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# Comptes Rendus
# Mathématique

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Volume 361 (2023), p. 679-683

Published online: 2 March 2023

https://doi.org/10.5802/crmath.437



*Les* Comptes Rendus. Mathématique *sont membres du*
*Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

*Comptes Rendus*
*Mathématique*
**2023**, Vol. 361, p. 679-683
https://doi.org/10.5802/crmath.437

Complex analysis and geometry / *Analyse et géométrie complexes*

# Optimal $L^2$ Extensions of Openness Type and Related Topics

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**Abstract.** We establish several optimal $L^2$ extension theorems of openness type on weakly pseudoconvex Kähler manifolds. We prove a product property for certain minimal $L^2$ extensions, which generalizes the product property of Bergman kernels. We describe a different approach to the Suita conjecture and its generalizations, which is based on a log-concavity for certain minimal $L^2$ integrals.

**Résumé.** Nous établissons quelques théorèmes d'extension optimaux $L^2$ pour les formes ouvertes sur les variété Kähler faiblement pseudoconvexes. Nous prouvons les propriétés de produit de certaines extensions minimales de $L^2$, qui généralisent les propriétés de produit du noyau Bergman. Sur la base de la concavité logarithmique de certaines intégrales minimales de $L^2$, nous donnons une méthode différente pour la conjecture de Suita et son extension.

**2020 Mathematics Subject Classification.** 32D15, 32A36, 32L05, 32W05, 32E10, 14C30, 30C40.

**Funding.** The second author is partially supported by the NSFC grant (11688101 and 12288201).

*Manuscript received 5 February 2022, accepted 16 October 2022.*

## 1. Backgrounds

In this note, we study the following problem: let $S$ be a closed submanifold of a complex manifold $M$ and $E$ be a Hermitian holomorphic vector bundle on $M$; suppose $f$ is an $L^2$ holomorphic section of $E$ defined in a neighborhood $U$ of $S$, find a holomorphic section $F \in \Gamma(M,E)$ so that $F|_S = f|_S$ and the $L^2$-norm $\|F\|_M$ is uniformly and optimally controlled by the $L^2$-norm $\|f\|_U$. This problem is closely related to but different from the usual $L^2$ extension problem (where $f$ is defined on $S$ and $\|f\|_U$ is replaced by $\|f\|_S$), which is called optimal $L^2$ extension problem of *openness type*, in order to distinguish with the usual one.

The existence part of the $L^2$ extension problem of openness type is well studied, e.g. Jennane [15] and Demailly [4, 5] (see also [6, Chapter VIII.7]). This note is devoted to the optimal part of the problem and some related topics. Details of the proofs will appear in [17].

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ISSN (electronic) : 1778-3569 

## 2. Optimal $L^2$ Extension Theorems of Openness Type

Recall that, a complex manifold $\Omega$ is said to be *weakly pseudoconvex* if there exists a smooth psh exhaustion function for $\Omega$; an upper semi-continuous function $\varphi : \Omega \to [-\infty, +\infty)$ is said to be *quasi-psh* if $\varphi$ is locally the sum of a psh function and a smooth function. Given a quasi-psh function $\varphi$ on a complex manifold $\Omega$, we denote by $\mathscr{I}(\varphi) \subset \mathscr{O}_\Omega$ the *multiplier ideal sheaf* associated to $\varphi$, i.e. $\mathscr{I}(\varphi)_x = \{f_x \in \mathscr{O}_{\Omega,x} : |f_x|^2 e^{-\varphi}$ is integrable in some neighborhood of $x\}$.

