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Comptes Rendus
Mathématique

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Volume 361 (2023), p. 803-806

Published online: 11 May 2023

https://doi.org/10.5802/crmath.445

Les Comptes Rendus. Mathématique *sont membres du Centre Mersenne pour l'édition scientifique ouverte*
www.centre-mersenne.org
e-ISSN : 1778-3569

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Group theory / *Théorie des groupes*

On the occurrence of elementary abelian p-groups as the Schur multiplier of non-abelian p-groups

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Abstract. We prove that every elementary abelian p-group, for odd primes p, occurs as the Schur multiplier of some non-abelian finite p-group.

Keywords. Schur multiplier, finite p-group.
Mathematical subject classification (2010). 20J99, 20D15.
Funding. This research is supported by INSPIRE Faculty Research Grant DST/INSPIRE/04/2016/001594.
Manuscript received 25 August 2022, revised and accepted 21 November 2022.

1. Introduction

The Schur multiplier $M(G)$ of a finite group G is defined as the second cohomology group of G with coefficients in $\mathbb{C}^*$. It plays an important role in the theory of extensions of groups. Finding bounds on the order, exponents, and ranks of the Schur multiplier of prime power groups, as well as classifying groups with trivial Schur multipliers, have been main lines of investigation in the past. Another problem where it appears that not much progress has been made is the following:

Question 1. Is it true that every finite abelian p-group is isomorphic to the Schur multiplier of some non-abelian finite p-group? [5, Question 15.30].

More generally,

Question 2. Which abelian groups occur as the Schur multipliers of non-abelian finite groups? [6].

The immediate questions that arise from these questions are the following:

Question 3. For all natural numbers n does there exist a non-abelian p-group G such that $|M(G)| = p^n$?

Question 4. For all natural numbers e does there exist a non-abelian p-group G such that exponent of $M(G)$ is p^e?

Question 5. For all natural numbers d does there exist a non-abelian p-group G such that the minimal number of generators of $M(G)$ is d?

It is easy to see that the answers to the Questions 3 and 4 are affirmative. To see this, consider

$$G = E_p \times (\mathbb{Z}_{p^{m+1}} \times \mathbb{Z}_{p^m}),$$

where E_p is the extra-special p-group of order p^3 and exponent p, and $\mathbb{Z}_n$ denote the cyclic group of order n. For a group G, let $\gamma_2(G)$ denote the derived subgroup of G. Applying [4, Theorem 2.2.10] we get that

$$M(G) \cong M(E_p) \times M(\mathbb{Z}_{p^{m+1}} \times \mathbb{Z}_{p^m}) \times \left(\frac{E_p}{\gamma_2(E_p)} \otimes (\mathbb{Z}_{p^{m+1}} \times \mathbb{Z}_{p^m})\right).$$

Therefore $|M(G)| = p^2 p^m p^4 = p^{m+6}$ and exponent of $M(G)$ is p^m. See [2, 3] for the existence of non-abelian groups having Schur multiplier of order p^n for each $n \le 6$.

We now state the main result of this article, which gives an affirmative answer to the Question 5:

Theorem 6. *Given a natural number n and an odd prime p, there exists a non-abelian p-group whose Schur multiplier is elementary abelian of order p^n.*

2. Proof

Before proceeding to the proof of Theorem 6, we set some notations that are mostly standard. Let G be a group and $x, y \in G$. Then $[x, y]$ denotes the commutator $x^{-1}y^{-1}xy$. By $d(G)$ we denote the minimal number of generators of G. The exponent of the group G is denoted by $\exp(G)$. We write $\gamma_i(G)$ for the i-th term in the lower central series of G. By $\mathbb{Z}_{p^n}$ we denote the cyclic group of order p^n.

The proof is founded on the following two results of Blackburn and Evens [1].

Proposition 7 ([4, Corollary 3.2.4]). *Let G be a finite nilpotent group of class $c \ge 2$ and let $G = F/R$ be a presentation of G as a factor group of a free group of finite rank. Then*

(1) *upon identifying $G/\gamma_2(G)$ with $F/\gamma_2(F)R$ and $\gamma_c(G)$ with $\gamma_c(F)R/R$, the map*

$$\begin{aligned} G/\gamma_2(G) \otimes \gamma_c(G) &\xrightarrow{\lambda} \gamma_{c+1}(F)/([F,R] \cap \gamma_{c+1}(F)) \\ f\gamma_2(F)R \otimes xR &\longmapsto [f, x]([F,R] \cap \gamma_{c+1}(F)) \end{aligned}$$

is a surjective homomorphism.

