INSTITUT DE FRANCE
Académie des sciences

# Comptes Rendus
# Mathématique

Timothée L. Labinet

**Projective varieties have countably many real forms**

Volume 361 (2023), p. 863-867

Published online: 18 July 2023

https://doi.org/10.5802/crmath.441



*Les* Comptes Rendus. Mathématique *sont membres du*
*Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

*Comptes Rendus Mathématique*
**2023**, Vol. 361, p. 863-867
https://doi.org/10.5802/crmath.441

Number theory / *Théorie des nombres*

# Projective varieties have countably many real forms

**Timothée L. Labinet**

**Abstract.** In this note, we check that a complex projective algebraic variety has (at most) countably many real forms. We more generally prove it when the field of reals is replaced with a field that has only countably many finite extensions up to isomorphism. The verification consists in gathering known results about automorphism groups and Galois cohomology. This contrasts with the recent discovery by A. Bot of an affine real variety with uncountably many real forms.

**Mathematical subject classification (2010).** 12G05, 12F10, 11S25, 11E72, 14J50.

*Manuscript received 11 September 2022, revised 13 October 2022, accepted 28 October 2022.*

## 1. Introduction

By countable we mean "at most countable", possibly finite or even empty.

We say that a field $k$, with separable closure $\bar{k}$, is tame if it has countably many finite Galois extensions up to $k$-isomorphism, or equivalently if its absolute Galois group $\mathrm{Gal}(\bar{k}/k)$ is metrizable, or equivalently if $\bar{k}$ has countable degree over $k$ (see Proposition 11). For instance, every non-discrete locally compact field of characteristic zero is tame. Also, every pseudofinite field is tame (indeed having at most $n$ finite extensions of degree $m$ can be characterized by a first-order formula), and so is every real-closed or algebraically closed field. The above examples actually have finitely many extensions, up to isomorphism, in each given degree. Furthermore, countable fields are tame, and if $k$ is a tame field of characteristic zero, then so is the field $k((t))$ of Laurent series. In contrast, the field $\mathbf{C}(t)$ is not tame.

If $k$ is a perfect field with an algebraic closure $k \subset \bar{k}$, by $k$-group we mean a $\mathrm{Gal}(\bar{k}/k)$-group (see Section 2). We say that a $k$-group is countable-by-algebraic if it has a normal $k$-subgroup of countable index that is isomorphic to an algebraic $k$-group.

**Theorem 1.** *Let $k$ be a perfect tame field. Let $G$ be a $k$-group that is countable-by-algebraic. Then the Galois cohomology set $\mathrm{H}^1(k,G)$ is countable.*

The main example of a $k$-group we have in mind is the group of automorphisms of a $k$-variety (see Section 4). Here, a $k$-variety is meant to be a separated noetherian scheme, locally of finite type over $k$ (no reducedness or irreducibility assumption). A classical result (see Theorem 12) is that the group of automorphisms of every complete $k$-variety is countable-by-algebraic.

**Corollary 2.** *Let $k$ be a perfect tame field and $k \subset \overline{k}$ an algebraic closure. Then every complete $\overline{k}$-variety has countably many $k$-forms (up to isomorphism).*

When $k$ is the real field (which was our starting point), this number can be infinite, even in the smooth projective case: the first examples are due to Lesieutre [8] and further ones were provided satisfying additional hypotheses [4–6].

Note that the corollary holds for other objects provided the automorphism group has a similar structure. For instance, it applies to $k$-forms of pairs $(X, Y)$ consisting of a complete variety $X$ and a closed subvariety $Y$.

However, it does not apply to arbitrary varieties, and actually fails for affine varieties. Indeed, Bot has constructed a smooth affine complex surface with continuum many real forms [2].

