\left\{1+\delta_{\alpha^{-1}},\frac{|\alpha q|}{|\alpha-1|}\right\},$$

Similarly to Case I, we obtain

$$D^-(\Lambda_x)\leq\frac{1}{\min\left\{1+\delta_{\alpha^{-1}},\frac{|\alpha q|}{|\alpha-1|}\right\}}\leq\max\left\{\frac{1}{1+\delta_{\alpha^{-1}}},\frac{|\alpha-1|}{|\alpha q|}\right\}<1.$$

**Case III.** $q=1$ **and** $\alpha\neq-1$**.** In this case,

$$\gamma=(\gamma_x,\gamma_y)=\left(\frac{(k_{j_2}-k_{j_1})\alpha}{p(\alpha-1)},\frac{(k_{j_2}-k_{j_1})}{p(\alpha-1)}\right).$$

If $\gamma\in\mathbb{Z}^2$, since $\alpha\neq\pm1$, this implies that

$$|k_{j_2}-k_{j_1}|\geq p\max\{|\alpha-1|,|1-\alpha^{-1}|\}>p\min\{|\alpha-1|,|1-\alpha^{-1}|\}.$$

For $\gamma \notin \mathbb{Z}^2$, by Lemma 7, there exists a positive constant $\delta_\alpha (0<|\alpha|<1)$ or $\delta_{\alpha^{-1}}(|\alpha|>1)$ such that $|\gamma_y| \ge 1+\delta_\alpha$ or $|\gamma_x| \ge 1+\delta_{\alpha^{-1}}$. Denote

$$g(\alpha)=\begin{cases} \frac{1}{1+\delta_\alpha}, & 0<|\alpha|<1;\\ \frac{1}{1+\delta_{\alpha^{-1}}}, & |\alpha|>1.\end{cases}$$

Hence

$$\min\{D^-(\Lambda_x), D^-(\Lambda_y)\} \le \max\left\{g(\alpha), \frac{\min\{|\alpha-1|,|1-\alpha^{-1}|\}}{\max\{|\alpha-1|,|1-\alpha^{-1}|\}}\right\} < 1.$$

In a word, if $\alpha \ne -1$, we always have $D^-(\Lambda_x)<1$ or $D^-(\Lambda_y)<1$. Then Theorem 3 and Landau's theorem [11] tell us that $\Lambda$ is not a frame of $\rho$, a contradiction. Hence $\alpha=-1$.

**Step 2. We will give the expression of $\Lambda$.** Since $\alpha=-1$ and $(0,0)\in\Lambda$, for any nonzero $(\lambda,-\lambda)\in\Lambda$, by the Orthogonality Equation (2), we have

$$(e^{\pi i \frac{2\lambda p}{q}}+1)\sin(\pi\lambda)=0.$$

Hence, $\lambda\in\mathbb{Z}$ or $\frac{2\lambda p}{q}\in 2\mathbb{Z}+1$, i.e.,

$$\Lambda \subset \{(n,-n): n\in\mathbb{Z}\} \cup \left\{\left(\frac{qk}{2p}, -\frac{qk}{2p}\right): k\in 2\mathbb{Z}+1\right\}.$$

Let $\left(\frac{qk_i}{2p}, -\frac{qk_i}{2p}\right)\in\Lambda$, $k_i=ps_i+r_i\in 2\mathbb{Z}+1$ for some integers $s_i\ge 0, 0\le r_i\le p-1$. For any $k_1\ne k_2$, then $k_2-k_1\in 2\mathbb{Z}$. The Orthogonality Equation (2) tells us

$$\left(e^{\pi i(k_2-k_1)}+1\right)\sin\left(\pi\frac{q(k_2-k_1)}{2p}\right)=0,$$

this yields $\frac{q(k_2-k_1)}{2p}\in\mathbb{Z}$, i.e., $r_1=r_2, 2|q(s_2-s_1)$.