**Theorem 1.** *Let $(\Omega, \omega)$ be a weakly pseudoconvex Kähler manifold and $(E, h)$ be a Hermitian holomorphic vector bundle over $\Omega$. Suppose there are quasi-psh functions $\psi < 0$ and $\varphi$, continuous real $(1,1)$-forms $\gamma \geq 0$ and $\rho$ on $\Omega$ such that*

$$\sqrt{-1}\partial\bar{\partial}\psi \geq \gamma, \quad \sqrt{-1}\partial\bar{\partial}\varphi \geq \rho \quad \text{and} \quad \sqrt{-1}\Theta(E,h) + (\gamma+\rho) \otimes \mathrm{Id}_E \geq_{\mathrm{Nak}} 0.$$

*Let $\Omega_a := \{z \in \Omega : \psi(z) < -a\}$, where $a \in \mathbb{R}_+$. Then for any holomorphic section $f \in \Gamma(\Omega_a, K_\Omega \otimes E)$ satisfying $\int_{\Omega_a} |f|^2_{\omega,h} e^{-\varphi} \mathrm{d}V_\omega < +\infty$, there exists $F \in \Gamma(\Omega, K_\Omega \otimes E)$ such that*

$$F|_{\Omega_a} - f \in \Gamma(\Omega_a, \mathscr{O}(K_\Omega \otimes E) \otimes \mathscr{I}(\varphi+\psi)) \tag{1}$$

*and*

$$\int_\Omega |F|^2_{\omega,h} e^{-\varphi} \mathrm{d}V_\omega \leq e^a \int_{\Omega_a} |f|^2_{\omega,h} e^{-\varphi} \mathrm{d}V_\omega. \tag{2}$$

The proof of Theorem 1 uses the $L^2$ techniques developed by Guan–Zhou [13] and Zhou–Zhu [18], together with a log-concavity for certain minimal $L^2$ integrals (which is essentially due to Guan [8]). Since $\Omega$ is a weakly pseudoconvex Kähler manifold, we approximate the quasi-psh functions on relatively compact subdomains by the methods of [7] and then solve $\bar{\partial}$-equations with error terms. Using the $L^2$ techniques, we prove the theorem with the uniform constant replaced by $e^a + 1$. Using the log-concavity, we then obtain the uniform constant $e^a$.

**Definition 2.** *Let $U \subset \mathbb{C}^n$ be an open set and $f, g \in \mathscr{O}(U)$. Given $x \in U$ and $k \in \mathbb{N}$, if $\partial^\alpha f(x) = \partial^\alpha g(x)$ for all multi-order $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, then we say "$f$ coincides with $g$ up to order $k$ at $x$". In particular, if $f$ coincides with the zero function up to order $k$ at $x$, then we say "$f$ vanishes up to order $k$ at $x$". Clearly, $f$ vanishes up to order $k$ at $x$ if and only if $[f]_x \in \mathfrak{m}_x^{k+1}$. These concepts can be easily extended to holomorphic sections of holomorphic vector bundles.*

If there exists a closed subset $S \subset \Omega$ so that $\mathscr{I}(\varphi+\psi)_x \subset \mathfrak{m}_x^{k+1}$ for any $x \in S$, then the condition (1) implies that $F$ coincides with $f$ up to order $k$ along $S$. In particular, we have the following corollary.

**Corollary 3.** *Let $\Omega$ be a bounded pseudoconvex domain in $\mathbb{C}^n$, $\varphi$ be a psh function on $\Omega$, and $G_\Omega(\cdot, w)$ be the pluricomplex Green function of $\Omega$ with a pole at $w \in \Omega$. Let $U = \{G_\Omega(\cdot, w) < -a\}$ for some $a \in \mathbb{R}_+$. For any $f \in \mathscr{O}(U)$ satisfying $\int_U |f|^2 e^{-\varphi} \mathrm{d}\lambda < +\infty$, there exists a holomorphic function $F \in \mathscr{O}(\Omega)$ such that $F$ coincides with $f$ up to order $k \in \mathbb{N}$ at $w$ and*

$$\int_\Omega |F|^2 e^{-\varphi} \mathrm{d}\lambda \leq e^{2(n+k)a} \int_U |f|^2 e^{-\varphi} \mathrm{d}\lambda. \tag{3}$$

The case of $k = 0$ and $\varphi \equiv 0$ has been obtained by Błocki [2], whose proof used a tensor power trick. It is clear that the uniform estimates of Theorem 1 and Corollary 3 are optimal.