An anecdotic remark is that a perfect field $k$ is tame if and only if the product $\overline{k}$-algebra $\overline{k}^n$ has countably many $k$-forms for every $n$. This being observed, one can naturally wonder whether a stronger converse of Corollary 2 is true, namely:

**Question 3.** *Is it true that for every non-tame perfect field $k$, there exists an absolutely irreducible (smooth?) projective $k$-variety with uncountably many $k$-forms?*

If one drops the irreducibility requirement, this is therefore true: just considering the finite $k$-variety $\mathrm{Spec}(k^n)$ for an appropriate $n$. Question 3 has a positive answer in many cases: for instance when $k^*/k^{*n}$ is uncountable for some $n \geq 2$. Indeed, the set of $k$-forms of the projective space $\mathbf{P}^{n-1}$ is in bijection with $H^1(k, \mathrm{PGL}_n)$, which is in bijection with $k^*/k^{*n}$ (this example was suggested by the referee).

Another related question is the following:

**Question 4.** *Does there exist a complex projective variety $X$ with infinitely many real forms, for which the group $\mathrm{Aut}(X)/\mathrm{Aut}(X)^0$ is finitely generated (it has to be infinite anyway)?*

Indeed, it is not finitely generated in the examples found so far [4–6, 8].

For completeness, let us formulate a more general result that follows from the proof. For $\alpha$ an infinite cardinal, say that a field $k$ is $\alpha$-tame if it has at most $\alpha$ Galois extensions up to $k$-isomorphism, or equivalently if $\mathrm{Gal}(\overline{k}/k)$ has $\leq \alpha$ open subgroups, or equivalently if the degree of $\overline{k}$ over $k$ is $\leq \alpha$. We say that a $k$-group is (cardinal $\leq \alpha$)-by-algebraic if it has a normal algebraic $k$-subgroup of index $\leq \alpha$.

**Theorem 5.** *Let $\alpha$ be an infinite cardinal, $k$ an $\alpha$-tame field and let $G$ be a $k$-group that is (cardinal $\leq \alpha$)-by-algebraic. Then the Galois cohomology set $\mathrm{H}^1(k, G)$ has cardinal $\leq \alpha$.*

*Let $k$ be a perfect field with at most $\alpha$ extensions up to $k$-isomorphism. Then every complete $\overline{k}$-variety has at most $\alpha$ $k$-forms.*

## 2. Group-theoretic background

We use standard conventions in Galois cohomology (as in [13]). If $P$ is a profinite group, by $P$-group $G$ we mean a group $G$ endowed with an action of $P$ by group automorphisms, such that the stabilizer $P_g$ of each $g \in G$ is open in $P$. The action is denoted $(s, g) \mapsto s \cdot g$. By $P$-subgroup we just mean a $P$-invariant subgroup. By $G^P$ we mean the subgroup of points fixed by $P$.

For reference we recall:

**Definition 6.** *Let $F$ be a finite group and $G$ an $F$-group. By definition, $\mathrm{H}^1(F, G)$ is the set of maps $b : F \to G$ satisfying $b(st) = b(s)(s \cdot b(t))$ for all $s, t \in F$, modulo identifying $b$ and $b'$ if there exists $g \in G$ such that $b'(s) = g^{-1} b(s)(s \cdot g)$ for all $s \in F$.*

*If $P$ is a profinite group and $G$ is a $P$-group, $\mathrm{H}^1(P, G)$ is the inductive limit of $\mathrm{H}^1(P/U, G^U)$ where $U$ ranges over open normal subgroups of $P$.*

**Lemma 7.** *Let $P$ be a metrizable profinite group and $G$ a countable $P$-group. Then $\mathrm{H}^1(P,G)$ is countable.*

**Proof.** In this case the number of open subgroups $U$ of $G$ is countable, so it is enough to check that each $\mathrm{H}^1(P/U, G^U)$ is countable. This is clear since the set of maps $P/U \to G^U$ is then countable. □

To prove Theorem 1 we need a dévissage result; namely there is one in [13, I.§5.5], and we need its corollary:

**Lemma 8 ([13, I.§5.5, Cor. 3]).** *Let $P$ be a profinite group, let $H$ be a $P$-group, $N$ a normal $P$-subgroup, $Q = H/N$. Then $\mathrm{H}^1(P,H)$ is finite (resp. countable) if and only if its image in $\mathrm{H}^1(P,Q)$ is finite (resp. countable), and that for every cocycle $b \in Z^1(P,H)$ the quotient $\mathrm{H}^1(P, {}_bN)/({}_bQ)^P$ is finite (resp. countable).*

Here ${}_bN$ means $N$ with the $P$-action twisted by the cocycle $b$ (see [13, I.§5.3]).

