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Uniqueness of bounded solutions to p-Laplace problems in strips

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Abstract. We consider a p-Laplace problem in a strip with two-constant boundary Dirichlet conditions. We show that if the width of the strip is smaller than some $d_0 \in (0, +\infty]$, then the problem admits a unique bounded solution, which is strictly monotone. Hence this unique solution is one-dimensional symmetric and belongs to the C^2 class. We also show that the problem has no bounded solution in the case that $d_0 < +\infty$ and the width of the strip is larger than or equal to d_0. An analogous rigidity result in the whole space was obtained recently by Esposito et al. [8]

Keywords. p-Laplace equation, uniqueness, monotonicity, 1D symmetry.

Mathematical subject classification (2010). 35J92, 35A01, 35A02, 35B06.

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1. Introduction

Given $N \geq 1$, $p > 1$, $a < b$, $\Omega = \mathbb{R}^{N-1} \times (0, h)$ and f is a locally Lipschitz continuous function, we study the uniqueness, symmetry and monotonicity of solutions to the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega, \\ u = a & \text{on } \mathbb{R}^{N-1} \times \{0\}, \\ u = b & \text{on } \mathbb{R}^{N-1} \times \{h\}, \end{cases} \tag{1}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian of u. When $p = 2$, this is a classical topic which was investigated in the works of Berestycki, Caffarelli and Nirenberg [2,3], Angenent [1], Clément and Sweers [4] for various types of domains.

One difficulty that arises in the case $p \neq 2$ is that the p-Laplacian is nonlinear. Moreover, this operator is singular or degenerate depending on $1 < p < 2$ or $p > 2$, respectively. Hence, solutions to (1) should be understood in the weak sense, see [6, 10]. Throughout our paper, solutions to p-Laplace problems with boundary conditions will be always understood in the following weak sense.

Definition 1. *Let $f:\mathbb{R}\to\mathbb{R}$ and $g:\partial\mathcal{U}\to\mathbb{R}$ be continuous functions, where $\mathcal{U}$ is a domain of $\mathbb{R}^N$. A function $u\in W^{1,p}_{\mathrm{loc}}(\mathcal{U})\cap C(\overline{\mathcal{U}})$ is called a (weak) solution to the problem*

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \mathcal{U}, \\ u = g & \text{on } \partial\mathcal{U} \end{cases}$$

if

$$u(x)=g(x) \quad \text{for all } x\in\partial\mathcal{U}$$

and

$$\int_{\mathcal{U}} |\nabla u|^{p-2}\nabla u\cdot\nabla\psi = \int_{\mathcal{U}} f(u)\psi \quad \text{for all } \psi\in C_0^1(\overline{\mathcal{U}}). \tag{2}$$

Throughout this paper, we denote

$$Z_f := \{z\in[a,b] \mid f(z)=0\}.$$

Under some mild assumptions on f, we find a necessary and sufficient condition on h to ensure that problem (1) has a bounded solution (by bounded solution, we mean weak solutions in $L^\infty(\Omega)$). We also show that the solution is unique and monotone increasing in the x_N-direction. Our main result is the following.

Theorem 2. *Suppose that f is a locally Lipschitz continuous function such that $f(a)=f(b)=0$, $f(s)>0$ for $s<a$, $f(s)<0$ for $s>b$, and*

$$\liminf_{t\to z^+}\frac{f(t)}{(t-z)^{p-1}} > -\infty, \quad \limsup_{t\to z^-}\frac{f(t)}{(z-t)^{p-1}} < +\infty \quad \text{for all } z\in Z_f. \tag{3}$$

Then problem (1) has a bounded solution if and only if

$$h < \left(\frac{p-1}{p}\right)^{\frac{1}{p}} \int_a^b \frac{ds}{\left[\max_{[a,b]} F - F(s)\right]^{\frac{1}{p}}}, \tag{4}$$

where $F(s)=\int_a^s f(t)\mathrm{d}t$. Moreover, such a solution u is unique and $\partial_{x_N}u>0$ in Ω. Consequently, $u\in C^2(\Omega)$ and u depends only on x_N, i.e., $u(x)=u(x_N)$.