Using similar arguments as Theorem 1, we prove another $L^2$ extension theorem of openness type, whose prototype comes from [6, Chapter VIII.7]. Demailly's theorem corresponds to the case of $m = 0$ and the right hand side of (4) is replaced by $(1 + \frac{(p+1)^2}{\varepsilon}) \int_U |f|^2_{\omega,h} e^{-\varphi} \mathrm{d}V_\omega$.

**Theorem 4.** *Let $(\Omega, \omega)$ be a weakly pseudoconvex Kähler manifold of dimension $n$ and $(E, h)$ be a Hermitian holomorphic vector bundle over $\Omega$. Suppose there exists a quasi-psh function $\varphi$ and a continuous real $(1,1)$-form $\rho$ on $\Omega$ such that*

$$\sqrt{-1}\partial\bar{\partial}\varphi \geq \rho \quad \text{and} \quad \sqrt{-1}\Theta(E,h) + \rho \otimes \mathrm{Id}_E \geq_{\mathrm{Nak}} 0.$$

*Suppose* $w = (w_1, \dots, w_p)$ *is a tuple of holomorphic functions on* $\Omega$ $(1 \le p \le n)$*, let*

$$S = \{x \in \Omega : w(x) = 0\} \quad \textit{and} \quad U = \{x \in \Omega : |w(x)| < 1\}.$$

*Assume that* $\mathrm{d}w_1 \wedge \cdots \wedge \mathrm{d}w_p \neq 0$ *generically on* $S$*. Given* $m \in \mathbb{N}$ *and* $0 < \varepsilon < m + p$*, for any holomorphic section* $f \in \Gamma(U, K_\Omega \otimes E)$ *satisfying* $\int_U |f|^2_{\omega,h} e^{-\varphi} \mathrm{d}V_\omega < +\infty$*, there exists* $F \in \Gamma(\Omega, K_\Omega \otimes E)$ *such that* $F$ *coincides with* $f$ *up to order* $m$ *along* $S$ *and*

$$\int_\Omega \frac{|F|^2_{\omega,h} e^{-\varphi}}{(1+|w|^2)^{m+p+\varepsilon}} \mathrm{d}V_\omega \le \frac{\int_0^1 \tau^{m+p-1}(1-\tau)^{\varepsilon-1}\mathrm{d}\tau}{\int_0^{1/2} \tau^{m+p-1}(1-\tau)^{\varepsilon-1}\mathrm{d}\tau} \int_U \frac{|f|^2_{\omega,h} e^{-\varphi}}{(1+|w|^2)^{m+p+\varepsilon}} \mathrm{d}V_\omega. \tag{4}$$

Let $\Omega = \mathbb{C}^n$ and $\omega$ be the standard Kähler form. Let $w = (z_1, \dots, z_p)$ be the first $p$ coordinates of $\mathbb{C}^n$, then $S \cong \mathbb{C}^{n-p}$ is a linear subspace and $U \cong \mathbb{B}^p \times \mathbb{C}^{n-p}$. Let $(E, h) = (K_\Omega^{-1}, \det\omega)$ and $\varphi \equiv 0$. Let $u$ be a homogeneous polynomial in $z_1, \dots, z_p$ of degree $m$ and $v \in \mathscr{O}(\mathbb{C}^{n-p})$ be a holomorphic function of $z_{p+1}, \dots, z_n$. Let $f := uv \in \mathscr{O}(U)$ and let $F$ be the unique holomorphic function on $\mathbb{C}^n$ such that $F$ coincides with $f$ up to order $m$ along $S$ and the left hand side of (4) is minimized. It is easy to show that $F \equiv uv$ and the inequality (4) is sharp.