**Corollary 9.** *Under the same assumption, if $\mathrm{H}^1(P,Q)$ is finite (resp. countable) as well as $\mathrm{H}^1(P, {}_bN)$ for every $b \in Z^1(P,H)$ then so is $\mathrm{H}^1(P,H)$.*

Recall that an abelian group is divisible if $g \mapsto g^n$ is surjective for every positive integer $n$ (or equivalently every prime $n$).

**Lemma 10.** *Let $P$ be a metrizable profinite group. Let $A$ be a divisible abelian $P$-group, whose torsion subgroup $T$ is countable. Then $\mathrm{H}^1(P,A)$ is countable.*

**Proof.** First we claim that $\mathrm{H}^1(P,A) = \{0\}$ if $A$ is torsion-free. In this case, $A$ is naturally a vector space over $\mathbf{Q}$. Let $U$ be an open subgroup of $P$. Since the action of $P$ on $A$ is $\mathbf{Q}$-linear, we see that $A^U$ is also divisible. Hence by the usual averaging argument, $\mathrm{H}^1(P/U, A^U) = \{0\}$. By Definition 6 we deduce $\mathrm{H}^1(P,A) = \{0\}$.

Under the more general assumption on $A$, we deduce $\mathrm{H}^1(P, A/T) = \{0\}$, so by dévissage (Corollary 9) we are reduced to checking that $\mathrm{H}^1(P,T)$ is countable, which holds by Lemma 7. □

## 3. Tameness

We write the following proposition for the record.

**Proposition 11.** *Let $k$ be a field with separable closure $\bar{k}$. The following are equivalent:*

(1) *the absolute Galois group $G = \mathrm{Gal}(\bar{k}/k)$ is metrizable;*
(2) *$\bar{k}$ has countable degree over $k$;*
(3) *$\bar{k}$ has countably many $k$-subfields that are finite-dimensional over $k$;*
(4) *$\bar{k}$ has countably many Galois $k$-subfields that are finite-dimensional over $k$;*
(5) *$k$ has countably many finite separable extensions up to $k$-isomorphism (i.e., $k$ is tame);*
(6) *$k$ has countably many finite Galois extensions up to $k$-isomorphism.*

**Proof.** A totally disconnected compact Hausdorff topological space is metrizable if and only if it has countably many clopen subsets. Therefore, a profinite group is metrizable if and only if it has countably many clopen subsets, if and only if it has countably many normal open subgroups. This already implies the equivalence of (1) with each of (3) and (4).

Suppose (2). Let $(k_n)$ be an increasing sequence of subfields of $\bar{k}$ containing $k$, with union $\bar{k}$. These correspond to a decreasing sequence of open normal subgroups of $G$, with trivial intersection. Hence $G$ embeds into $\prod_n G_n$, and hence is metrizable. Conversely if $G$ is metrizable, it has such a sequence of open normal subgroups (enumerate open normal subgroups as $(H_n)$ and define $G_n = \bigcap_{m \le n} H_m$) and we deduce the equivalence with (1).

(5) clearly implies (6) and the converse is easy since each finite Galois extension contains only finitely many subextensions and each finite separable extension is contained in a finite Galois one.