If we further assume that $f(t)=-f(a+b-t)$ for $t\in[a,\frac{a+b}{2}]$, then

$$u(x_N) = a+b-u(h-x_N) \quad \text{for } x_N\in\left[0,\frac{h}{2}\right].$$

Remark 3. Locally Lipschitz continuous functions satisfy (3) automatically when $1<p\le 2$. Notice also that the integral in (4) may be infinity. In particular, this happens to the Allen–Cahn nonlinearity $f(t)=t-t^3$ when $(a,b)=(-1,1)$ and $1<p\le 2$.

Theorem 2 is motivated by a recent result by Esposito et al. [8], where they studied the problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \mathbb{R}^N, \\ |u|\le 1 & \text{in } \mathbb{R}^N, \\ \lim_{x_N\to\pm\infty} u(x',x_N) = \pm 1 & \text{uniformly in } x'\in\mathbb{R}^{N-1}, \end{cases} \tag{5}$$

under assumptions $\frac{2N+2}{N+2}<p<2$ and $f\in C^1([-1,1])$ such that $f(-1)=f(1)=0$, $f'_+(-1)<0$, $f'_-(1)<0$ and Z_f is finite. They prove that every solution u of (5) depends only on x_N and $\frac{\partial u}{\partial x_N}>0$ in $\mathbb{R}^N$.

The proof in [8] is based on the method of moving planes, in which various types of comparison principles for the p-Laplacian are exploited. This in turn requires the technical assumption $\frac{2N+2}{N+2}<p<2$ and some restrictions on f.

The moving plane method can also be applied to problem (1), which can be regarded as a finite version of problem (5), to yield the one-dimensional symmetry of solutions under similar restrictions on p and f. To cover the full range $1<p<+\infty$ and to deal with weaker assumptions on f in Theorem 2, we follow another approach based on the ideas in [5, 7]. First, we construct the minimal and maximal solutions to the problem. These solutions depend only on x_N. Then a careful ODE analysis of (1) with $N = 1$ reveals that the problem has a unique solution in dimension one, and it is monotone increasing. This implies that solutions in higher dimensions are also unique, one-dimensional symmetric and monotone increasing in x_N.

In Section 2, we prove Theorem 2 for dimension one. In Section 3, we prove it for more general cases. We always use C to denote a generic positive constant whose concrete value is not important.

2. ODE analysis

In this section, we study the ODE problem

$$\begin{cases} -(|u'|^{p-2}u')' = f(u) & \text{in } (0,h), \\ a \le u \le b & \text{in } (0,h), \\ u(0) = a, \ u(h) = b. \end{cases} \tag{6}$$

Theorem 4. *Let f be a locally Lipschitz continuous function such that $f(a) = f(b) = 0$ and* (3) *holds. Then problem* (6) *has a solution if and only if* (4) *holds. Moreover, such a solution u is unique, $u' > 0$ in $(0,h)$ and $u \in C^2((0,h))$. If we further assume that $f(t) = -f(a+b-t)$ for $t \in [a, \frac{a+b}{2}]$, then*

$$u(t) = a + b - u(h-t) \quad \textit{for } t \in \left[0, \frac{h}{2}\right]. \tag{7}$$

Proof. Assume that problem (6) admits a solution u. By Hopf's lemma [11], we have $u'(0) > 0$. Let $(0, t_0)$ be the largest interval such that $u'(t) > 0$ for all $t \in (0, t_0)$. By the standard regularity, u is C^2 in $(0, t_0)$ and in this interval

$$(p-1)|u'|^{p-2}u'' = -f(u). \tag{8}$$

We show that $t_0 = h$.

Suppose on the contrary that $t_0 < h$. Then $u'(t_0) = 0$.

First, we claim that $f(u(t_0)) > 0$. Indeed, if $f(u(t_0)) = 0$, then we obtain from (3) that

$$-\Delta_p(u(t_0) - u(t)) + C(u(t_0) - u(t))^{p-1} = -f(u(t)) + C(u(t_0) - u(t))^{p-1} \ge 0$$

for some constant $C > 0$ and all $t < t_0$ close to t_0. We can now apply Hopf's lemma [11] to conclude that $u'(t_0) > 0$, a contradiction to $u'(t_0) = 0$. Now we assume $f(u(t_0)) < 0$, then from (8) we obtain $u''(t) > 0$ for all $t < t_0$ and close to t_0. It follows that $u'(t_0) > u'(t) > 0$ for $t < t_0$ and close to t_0. This contradiction shows that $f(u(t_0)) > 0$.