## 3. Sharper Estimates in Optimal $L^2$ Extension Theorems

Let $\Omega$ be a weakly pseudoconvex Kähler manifold of dimension $n$. Let $(E, h)$ be a Hermitian holomorphic vector bundle over $\Omega$ whose curvature is Nakano semi-positive. Assume that $\psi < 0$ is a psh function on $\Omega$ having a logarithmic pole at $x \in \Omega$: if $(U, z)$ is a coordinate chart so that $z(x) = 0$, then $\psi - \log|z|$ is bounded near $x$. By the optimal $L^2$ extension theorem ( [13, 18]): *for any* $\xi \in E_x$*, there exists a holomorphic section* $F \in \Gamma(\Omega, K_\Omega \otimes E)$ *such that* $F(x) = \mathrm{d}z \otimes \xi$ *and* $\int_\Omega (\sqrt{-1})^{n^2} F \wedge_h \overline{F} \le \frac{(2\pi)^n}{n!} e^{-2nc} |\xi|^2_h$*, where* $\mathrm{d}z := \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ *and* $c := \underline{\lim}_{z \to 0}(\psi(z) - \log|z|)$.

The uniform estimate in the above statement is optimal: there are admissible situations where the constant $\frac{(2\pi)^n}{n!} e^{-2nc}$ cannot be replaced by a smaller one. Under some additional conditions, we may obtain a sharper estimate. For example, if the curvature of $(E, h)$ is Nakano positive somewhere, then there exists an extension $F$ of $\mathrm{d}z \otimes \xi$ so that $\int_\Omega (\sqrt{-1})^{n^2} F \wedge_h \overline{F} < \frac{(2\pi)^n}{n!} e^{-2nc} |\xi|^2_h$. Hosono's [14] result is a special case. Using Theorem 1, we have one more example.

**Theorem 5.** *Let* $\Omega \ni x$*,* $(E, h)$ *and* $\psi$ *be the same as above. Moreover, we assume that:*

(i) *the curvature of* $(E, h)$ *is Griffiths positive at* $x$*;*

(ii) $(U, z)$ *can be chosen so that* $z(x) = 0$ *and* $\psi(z) = c + \log|z| + o(|z|^2)$ *as* $z \to 0$.

*There exists a constant* $\tau \in (0, 1)$ *depends on* $h$ *and* $\psi$*, for any* $\xi \in E_x$*, we can find a holomorphic section* $F \in \Gamma(\Omega, K_\Omega \otimes E)$ *such that* $F(x) = \mathrm{d}z \otimes \xi$ *and*

$$\int_\Omega (\sqrt{-1})^{n^2} F \wedge_h \overline{F} \le (1 - \tau) \frac{(2\pi)^n}{n!} e^{-2nc} |\xi|^2_h. \tag{5}$$

Since $(E, h)$ is Griffiths positive at $x$, we can construct a local $L^2$ holomorphic extension with sharper estimate, then we apply Theorem 1 to obtain a global extension with sharper estimate. The technical condition (ii) guarantees that the sublevel set $\{\psi < -a\}$ (with $a \gg 1$) is almost an Euclidean ball in $U$. Moreover, if $\Omega$ is a hyperbolic Riemann surface and $\psi = G_\Omega(\cdot, x)$ is the Green function, then the condition (ii) is automatically satisfied.

## 4. A Product Property for Minimal $L^2$ Extensions

The tensor power trick of [2] relies on the product property of Bergman kernels. Notice that, Bergman kernel records the norm of certain minimal $L^2$ extension: let $\Omega \subset \mathbb{C}^n$ be an open set, then $B_\Omega(w)^{-1} = \inf\{\int_\Omega |f|^2 \mathrm{d}\lambda : f \in \mathscr{O}(\Omega), f(w) = 1\}$. As a generalization, we prove a product property for general minimal $L^2$ extensions.