Let $(n,-n)\in\Lambda, \left(\frac{qk_i}{2p},-\frac{qk_i}{2p}\right)\in\Lambda$, and $n-\frac{qk_i}{2p}\notin\mathbb{Z}$. Taking $\left(n-\frac{qk_i}{2p}, -n+\frac{qk_i}{2p}\right)$ into the Orthogonality Equation (2), we have

$$\frac{2pn}{q}-k_i\in 2\mathbb{Z}+1, \text{ i.e., } n\in q\mathbb{Z}.$$

Note that we have actually proved that there exist some $\mathscr{S}_i\subset\mathbb{Z}, i=1,2$ and a $r\in[0,p-1]$ such that

$$\Lambda=\{(qn,-qn): n\in\mathscr{S}_1\}\cup\left\{\left(\frac{qs_i}{2}+\frac{qr}{2p}, -\frac{qs_i}{2}-\frac{qr}{2p}\right): s_i\in\mathscr{S}_2\right\} := \Lambda_1\cup\Lambda_2, \tag{9}$$

where $k_i=ps_i+r\in 2\mathbb{Z}+1$ and $2|q(s_j-s_i)$. Since $(0,0)\in\Lambda$, we know that $\Lambda_1\ne\emptyset$. By Theorem 4 (ii), we know that $\Lambda_2\ne\emptyset$ and $\Lambda_2\not\subset\mathbb{Z}^2$.

Furthermore, combining with $k_i=ps_i+r\in 2\mathbb{Z}+1$ and $2|q(s_j-s_i)$, if $p\ne 1$ and $r\ne 0$, by simple discussing, we always have

$$\Lambda_2=\left\{\left(qm+\frac{qr}{2p}, -qm-\frac{qr}{2p}\right): m\in\mathscr{S}_3\subset\mathbb{Z}\right\} \text{ for some odd } r\in[1-p,p-1]. \tag{10}$$

If $p=1$ or $r=0$, then $q, s_i$ are both odd. By the maximal orthogonality of $\Lambda$, we have

$$\Lambda=\left\{\left(\frac{qm}{2}, -\frac{qm}{2}\right): m\in\mathbb{Z}\right\}. \tag{11}$$

**Step 3.** Our next goal is to prove that $\Lambda$ can't be the spectrum of $\rho$ if $t\in\mathbb{Q}\setminus(\frac{1}{2}\mathbb{Z})$.

If $t\in\mathbb{Q}\setminus(\frac{1}{2}\mathbb{Z})$, then $2t+1=\frac{p}{q}\in\mathbb{Q}\setminus\mathbb{Z}$, i.e., $q\ge 2$.

When $q>2$ (or $p=1$), by the expression of $\Lambda$ in (9),(10),(11), and Theorem 6,

$$D^-(\Lambda_x)\le\frac{2}{q}<1.$$

Landau's theorem [11] tells us that $\Lambda$ is not a spectrum of $\rho$.

When $q = 2$, then $p \geq 3$ is odd. By the expression of (9),(10), without loss of generality, we assume that

$$\Lambda = \{(n,-n) : n \in 2\mathbb{Z}\} \cup \left\{\left(m + \frac{r}{p}, -m - \frac{r}{p}\right) : m \in 2\mathbb{Z}\right\}$$

for some odd $r \in [1-p, p-1]$. Let $F(x,y) = e^{2\pi i(x-y)b}$ with $b := \frac{r+2}{p}$, $F_x = F(x,0) = e^{2\pi i b x}$ and $F_y = F(0,y) = e^{-2\pi i b y}$. Then

$$\|F(x,y)\|^2_{L^2(\rho)} = \frac{1}{2}\|F_x\|^2 + \frac{1}{2}\|F_y\|^2 \equiv 1.$$

Let $a = b - \frac{r}{p}$. Since $2t+1 = \frac{p}{2}$, we obtain

$$\begin{aligned}
&\sum_{(u,v)\in\Lambda} |\langle F, e_{u,v}\rangle_{L^2(\rho)}|^2 \\
&\quad = \frac{1}{4}\sum_{(u,v)\in\Lambda} |\langle F_x, e_u\rangle + \langle F_y, e_v\rangle|^2 \\
&\quad = \frac{1}{4}\sum_{\lambda\in 2\mathbb{Z}} \left( \left| \int_t^{t+1} \left(e^{2\pi i(b-\lambda)x} + e^{2\pi i(\lambda-b)x}\right) \mathrm{d}x \right|^2 + \left| \int_t^{t+1} \left(e^{2\pi i(a-\lambda)x} + e^{2\pi i(\lambda-a)x}\right) \mathrm{d}x \right|^2 \right) \\
&\quad = \frac{1}{\pi^2}\sum_{\lambda\in 2\mathbb{Z}} \left( \frac{|\cos\pi\frac{bp}{2}\sin\pi b|^2}{(b-\lambda)^2} + \frac{|\cos\pi\frac{ap}{2}\sin\pi a|^2}{(a-\lambda)^2} \right) \\
&\quad = \left|\cos\pi\frac{bp}{2}\cos\pi\frac{b}{2}\right|^2 + \left|\cos\pi\frac{ap}{2}\cos\pi\frac{a}{2}\right|^2 \\
&\quad = \cos^2\frac{\pi}{p} < 1.
\end{aligned}$$