The set of Galois $k$-subfields of $\bar{k}$ is in natural bijection with the set of $k$-isomorphism classes of finite Galois extensions, whence the equivalence between (6) and (4). $\square$

## 4. Automorphism groups

If $k$ is a perfect field with an algebraic closure $k \subset \bar{k}$, by $k$-group we mean a $\mathrm{Gal}(\bar{k}/k)$-group. If $H$ is an algebraic $k$-group, then $H(\bar{k})$ is naturally a $k$-group in this sense. Say that a $k$-group $A$ is countable-by-algebraic if it has a normal $k$-subgroup $N$ such that $H/N$ is countable and such that $N$ is isomorphic as $k$-group to the group of $\bar{k}$-points associated to some algebraic $k$-group.

We need the following classical result.

**Theorem 12 (Matsusaka, Grothendieck, Matsumura–Oort).** *Let $k$ be a field. For every complete $k$-variety $X$, the group of $\bar{k}$-automorphisms of $X$ is a countable-by-algebraic $k$-group.*

This is mostly due to Matsusaka [10, §11], who essentially proved it assuming $X$ smooth projective. The general case was formulated by Matsumura and Oort [9, Thm. 3.7], relying on Grothendieck's work in [7]. Note that "complete" means geometrically complete, i.e., without reference to the field of definition.

**Deduction of Corollary 2.** This is basic Galois cohomology (see [13, III.§1]). For a $\bar{k}$-variety with no $k$-form, there is nothing to prove. Hence, let $X$ be a complete $k$-variety. Then mapping a $k$-form to its cohomology class induces an injective map from the set of $k$-isomorphism types of $k$-forms of $X$ into $\mathrm{H}^1(k, \mathrm{Aut}_{\bar{k}}(X))$. Since the $k$-group $\mathrm{Aut}_{\bar{k}}(X)$ is countable-by-algebraic (Theorem 12), Theorem 1 applies. $\square$

## 5. Proof of the theorem

The following is well-known and holds regardless of the characteristic.

**Lemma 13.** *Let $G$ be the group of $k$-points of a torus or abelian variety over an algebraically closed field $k$. Then $G$ is divisible with countable torsion subgroup.*

**Proof.** The case of tori is straightforward (as $G$ is then isomorphic to $(k^*)^{\dim G}$). For abelian varieties, this follows from [12, §II.6, Appl. 2], which says that $G$ is divisible with finite $n$-torsion for every $n$. The countability of the torsion subgroup immediately follows. $\square$

We also use the following result about dévissage of algebraic group themselves.

**Theorem 14 (Chevalley, see [3]).** *Let $k$ be a perfect field. Every smooth connected algebraic $k$-group $G$ has a smallest normal algebraic subgroup $L$ such that $G/L$ is complete. Moreover, $G/L$ is an abelian variety, $L$ is connected affine and smooth, and this formation commutes with taking field extensions.*

**Proof of Theorem 5.** We assume that $k$ is a perfect tame field and $G$ is a countable-by-algebraic $k$-group, and we have to prove that $\mathrm{H}^1(k, G)$ is countable. Let us start with three special cases:

- Suppose that $G$ is countable. Then Lemma 7 does the job.
- Suppose that $G$ is a torus or an abelian variety. Then Lemma 10 yields the conclusion (we can apply it thanks to Lemma 13).
- Suppose that $G$ is unipotent. Then $\mathrm{H}^1(k, G) = \{0\}$ (see [15] for references and for counterexamples when $k$ is not perfect).

By the dévissage Corollary 9, we can suppose that the $k$-group $G$ is either countable (this case is settled), or connected algebraic. So we now assume that $G$ is a connected algebraic $k$-group.

Using Theorem 14 and again using dévissage, we can suppose that the connected algebraic $k$-group is either an abelian variety (this case has been settled), or affine. So we now assume that $G$ is a connected affine algebraic $k$-group.

