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q-Rationals and Finite Schubert Varieties

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Abstract. The classical q-analogue of the integers was recently generalized by Morier-Genoud and Ovsienko to give q-analogues of rational numbers. Some combinatorial interpretations are already known, namely as the rank generating functions for certain partially ordered sets. We give a new interpretation, showing that the numerators of q-rationals count the sizes of certain varieties over finite fields, which are unions of open Schubert cells in some Grassmannian.

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1. Background

The classical "*q-analogue*" of a positive integer n is the following polynomial in $\mathbb{Z}[q]$

$$[n]_q := 1 + q + q^2 + \cdots + q^{n-1}.$$

It satisfies the property that evaluation at $q = 1$ yields the number n. In general, any "*q-analogue*" should satisfy at least this simple property — that evaluating $q = 1$ (or more generally taking a limit $q \to 1$) should recover the classical case.

Some simple well-known examples are built from the q-integers in the obvious way. First is the q-factorial:

$$[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q.$$

Also we have the q-binomial coefficients:

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

However, the significance of a q-analogue would be superficial if the only connection were this "$q = 1$" property. Usually, a q-analogue satisfies some other interesting properties, and has some deeper significance. In particular, q-analogues often have two other interesting combinatorial interpretations.

(1) They count the size of some algebraic variety defined over a finite field $\mathbb{F}_q$.
(2) They appear as weight generating functions of some combinatorial set for some natural statistic.

Recently, Morier-Genoud and Ovsienko defined q-analogues of rational numbers [9]. Given a rational number $\frac{r}{s}$, they define a rational function $\left[\frac{r}{s}\right]_q = \frac{\mathscr{R}(q)}{\mathscr{S}(q)}$, which is defined in terms of the continued fraction expansion of $\frac{r}{s}$. The formal definition will be given later, but here are some examples:

$$\left[\frac{5}{2}\right]_q = \frac{1+2q+q^2+q^3}{1+q}, \qquad \left[\frac{10}{7}\right]_q = \frac{1+q+2q^2+3q^3+2q^4+q^5}{1+q+2q^2+2q^3+q^4}.$$

Morier-Genoud and Ovsienko give some combinatorial interpretations and formulas for the polynomials $\mathscr{R}(q)$, but they pose the open problem of finding more. They say the following in [9]:

> *"It is a challenging problem to find more different combinatorial and geometric interpretations of the polynomials $\mathscr{R}$ and $\mathscr{S}$. A natural question is to connect the polynomials $\mathscr{R}$ and $\mathscr{S}$ with counting of points in varieties defined over the finite fields $\mathbb{F}_q$. This property would be similar to that of the Guassian q-binomial coefficients."*

The aim of this paper is to address this question. We will review some known interpretations of the polynomials $\mathscr{R}(q)$, as well as present some new ones. In particular, we answer the question of giving an interpretation in terms of varieties over finite fields. The organization of the rest of the paper is as follows.

In the remainder of Section 1, we discuss, for $[n]_q!$ and $\binom{n}{k}_q$, the two points mentioned above, how q-analogues often have interpretations in terms of finite algebraic varieties as well as weight generating functions.

In Section 2, we give the definition of $\left[\frac{r}{s}\right]_q$, the q-analogue of the rational number $\frac{r}{s}$, and discuss some combinatorial interpretations in terms of weight generating functions which have already appeared in the literature. In this section we define "*snake graphs*", and the main combinatorial interpretation of q-rationals for our purposes will be as weight generating functions for certain paths on these graphs. In this section we also define two partitions λ and μ (with $\mu < \lambda$) associated with a rational number, which depend on the continued fraction expansion. These partitions play an important part in the statement of the main theorem later.

In Section 3, we begin by reviewing the decomposition of the Grassmannian $\mathrm{Gr}_k(n)$ into "*open Schubert cells*". These cells are indexed by partitions which fit inside a $k \times (n-k)$ rectangle. For each partition λ, we write Ω_λ° for the open Schubert cell. The main result (Theorem 25) is the following.

Theorem. *Let $\frac{r}{s}$ be a rational number, and $\left[\frac{r}{s}\right]_q = \frac{\mathscr{R}(q)}{\mathscr{S}(q)}$. Also let μ and λ be the partitions defined in Section 2 associated with $\frac{r}{s}$. Then*

$$q^{|\mu|}\mathscr{R}(q) = \left| \bigcup_{\mu \le \nu \le \lambda} \Omega_\nu^\circ \right|.$$

1.1. *Geometry Over Finite Fields*

Now let us recall how the above-mentioned examples (the q-factorial and q-binomial coefficients) can be interpreted as the sizes of certain algebraic varieties over finite fields. The statements and results in this section are well-known (see [17] for example), but we recall them for the sake of presentation.