By continuity, we have $f(u(t)) > 0$ for $t \in [t_0, t_0 + \varepsilon]$ for some small $\varepsilon > 0$.

We claim that $u'(t) \le 0$ or all $t \in (t_0, t_0 + \varepsilon)$. Otherwise we can find $r_1 \in (t_0, t_0 + \varepsilon)$ such that $u'(r_1) > 0$. Consider the maximal interval $(r_2, r_1] \subset (t_0, r_1]$ such that

$$u'(t) > 0 \text{ in } (r_2, r_1], \quad u'(r_2) = 0.$$

From the standard elliptic regularity, we know that u is C^2 in $(r_2, r_1]$. Hence (8) holds in this interval and $u''(t) < 0$ for $t \in (r_2, r_1]$. This implies $u'(r_2) > u'(r_1) > 0$, which is a contradiction.

We further claim that $u'(t) < 0$ or all $t \in (t_0, t_0 + \varepsilon)$. Suppose on the contrary that $u'(r_1) = 0$ for some $r_1 \in (t_0, t_0 + \varepsilon)$. If $u'(t) = 0$ for all $t \in [t_0, r_1]$ then from the equation we deduce $f(u(t)) = 0$ in this interval, a contradiction. Hence there exists $r_2 \in (t_0, r_1)$ such that $u'(r_2) < 0$. Let $(r_2, r_3) \subset (r_2, r_1)$ be the maximal interval such that

$$u'(t) < 0 \text{ in } [r_2, r_3), \quad u'(r_3) = 0.$$

Now, (8) holds in $[r_2, r_3)$, which implies $u''(t) < 0$ in this interval. Therefore, $u'(r_2) > u'(r_3) = 0$, a contradiction.

From (8), we have

$$\frac{p-1}{p}|u'|^p + F(u) = \frac{p-1}{p}|u'(0)|^p \quad \text{for } t \in [0, t_0]. \tag{9}$$

Taking $t = t_0$ we deduce $F(u(t_0)) = \frac{p-1}{p}|u'(0)|^p > 0$. From this and the fact that $u' > 0$ in $(0, t_0)$, formula (9) becomes

$$\frac{p-1}{p}(u')^p = F(u(t_0)) - F(u),$$

or equivalently,

$$\frac{u'}{[F(u(t_0)) - F(u)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}}.$$

It follows that $u(t)$ is uniquely determined in $(0, t_0)$ by

$$\int_a^{u(t)} \frac{\mathrm{d}s}{[F(u(t_0)) - F(s)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} t \quad \text{for all } t \in [0, t_0]. \tag{10}$$

Now let $(t_0, t_1) \subset (t_0, h)$ be the maximal interval such that $u'(t) < 0$ or all $t \in (t_0, t_1)$. We show that $t_1 \geq 2t_0$ and $u(t) = u(2t_0 - t)$ for $t \in [0, t_0]$. Clearly, (9) holds in $[t_0, t_1]$ and we deduce

$$\int_{u(t)}^{u(t_0)} \frac{\mathrm{d}s}{[F(u(t_0)) - F(s)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} (t - t_0) \quad \text{for all } t \in [t_0, t_1].$$

Combining this with (10), we have

$$\int_a^{u(t)} \frac{\mathrm{d}s}{[F(u(t_0)) - F(s)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} (2t_0 - t) \quad \text{for all } t \in [t_0, t_1],$$

or equivalently,

$$\int_a^{u(2t_0-t)} \frac{\mathrm{d}s}{[F(u(t_0)) - F(s)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} t \quad \text{for all } t \in [2t_0 - t_1, t_0],$$

Comparing this with (10) we immediately obtain $t_1 \geq 2t_0$ and $u(2t_0 - t) = u(t)$ in $[0, t_0]$. Hence $u'(2t_0) = -u'(0) < 0$. This implies $u(t) < a$ for $t > 2t_0$ and close to $2t_0$, a contradiction.

Therefore, we must have $t_0 = h$. By Hopf's lemma [11], we also have $u'(h) > 0$.