The penultimate equality follows from the fact that

$$\sum_{n\in\mathbb{Z}} \frac{|\sin\pi z|^2}{(z-n)^2} = \pi^2 \quad \text{for } z \in \mathbb{C}\setminus\mathbb{Z}.$$

The last equality follows from $r$ is odd and $a = \frac{2}{p}$. Hence $\Lambda$ is not a spectrum of $\rho$.

Then we have proved that $\Lambda$ can't be the spectrum of $\rho$ if $t \in \mathbb{Q}\setminus(\frac{1}{2}\mathbb{Z})$.

**Completion of the proof of Theorem 1.** In the final step, we construct a spectrum of $\rho$ under the condition that $t \in \frac{1}{2}\mathbb{Z}$, i.e., $q = 1$. Combining with the expression of $\Lambda$ in (9), (10), (11), we know if $p$ is odd, then there exists an odd $r \in [1-p, p-1]$ such that

$$\Lambda = \{(n,-n) : n \in \mathbb{Z}\} \cup \left\{\left(m + \frac{r}{2p}, -m - \frac{r}{2p}\right) : m \in \mathbb{Z}\right\},$$

or

$$\Lambda = \left\{\left(\frac{n}{2}, -\frac{n}{2}\right) : n \in \mathbb{Z}\right\}.$$

If $p$ is even, then there exists an odd $r \in [1-p, p-1]$ such that

$$\Lambda = \{(n,-n) : n \in \mathbb{Z}\} \cup \left\{\left(m + \frac{r}{2p}, -m - \frac{r}{2p}\right) : m \in \mathbb{Z}\right\}.$$

The proof of [10, Theorem 1.3] (see [10, Remark 6.2]) shows that if $\Lambda_0$ is the spectrum of the Lebesgue measure supported on $[-t-1, -t] \cup [t, t+1]$, then $\Lambda = \{(\lambda, -\lambda) : \lambda \in \Lambda_0\}$ is the spectrum of $\rho$. This together with [10, Theorem 1.1] completes the proof. $\square$

## 4. Concluding remarks

In this paper, we only consider the case $t \in \mathbb{Q} \setminus \{-\frac{1}{2}\}$. There are many meaningful problems left in the other cases. For the case $t = -\frac{1}{2}$, i.e., the Plus space called by Lai, Liu and Prince [10], since we can't solve the equation

$$\frac{\sin(\pi\lambda_1)}{\pi\lambda_1} = -\frac{\sin(\pi\lambda_2)}{\pi\lambda_2},$$

it is interesting but very difficult. We only have the following partial results.

**Lemma 9.** *Let $\rho$ be a symmetric measure of Lebesgue type, where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$ and $t \in \mathbb{Q}^c \cup \{-\frac{1}{2}\}$. If $\rho$ has a spectrum $\Lambda$, then $\#(\Lambda \cap \mathbb{Z}^2) < \infty$.*

**Proof.** Suppose that $\Lambda$, with $(0,0) \in \Lambda$, is the spectrum of $\rho$ and $\#(\Lambda \cap \mathbb{Z}^2) = \infty$. Set $\theta = 2t+1$. We first consider the case $t \in \mathbb{Q}^c$, i.e., $\theta \in \mathbb{Q}^c$. According to Theorem 4 (ii), $\Lambda \setminus \mathbb{Z}^2 \neq \emptyset$. Choose an element $(\lambda_1, \lambda_2) \in \Lambda \setminus \mathbb{Z}^2$, which satisfies the Orthogonality Equation

$$e^{\pi i(\lambda_1-\lambda_2)\theta} \frac{\sin(\pi\lambda_1)}{\pi\lambda_1} = -\frac{\sin(\pi\lambda_2)}{\pi\lambda_2}. \tag{12}$$