(2) *Let $X = \operatorname{Ker}\lambda$. Then there is an exact sequence*

$$1 \longmapsto X \longmapsto G/\gamma_2(G) \otimes \gamma_c(G) \longmapsto M(G) \longmapsto M(G/\gamma_c(G)) \longmapsto \gamma_c(G) \longmapsto 1.$$

Lemma 8 ([1, Remark, Section 3]). *Let G be a group of nilpotency class 2 with $G/\gamma_2(G)$ elementary abelian. Then X, as defined in Proposition 7, is generated by all elements of the form $x\gamma_2(G) \otimes [y, z] + y\gamma_2(G) \otimes [z, x] + z\gamma_2(G) \otimes [x, y]$ and all elements of the form $x\gamma_2(G) \otimes x^p$.*

The proof of the Theorem 6 also makes use of the following lemma. The proof of the lemma can be extracted from the proof of [7, Proposition 3.2].

Lemma 9. *Let G be a special p-group minimally generated by $x_1, x_2, \dots, x_d$, and W_G be the subgroup of $G/\gamma_2(G) \otimes \gamma_2(G)$ generated by all elements of the form $x\gamma_2(G) \otimes [y, z] + y\gamma_2(G) \otimes [z, x] + z\gamma_2(G) \otimes [x, y]$, $x, y, z \in G$. Then W_G is generated by B_G where*

$$B_G = \{x_i\gamma_2(G) \otimes [x_j, x_k] + x_j\gamma_2(G) \otimes [x_k, x_i] + x_k\gamma_2(G) \otimes [x_i, x_j] \mid 1 \le i < j < k \le d\}.$$

For a special p-group G notice that $G/\gamma_2(G)$ and $\gamma_2(G)$ are elementary abelian and therefore can be regraded as vector spaces over $\mathbb{Z}_p$. With this in mind, we are now ready to present the following two propositions, that are the main ingredients of the proof of the Theorem 6.

Proposition 10. *For an odd prime p and natural numbers d and j, where $d \geq 4$ and $3 \leq j+2 \leq d$, let G_j be the special p-group of exponent p minimally generated by $x_1, x_2, \ldots, x_d$, such that all commutators $[x_r, x_s]$, other than $[x_1,x_2],[x_2,x_3],\ldots [x_{j+1},x_{j+2}]$, are identity and the set $\{[x_1,x_2],[x_2,x_3],\ldots [x_{j+1},x_{j+2}]\}$ is linearly independent. Then $M(G_j)$ is elementary abelian of order $p^{\frac{1}{2}d(d-1)+2j+1}$.*

Proof. Since G_j is a group of nilpotency class 2, $\exp(M(G_j))$ divides $\exp(G)$. Therefore $M(G_j)$ is an elementary abelian group. Now, using the exact sequence given in Proposition 7, we get that

$$\left|M(G_j)\right| = \frac{\left|M\left(\frac{G_j}{\gamma_2(G_j)|}\right)\right|}{\left|\gamma_2(G_j)\right|}\frac{\left|\frac{G_j}{\gamma_2(G_j)}\otimes\gamma_2(G_j)\right|}{\left|X\right|}. \tag{1}$$

Let W_{G_j} and B_{G_j} be the sets defined in Lemma 9. To obtain $|X|$, note from Lemma 8 that $|X| = |W_{G_j}|$ because the group G_j is of exponent p. Let (x,y,z) denotes the element $x\gamma_2(G_j)\otimes[y,z]+y\gamma_2(G_j)\otimes[z,x]+z\gamma_2(G_j)\otimes[x,y] \in G_j/\gamma_2(G_j)\otimes\gamma_2(G_j)$ for $x,y,z \in G_j$. Consider the following subset D_{G_j} of B_{G_j}

$$D_{G_j} = \left\{(x_i, x_{i+1}, x_k)\,\middle|\, i = 1,2,\ldots j+1,\ k = 1,2,\ldots d \text{ and } k \neq i-1, i, i+1\right\},$$

and notice that all the elements of B_{G_j} can be generated by elements of D_{G_j}. As a result D_{G_j} generates W_{G_j}. We further claim that D_{G_j} is a basis for W_{G_j}. To see this, let

$$\sum_{i=1}^{j+1}\sum_{\substack{k=1\\k\neq i-1,i,i+1}}^{d}\alpha_{ik}(x_i,x_{i+1},x_k) = 0.$$