Let $H$ be a Cartan subgroup of $G$ that is defined over $k$. Then $\mathrm{H}^1(k,H) \to \mathrm{H}^1(k,G)$ is surjective [1, Cor. 2.14]. This reduces to the case when $G^0$ is affine solvable, and hence, by dévissage again, to the case when $G$ is affine, solvable and connected. In turn, the latter reduces, once more by dévissage, to when $G$ is either unipotent or a torus, and these cases have been settled. □

**Remark 15.** If $k = \mathbf{R}$ one easily checks that $\mathrm{H}^1(k,G)$ is finite for every abelian $k$-variety $G$ (and hence for every algebraic $k$-group $G$). For $k = \mathbf{Q}_p$ this is not true: $\mathrm{H}^1(k,G)$ is infinite for every positive-dimensional abelian $k$-variety $G$. (Indeed, by [14], $\mathrm{H}^1(k,G)$ is then isomorphic to the Pontryagin dual of the compact $p$-adic Lie group $\widehat{G}(\mathbf{Q}_p)$, where $\widehat{G}$ is the dual abelian variety; in addition $\widehat{G}(\mathbf{Q}_p)$ has dimension equal to $\dim(\widehat{G}) = \dim(G)$ [11] and in particular is infinite, so the Pontryagin dual is infinite.)

**On the proof of Theorem 5.** The main difference is Lemma 7, in which case we get the bound $\alpha$ instead of $\aleph_0$. Then we need the obvious analogue of Corollary 9, which is not stated in [13] but can be derived in the same fashion. Note that the countability result in Lemma 13, which is used at several steps in the dévissage, does not require any tameness assumption on $k$. □

## References


[1] A. Borel, J.-P. Serre, "Théorèmes de finitude en cohomologie galoisienne", *Comment. Math. Helv.* **39** (1964), p. 111-169.

[2] A. Bot, "A smooth complex rational affine surface with uncountably many real forms", https://arxiv.org/abs/2105.08044, 2021.

[3] M. Brion, "Some structure theorems for algebraic groups", in *Algebraic groups: structure and actions*, Proceedings of Symposia in Pure Mathematics, vol. 94, American Mathematical Society, 2017, p. 53-126.

[4] T.-C. Dinh, K. Oguiso, "A surface with discrete and nonfinitely generated automorphism group", *Duke Math. J.* **168** (2019), no. 6, p. 941-966.

[5] T.-C. Dinh, K. Oguiso, X. Yu, "Smooth complex projective rational surfaces with infinitely many real forms", https://arxiv.org/abs/2106.05687, to appear in *J. Reine Angew. Math.*, 2021.

[6] ———, "Smooth rational projective varieties with non-finitely generated discrete automorphism group and infinitely many real forms", *Math. Ann.* **383** (2022), p. 399-414.

[7] A. Grothendieck, "Techniques de construction et théorèmes d'existence en géométrie algébrique IV : les schémas de Hilbert", in *Séminaire Bourbaki. Vol. 6. Années 1960/61*, Société Mathématique de France, 1960, p. 249-276.

[8] J. Lesieutre, "A projective variety with discrete, non-finitely generated automorphism group", *Invent. Math.* **212** (2018), no. 1, p. 189-211.

[9] H. Matsumura, F. Oort, "Representability of group functors, and automorphisms of algebraic schemes", *Invent. Math.* **4** (1967), p. 1-25.

[10] T. Matsusaka, "Polarized varieties, fields of moduli and generalized Kummer varieties of polarized abelian varieties", *Am. J. Math.* **80** (1958), p. 45-82.

[11] A. Mattuck, "Abelian varieties over $p$-adic ground fields", *Ann. Math.* **62** (1955), p. 92-119.

[12] D. Mumford, *Abelian varieties*, Tata Institute of Fundamental Research Studies in Mathematics, vol. 5, Oxford University Press, 1970.

[13] J.-P. Serre, *Cohomologie galoisienne*, Lecture Notes in Mathematics, Springer, 1994.

[14] J. Tate, "WC-groups over $\mathfrak{p}$-adic fields", in *Bourbaki Seminar. Vol. 4. Years 1956/57-1957/58*, Société Mathématique de France, 1995, p. 265-277.

[15] N. Q. Thắng, N. D. Tân, "On the Galois and flat cohomology of unipotent algebraic groups over local and global function fields. I", *J. Algebra* **319** (2008), no. 10, p. 4288-4324.