Hence $\lambda_1 - \lambda_2 = m\theta^{-1}$ for some integer $m$. Let $(k_1, k_2) \in \Lambda \cap \mathbb{Z}^2$, then

$$e^{\pi i(k_1-\lambda_1-k_2+\lambda_2)\theta} \frac{\sin\pi(k_1-\lambda_1)}{\pi(k_1-\lambda_1)} = -\frac{\sin\pi(k_2-\lambda_2)}{\pi(k_2-\lambda_2)}. \tag{13}$$

This implies that $k_1 - \lambda_1 - k_2 + \lambda_2 = k_1 - k_2 - m\theta^{-1} \in \theta^{-1}\mathbb{Z}$. This yields that $k_1 = k_2$, as $\theta \in \mathbb{Q}^c$. Combining with (12), (13), one has $\lambda_1/\lambda_2 = k_1/k_2 = 1$. Then $m = 0$. Using (12) again, then $(\lambda_1, \lambda_2) \in \mathbb{Z}^2$, a contradiction.

For the case $t = -\frac{1}{2}$, i.e., $\theta = 0$. Let $(\lambda_1, \lambda_2) \in \Lambda \setminus \mathbb{Z}^2$ and $(k_1, k_2) \in \Lambda \cap \mathbb{Z}^2$, we obtain

$$\frac{\sin(\pi\lambda_1)}{\pi\lambda_1} = -\frac{\sin(\pi\lambda_2)}{\pi\lambda_2}, \quad \frac{\sin\pi(k_1-\lambda_1)}{\pi(k_1-\lambda_1)} = -\frac{\sin\pi(k_2-\lambda_2)}{\pi(k_2-\lambda_2)}.$$

Hence,

$$\frac{\lambda_1}{\lambda_2} = (-1)^{k_1-k_2} \frac{k_1-\lambda_1}{k_2-\lambda_2}.$$

If $k_1 - k_2 \in 2\mathbb{Z}$, then $\lambda_1/\lambda_2 = k_1/k_2 = a$ for fixed $a$. If $k_i - k_j \in 2\mathbb{Z}$ for all $(k_i, k_j) \in \Lambda \cap \mathbb{Z}^2$, then by Proposition 7.1 in [10], $\Lambda$ is not the spectrum of $\rho$. If not, similar to the proof of Proposition 8, $\Lambda$ is not the spectrum of $\rho$ either.

Hence, the proof is complete. □

At last, we post the following conjecture.

**Conjecture 10.** *Assume that $\Lambda \cap \mathbb{Z}^2 = \{(0,0)\}$. For any $(\lambda_1, \lambda_2) \in (\Lambda - \Lambda) \setminus \{(0,0)\}$, if*

$$\frac{\sin\pi\lambda_1}{\pi\lambda_1} = -\frac{\sin\pi\lambda_2}{\pi\lambda_2},$$

*then $D^-(\Lambda_x)$ or $D^-(\Lambda_y)$ is less than 1.*

If above conjecture is true, we have the following theorem.

**Theorem 11.** *Let $\rho$ be a symmetric measure of Lebesgue type, where the component measure $\mu$ is the Lebesgue measure supported on $[t, t+1]$. Let $t \notin \frac{1}{2} + \mathbb{Z}$ or $t = -\frac{1}{2}$. If Conjecture 10 is true, then $\rho$ is not spectral.*

**Proof.** Let $\Lambda$ be the spectrum of $\rho$. By Lemma 9, we can assume that $(0,0) \in \Lambda$ and $\#(\Lambda \cap \mathbb{Z}^2) < \infty$. Write $\Lambda := \Gamma \cup \widetilde{\Gamma}$ with $\Gamma \subset \mathbb{Z}^2$ and $\widetilde{\Gamma} \cap \mathbb{Z}^2 = \emptyset$. By Conjecture 10, without loss of generality, we suppose that $D^-(\widetilde{\Gamma}_x) < 1$. It follows from Lemma 9 that

$$D^-(\Lambda_x) = D^-(\widetilde{\Gamma}_x) < 1.$$

Then, by Theorem 3 and Landau's theorem [11], $E(\Lambda)$ is not a frame of $\rho$. This yields that $\Lambda$ is not the spectrum of $\rho$, a contradiction. □

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