For $i=1$ and 2, let $(\Omega_i, \mathrm{d}V_i)$ be a complex manifold with continuous volume form, $(E_i, h_i) \to \Omega_i$ be a holomorphic vector bundle with continuous Hermitian metric, $S_i \subset \Omega_i$ be an arbitrary closed subset, and $\psi_i$ be a measurable function on $\Omega_i$ which is locally bounded above. We define

$$A^2(\Omega_i, E_i) := \left\{ f_i \in \Gamma(\Omega_i, E_i) : \|f_i\|^2 = \int_{\Omega_i} |f_i|^2_{h_i} e^{-\psi_i} \mathrm{d}V_i < +\infty \right\}, \quad i = 1 \text{ and } 2. \tag{6}$$

Let $\Omega := \Omega_1 \times \Omega_2$ be the product manifold and let $p_i : \Omega \to \Omega_i$ be the natural projections. Let $\mathrm{d}V := p_1^* \mathrm{d}V_1 \times p_2^* \mathrm{d}V_2$, $E := p_1^* E_1 \otimes p_2^* E_2$, $h := p_1^* h_1 \otimes p_2^* h_2$ and $\psi := p_1^* \psi_1 + p_2^* \psi_2$. Then we define $A^2(\Omega, E)$ in a similar way as (6). We can show that $A^2(\Omega, E) = A^2(\Omega_1, E_1) \widehat{\otimes} A^2(\Omega_2, E_2)$.

By constructing a special orthogonal decomposition which is compatible with the Hilbert tensor product $\widehat{\otimes}$, we can prove a product property for minimal $L^2$ holomorphic extensions. Since we need to assume the existence of global $L^2$ extensions, we simply assume that $f_i \in A^2(\Omega_i, E_i)$, and we are interested in high-order minimal $L^2$ extensions of $f_i|_{S_i}$.

**Theorem 6.** *Let $f_1 \in A^2(\Omega_1, E_1)$ and $f_2 \in A^2(\Omega_2, E_2)$ be given. For $i = 1$ and 2, let $F_i$ be the unique element with minimal norm in $A^2(\Omega_i, E_i)$ that coincides with $f_i$ up to order $m_i \in \mathbb{N}$ along $S_i$. Let $F$ be the unique element with minimal norm in $A^2(\Omega, E)$ that coincides with $f := f_1 \otimes f_1$ up to order $m_1 + m_2$ along $S := S_1 \times S_2$. Then*

$$\|F\|^2_{A^2(\Omega,E)} \geq \|F_1\|^2_{A^2(\Omega_1,E_1)} \|F_2\|^2_{A^2(\Omega_2,E_2)}. \tag{7}$$

*Moreover, if $m_i = 0$ or $f_i$ vanishes up to order $m_i - 1$ on $S_i$ for $i = 1$ and 2, then*

$$F = F_1 \otimes F_2 \quad \text{and} \quad \|F\|^2_{A^2(\Omega,E)} = \|F_1\|^2_{A^2(\Omega_1,E_1)} \|F_2\|^2_{A^2(\Omega_2,E_2)}. \tag{8}$$

If $S_i$ are singleton sets and $m_i = 0$, then (8) corresponds to the product property of Bergman kernels. Using Theorem 6 and imitating the tensor power trick of [2], one can prove a weak version of Theorem 1.

## 5. Generalized Suita Conjecture with Jets and Weights

A motivation for optimal $L^2$ extension theorems is the long-standing Suita conjecture. Let $\Omega$ be a potential-theoretically hyperbolic Riemann surface. Let $(U, z)$ be a coordinate chart of $\Omega$. Locally, we denote by $\kappa_\Omega = B_\Omega |\mathrm{d}z|^2$ the Bergman kernel of $\Omega$ and $c_\beta |\mathrm{d}z|$ the logarithmic capacity of $\Omega$. Suita [16] conjectured that $\pi B_\Omega \geq c_\beta^2$, and $\pi B_\Omega(w) = c_\beta(w)^2$ for some $w \in \Omega$ if and only if $\Omega$ is conformally equivalent to the unit disc less a possible closed polar set. The inequality part of Suita's conjecture was solved by Błocki [1] and Guan–Zhou [11], and the equality part was solved by Guan–Zhou [13]. Moreover, Guan–Zhou [12, 13] proved the extended Suita conjecture and Błocki-Zwonek [3] obtained a high-order generalization to the inequality part.

In proving Theorem 1, we use a log-concavity for minimal $L^2$ integrals. Tracing the proof, we obtain a necessary condition for the concavity degenerating to linearity (see also [9]). Using the concavity and the necessary condition, we have a different approach to Suita's conjecture and its generalizations.