First, let us establish some notation. Let V be a vector space (over a field $\mathbb{K}$), of dimension n. The *(complete) flag variety*, which we will denote by $\mathrm{Fl}(V)$, is the set of all complete flags in V. By a complete flag we mean a chain of nested subspaces

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V. \qquad (\dim(V_i) = i)$$

Recall also that the set of k-dimensional subspaces of a vector space is called a *Grassmannian*, and denoted $\mathrm{Gr}_k(V)$. Then the q-factorial and q-binomial coefficients are the sizes of these varieties over a finite field $\mathbb{F}_q$:

$$\left|\mathrm{Fl}(\mathbb{F}_q^n)\right| = [n]_q! \qquad \text{and} \qquad \left|\mathrm{Gr}_k(\mathbb{F}_q^n)\right| = \binom{n}{k}_q.$$

Example 1. The projective space $\mathbb{P}^{n-1}$ is the special case $\mathrm{Gr}_1(\mathbb{K}^n)$, consisting of lines in $\mathbb{K}^n$. In particular, we have that the q-integer $[n]_q = \binom{n}{1}_q$ is the size of $\mathbb{P}^{n-1}$ over $\mathbb{F}_q$. This is also easy to see using the usual definition of projective space: $\mathbb{P}^{n-1} = (\mathbb{K}^n \setminus \{0\})/\sim$, where $\sim$ is the equivalence relation $p \sim \alpha p$ for any non-zero $\alpha \in \mathbb{K}$. In the case $\mathbb{K} = \mathbb{F}_q$, we have $|\mathbb{K} \setminus \{0\}| = q^n - 1$, and the number of non-zero α's is $q-1$. And indeed we have $[n]_q = \frac{q^n-1}{q-1}$.

1.2. *Weight Generating Functions*

We will now recall the interpretations of q-factorials and q-binomial coefficients as weight generating functions. As in the previous section, all results presented here are well-known (see [17] for a standard reference).

First we will discuss the q-factorial. The natural set that comes to mind when one thinks of the number $n!$ is the symmetric group S_n, and indeed the q-analogue $[n]_q!$ is a weight generating function for a statistic on this set. Given a permutation $\sigma \in S_n$, an *inversion* of σ is a pair (i,j) such that $i<j$ and $\sigma(i) > \sigma(j)$. Define $\mathrm{inv}(\sigma)$ to be the total number of inversions. Then

$$[n]_q! = \sum_{\sigma \in S_n} q^{\mathrm{inv}(\sigma)}.$$

Next we turn to the q-binomial coefficients. The number $\binom{n}{k}$ is of course the size of the set $\binom{[n]}{k}$, of k-sized subsets of $[n] = \{1,2,\dots,n\}$. However, for the present purposes it is more natural to consider a couple other sets which are in bijection with $\binom{[n]}{k}$.

Consider the set $P_{n,k}$ of lattice paths from $(0,0)$ to $(n-k,k)$ which only take unit steps right or up. There is a natural bijection $\binom{[n]}{k} \to P_{n,k}$, given by indicating which steps in the lattice path are the "up" steps. For a path $p \in P_{n,k}$, let $|p|$ be the area between p and the horizontal line $y = k$. Then

$$\binom{n}{k}_q = \sum_{p \in P_{n,k}} q^{|p|}.$$

This can also be described nicely in terms of integer partitions. Recall that a *partition* λ is a sequence of weakly decreasing integers $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_j$. If $\sum \lambda_i = m$, we write $\lambda \vdash m$, and also $|\lambda| = m$. Partitions are visualized by their Young diagram, which is an array of boxes (left-aligned) with λ_i boxes in row i. Let $Y_{n,k}$ be the set of all partitions whose Young diagram fits inside the rectangle with height k and width $n-k$. There is a clear bijection $Y_{n,k} \to P_{n,k}$. The bottom boundary of any Young diagram is a lattice path in $P_{n,k}$, and the number of boxes in the Young diagram is precicely the area above the lattice path. We can then also express the q-binomial coefficient as