Hence $u'(t) > 0$ for all $t \in [0, h]$ and we have

$$\frac{p-1}{p}|u'|^p + F(u) = \frac{p-1}{p}|u'(0)|^p \quad \text{for } t \in [0, h]. \tag{11}$$

This implies $\frac{p-1}{p}|u'(0)|^p > \max_{[a,b]} F$ and

$$\int_a^{u(t)} \frac{\mathrm{d}s}{\left[\frac{p-1}{p}|u'(0)|^p - F(s)\right]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} t \quad \text{for all } t \in [0, h].$$

Choosing $t = h$, we have

$$\int_a^b \frac{\mathrm{d}s}{\left[\frac{p-1}{p}|u'(0)|^p - F(s)\right]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} h.$$

Therefore, the necessary condition for the existence of a solution of (6) is that (4) holds.

Conversely, if (4) holds, then every solution u to (6) satisfies

$$\int_a^{u(t)} \frac{\mathrm{d}s}{[c - F(s)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} t \quad \text{for all } t \in [0, h], \tag{12}$$

where $c > \max_{[a,b]} F$ is such that

$$\int_a^b \frac{\mathrm{d}s}{[c-F(s)]^{\frac{1}{p}}} = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} h.$$

This implies uniqueness. It also implies

$$u'(t) = \left(\frac{p}{p-1}\right)^{\frac{1}{p}} [c - F(u(t))]^{\frac{1}{p}}.$$

Hence $u' > 0$ in $[0,h]$, $c = \frac{p-1}{p}|u'(0)|^p$ and (11) holds. This means that u defined by (12) is actually a solution to problem (6). Property (7) can easily be checked using the above formulae. This completes the proof. □

3. Higher dimensions

This section is devoted to the proof of our main result.

Lemma 5. *Under assumptions of Theorem 2, we have*

$$a \le u \le b \quad in\, \Omega.$$

Proof. We follow an idea in [9]. Assume by contradiction that $\sup u = M > b$. Let $\gamma \in (b, M)$ and $\varphi_R \in C_c^\infty(\mathbb{R}^N)$ be a standard cut-off function. That is, $0 \le \varphi \le 1$ and

$$\begin{cases} \varphi_R = 1 & \text{in } B_R, \\ \varphi_R = 0 & \text{in } \mathbb{R}^{N-1} \setminus B_{2R}, \\ |\nabla \varphi_R| \le \frac{2}{R} & \text{in } B_{2R} \setminus B_R. \end{cases}$$

We denote $a^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. By approximation argument, we can use $\psi = (u-\gamma)^+ \varphi_R^p \chi_\Omega$ as a test function in (2) to obtain

$$\begin{aligned} -\int_{\{u>\gamma\}} f(u)(u-\gamma)\varphi_R^p &= -\int_{\{u>\gamma\}} |\nabla u|^p \varphi_R^p - p\int_{\{u>\gamma\}} |\nabla u|^{p-2}(u-\gamma)\varphi_R^{p-1} \nabla u \cdot \nabla \varphi_R \\ &\le -\int_{\{u>\gamma\}} |\nabla u|^p \varphi_R^p + p\int_{\{u>\gamma\}} |\nabla u|^{p-1}(u-\gamma)\varphi_R^{p-1} |\nabla \varphi_R| \\ &\le -\int_{\{u>\gamma\}} |\nabla u|^p \varphi_R^p + \int_{\{u>\gamma\}} \left\{ \left(|\nabla u|^{p-1}\varphi_R^{p-1}\right)^{\frac{p}{p-1}} + C_p \left[(u-\gamma)|\nabla \varphi_R|\right]^p \right\} \\ &\le C_p \int_{\{u>\gamma\}} [(u-\gamma)]^p |\nabla \varphi_R|^p. \end{aligned}$$

Taking into account that u is bounded and $p > 1$, we have

$$C_0 \int_{\{u>\gamma\}} (u-\gamma)\varphi_R^p \le C_p \int_{\{u>\gamma\}} [(u-\gamma)]^p |\nabla \varphi_R|^p \le C_1 R^{-p} \int_{\{u>\gamma\}\cap B_{2R}} (u-\gamma), \tag{13}$$

where $C_0 = \min_{[\gamma,M]}(-f) > 0$. By setting

$$\mathscr{L}(R) = \int_{\{u>\gamma\}\cap B_R} (u-\gamma),$$

then $\mathscr{L}(R) \le C_2 R^N$. Moreover, from (13), there exists $R_0 > 0$ such that

$$\mathscr{L}(R) \le 2^{-N-1} \mathscr{L}(2R) \quad \text{for all } R > R_0.$$

Hence Lemma 2.1 in [9] implies $\mathscr{L}(R) \equiv 0$, which means $u \le \gamma$ in Ω. This contradicts the assumption $\sup u = M > \gamma$.