This gives

$$\sum_{i=1}^{j+1}\left(\left(\sum_{\substack{k=1\\k\neq i-1,i,i+1}}^{d}\alpha_{ik}x_k\right)+\alpha_{(i-1)(i+1)}x_{i-1}\right)\otimes[x_i,x_{i+1}] = 0.$$

Now, since $\{x_1\gamma_2(G), x_2\gamma_2(G),\ldots x_d\gamma_2(G)\}$ and $\{[x_1,x_2],[x_2,x_3],\ldots,[x_{j+1},x_{j+2}]\}$ are linearly independent, it follows that $\alpha_{ik} = 0$ for all $i = 1,2,\ldots j+1,\ k = 1,2,\ldots d, k \neq i-1,i,i+1$. Next, notice that $|D_{G_j}| = (d-3)j+1$, and hence $|W_{G_j}| = p^{(d-3)j+1}$. From [4, Corollary 2.2.12] we also have that $|M(G/\gamma_2(G_j))| = p^{\frac{1}{2}d(d-1)}$, because $G_j/\gamma_2(G_j)$ is an elementary abelian p-group of order p^d. Putting these values and $|\gamma_2(G_j)| = p^{j+1}$ in Equation (2.1) we get that

$$|M(G_j)| = p^{\frac{1}{2}d(d-1)+2j+1}.$$ □

The following Proposition can be proved along the same lines.

Proposition 11. *For an odd prime p and a natural number j, where $4 \leq j+3 \leq d$, let G_j be the special p-group of exponent p minimally generated by $x_1, x_2, \ldots, x_d$, $d \geq 4$, such that all commutators $[x_r, x_s]$, other than $[x_1,x_2],[x_2,x_3],\ldots [x_j,x_{j+1}]$, $[x_{j+2},x_{j+3}]$, are identity and the set $\{[x_1,x_2],[x_2,x_3],\ldots [x_j,x_{j+1}],[x_{j+2},x_{j+3}]\}$ is linearly independent. Then $M(G_j)$ is elementary abelian of order $p^{\frac{1}{2}d(d-1)+2j}$.*

We are now ready to prove the Theorem 6.

Proof of Theorem 6. Let m be a natural number such that $m \geq 10$. Then there exist unique non-negative integers $k \geq 5$ and $l \leq k-1$ such that $m = \frac{1}{2}k(k-1)+l$. If l is a non-zero even number, i.e., $l = 2j$ for some $j \geq 1$, consider the group G_j constructed in Proposition 11 with $d = k$ so that $|M(G_j)| = p^m$. In case l is an odd number greater than or equal to 3, i.e., $l = 2j+1$ for some $j \geq 1$, consider the group G_j constructed in Proposition 10 with $d = k$. If $l = 0$ and k is an even number, consider the group G_j constructed in 10 with $d = k-1$ and $j = \frac{k}{2} - 1$, so that $|M(G_j)| = p^{\frac{1}{2}(k-1)(k-2)+k-1} = p^{\frac{1}{2}k(k-1)}$. If $l = 0$ and k is an odd number, consider the group G_j constructed in 11 with $d = k-1$ and $j = \frac{k-1}{2}$, so that $|M(G_j)| = p^{\frac{1}{2}(k-1)(k-2)+k-1} = p^{\frac{1}{2}k(k-1)}$. If $l = 1$ and k is an even number, consider the group G_j constructed in 11 with $d = k-1$ and $j = \frac{k}{2}$, so that $|M(G_j)| = p^{\frac{1}{2}(k-1)(k-2)+k} = p^{\frac{1}{2}k(k-1)+1}$. If $l = 1$ and k is an odd number, consider the group G_j constructed in 10 with $d = k-1$ and $j = \frac{k-1}{2}$, so that $|M(G_j)| = p^{\frac{1}{2}(k-1)(k-2)+k-1+1} = p^{\frac{1}{2}k(k-1)+1}$. For the existence of groups with elementary abelian Schur multiplier of order p^m, $m \leq 9$, see [2,3]. $\square$

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