**Proposition 7.** *Let $\varphi$ be a harmonic function on $\Omega$ and $\psi = 2(m+1)G_\Omega(\cdot, w)$. For each $t \geq 0$, let*

$$\Omega_t := \{z \in \Omega : \psi(z) < -t\} \quad \text{and} \quad \mathscr{A}_t := \left\{ F \in \Gamma(\Omega_t, K_\Omega) : \|F\|^2 = \int_{\Omega_t} \frac{\sqrt{-1}}{2} F \wedge \overline{F} e^{-\varphi} < +\infty \right\}.$$

*Let $f$ be a holomorphic 1-form defined in a neighborhood of $w$. For each $t \geq 0$, let $F_t \in \mathscr{A}_t$ be the unique element with minimal norm that coincides with $f$ up to order $m$ at $w$. Let $I(t) := \|F_t\|^2_{\mathscr{A}_t}$ and we assume that $I(t) > 0$. Then $r \mapsto I(-\log r)$ is a concave non-decreasing function on $(0,1]$ and $I(0) \leq I(t)e^t$ for all $t > 0$. Moreover, if $r \mapsto I(-\log r)$ is linear, then $F_t \equiv F_0|_{\Omega_t}$ for any $t > 0$.*

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathbb{S} = \{z \in \mathbb{C} : |z| = 1\}$. Let $p : \mathbb{D} \to \Omega$ be a universal covering. Then the fundamental group $\pi_1(\Omega)$ is identified with a subgroup of $\mathrm{Aut}(\mathbb{D})$, and $p \circ \sigma = p$ for any $\sigma \in \pi_1(\Omega)$. For any $w \in \Omega$, there exists an $f_w \in \mathscr{O}(\mathbb{D})$ so that $\log|f_w| = p^*(G_\Omega(\cdot, w))$. Then there is a multiplier $\chi_w \in \mathrm{Hom}(\pi_1(\Omega), \mathbb{S})$ so that $p^* f_w = \chi_w(\sigma) f_w$ for any $\sigma \in \pi_1(\Omega)$. For any harmonic function $\eta$ on $\Omega$, there exists an $f_\eta \in \mathscr{O}(\mathbb{D})$ so that $\log|f_\eta| = p^*\eta$. Then there is a multiplier $\chi_\eta \in \mathrm{Hom}(\pi_1(\Omega), \mathbb{S})$ so that $p^* f_\eta = \chi_\eta(\sigma) f_\eta$ for any $\sigma \in \pi_1(\Omega)$.

**Theorem 8.** *Given an integer $m \in \mathbb{N}$ and a harmonic function $\eta$ on $\Omega$, we define*

$$B^{(m)}_{\Omega,\eta}(w) := \sup\left\{ \left|\frac{\partial^m f}{\partial z^m}(w)\right|^2 : \begin{array}{l} F \in \Gamma(\Omega, K_\Omega) \text{ with } F|_U = f\,\mathrm{d}z, \\ \int_\Omega \frac{\sqrt{-1}}{2} F \wedge \bar{F} e^{-2\eta} \le 1, [f]_w \in \mathfrak{m}_w^m \end{array} \right\}, \quad w \in \Omega. \tag{9}$$

*Then*

$$\pi e^{-2\eta(w)} B^{(m)}_{\Omega,\eta}(w) \ge m!(m+1)! c_\beta(w)^{2m+2}. \tag{10}$$

*Moreover, the equality holds if and only if $\chi_\eta^{-1} = \chi_w^{m+1}$, if and only if there exists a holomorphic function $g \in \mathscr{O}(\Omega)$ such that* $\log|g| = (m+1)G_\Omega(\cdot, w) + \eta$.

The extended Suita conjecture is the case of $m = 0$, and Błocki-Zwonek's generalization is the inequality (10) with $\eta \equiv 0$. Recently, Guan–Mi–Yuan [10] obtain a result generalizing Theorem 8, but our proof is different.

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