Therefore, $u \le b$ in Ω. Similarly, we can show that $u \ge a$ in Ω. □

Proof of Theorem 2. The existence follows from Theorem 4.

Let us assume that problem (1) has a bounded solution u. By Lemma 5,

$$a \le u \le b \quad \text{in } \Omega.$$

Since $f(a) = f(b) = 0$ and by the boundary conditions, we can apply the strong maximum principle [11] to deduce that

$$a < u < b \quad \text{in } \Omega.$$

We show that there is a maximal solution u^* in the order interval $[u, b]$ in the sense that any solution v with $u \le v \le b$ satisfies $v \le u^*$. To this end, for each $n > 1$ we define

$$\Omega_n = \{(x', x_N) \in \Omega \mid |x'| < n\}.$$

Setting

$$\phi_n(x) = \begin{cases} b & \text{on } \partial\Omega_n \setminus \{x_N = 0\}, \\ a + (b-a)(|x'| - n + 1)^+ & \text{on } \partial\Omega_n \cap \{x_N = 0\}. \end{cases}$$

We now consider the auxiliary problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega_n, \\ u = \phi_n & \text{on } \partial\Omega_n. \end{cases} \tag{14}$$

Obviously, $u|_{\Omega_n}$ is a lower solution and b is an upper solution to (14). Hence by standard upper and lower solution argument, problem (14) has a maximal solution u_n in $[u, b]$. Since $u_{n+1}|_{\Omega_n}$ is a lower solution to (14) in this interval, we have $u_n \ge u_{n+1}$ in Ω_n. By a standard regularity argument, $u^* = \lim_{n\to\infty} u_n$ is well defined in Ω and it is a solution of (1).

Clearly, $u^* \in [u, b]$. If v is any solution of (1) in $[u, b]$, then it is evident that $v|_{\Omega_n}$ is a lower solution of (14) and it follows that $u_n \ge v$ (since the standard upper and lower solution argument implies the existence of a solution in $[v, b]$ which is less than or equal to u_n). It follows that $v \le u^*$. Thus u^* is the maximal solution in $[u, b]$.

Since the equation is invariant under translations in x', the maximality implies that u^* is a function of x_N only. Indeed, we assume by contradiction that there exist two points $x = (x^0, l)$ and $y = (y^0, l)$, $a < l < b$, such that $u^*(x) \ne u^*(y)$. We define $\tilde{u}^*(x', y) = u^*(x' + y^0 - x^0, x_N)$. Then $\tilde{u}^*$ is a solution to (1) and $v(x) := \max\{u^*(x), \tilde{u}^*(x)\} \ge u^*(x)$. Moreover, $v \not\equiv u^*$ and v is a lower solution to (1) satisfying $v \le b$. Hence (1) has a solution v^* in the order interval $[v, b]$. It follows that v^* is a solution in $[u, b]$ which satisfies $v^* \ge u^*$ and $v^* \not\equiv u^*$. This contradicts the maximality of u^*.

Thus u^* is a function of x_N only and hence it is a solution of (6).

In the same way, we can construct a minimal solution u_* in $[a, u]$ by considering the auxiliary problem

$$\begin{cases} -\Delta_p u = f(u) & \text{in } \Omega_n, \\ u = \psi_n & \text{on } \partial\Omega_n, \end{cases} \tag{15}$$

where

$$\psi_n(x) = \begin{cases} a & \text{on } \partial\Omega_n \setminus \{x_N = h\}, \\ b + (a-b)(|x'| - n + 1)^+ & \text{on } \partial\Omega_n \cap \{x_N = h\}. \end{cases}$$

Moreover, the minimality of u_* implies that u_* is a function of x_N only. Thus u_* is a solution of (6).

By Theorem 4, we necessarily have $u_* = u^* = U$, where U is the unique solution of (6). Moreover, condition (4) holds. Since $u_* \le u \le u^*$, this implies that $u(x) \equiv U(x_N)$. The conclusion then follows from the properties of U obtained in Theorem 4. □

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