$$\binom{n}{k}_q = \sum_{\lambda \in Y_{n,k}} q^{|\lambda|}.$$

Lastly, let us just mention that the previous expressions can also be thought of in terms of posets. In particular, the set of all Young diagrams (or partitions) has a natural partial order where $\mu \leq \lambda$ means that $\mu_i \leq \lambda_i$ for all i, or equivalently the Young diagram of μ fits inside λ. This poset is called *Young's lattice*. The poset has a natural *rank* function, given by the size of the partition (the number of boxes). If we let $\lambda = (n-k)^k$ denote the Young diagram given by the $k \times (n-k)$

rectangle, then the set $Y_{n,k}$ above is simply the interval of Young's lattice $[\varnothing, \lambda] = \{\mu \mid \mu \le \lambda\}$, and the generating function given above is the rank generating function for this poset. This simply means the coefficient of q^k is the number of elements of rank k.

Example 2. The poset structure for $Y_{4,2}$ and $P_{4,2}$ is pictured in Figure 1. The sizes of the ranks are the coefficients of $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$.

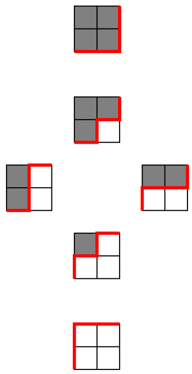


Figure 1. Poset structure for $Y_{4,2} \cong P_{4,2}$. The Young diagrams are shown by the shaded gray boxes, and the lattice paths are shown as red lines.

2. q-Rational Numbers

In [9], Morier-Genoud and Ovsienko extend the definition of $[n]_q$ to include the case when $n = \frac{r}{s} \in \mathbb{Q}$ is a rational number. Their definition uses the continued fraction expansion of $\frac{r}{s}$. Specifically, if $\frac{r}{s} = [a_1, a_2, \dots, a_{2m}]$ is the finite continued fraction expansion (and $a_1 > 1$), then they define

$$\left[\frac{r}{s}\right]_q := [a_1]_q + \cfrac{q^{a_1}}{[a_2]_{q^{-1}} + \cfrac{q^{-a_2}}{[a_3]_q + \cfrac{q^{a_3}}{[a_4]_{q^{-1}} + \cfrac{q^{-a_4}}{\ddots + \cfrac{q^{a_{2m-1}}}{[a_{2m}]_{q^{-1}}}}}}}$$

They also gave several other ways to compute these expressions, including some determinantal formulas, a recursive procedure involving triangulated polygons, and using products of q-deformed matrices in $\mathrm{PSL}_2(\mathbb{Z})$. We will briefly describe this last interpretation in terms of matrices (for more details, see [6] and [9]).

Define the following two matrices:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

They generate $\mathrm{PSL}_2(\mathbb{Z})$, and the group acts transitively on $\mathbb{Q} \cup \{\infty\}$ by the rule:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot x := \frac{ax+b}{cx+d}.$$

Define q-deformed versions of these matrices:

$$A_q = \begin{pmatrix} q & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B_q = \begin{pmatrix} q & 0 \\ q & 1 \end{pmatrix}.$$

Let $\mathrm{PSL}_2^q(\mathbb{Z})$ be the group generated by A_q and B_q, modulo scaling by monomials $q^{\pm n}$. In general, for any $M \in \mathrm{PSL}_2(\mathbb{Z})$, we get a corresponding element $[M]_q \in \mathrm{PSL}_2^q(\mathbb{Z})$. This group acts on the set of power series $\mathbb{Z}[[q]] \cup \{\infty\}$ by the same formula as above. The following result gives another, more conceptual, definition of q-rational numbers.

Theorem 3 ([6, Prop. 3.2]). *The q-deformation $x \mapsto [x]_q$ commutes with the* $\mathrm{PSL}_2(\mathbb{Z})$ *action. This means if $x \in \mathbb{Q}$ and $M \in \mathrm{PSL}_2(\mathbb{Z})$, then $[M \cdot x]_q = [M]_q \cdot [x]_q$.*

Example 4. The continued fraction expansion of $\frac{7}{3}$ is $[2,3]$. Using the definition, we get

$$\left[\frac{7}{3}\right]_q = [2]_q + \frac{q^2}{[3]_{q^{-1}}} = (1+q) + \frac{q^2}{1+q^{-1}+q^{-2}} = \frac{1+2q+2q^2+q^3+q^4}{1+q+q^2}.$$

Alternatively, we can use Theorem 3 to compute. Notice that the generators A and B act by $A \cdot x = x+1$ and $B \cdot x = \frac{x}{x+1}$, and we can obtain $\frac{7}{3}$ as $A^2B^2 \cdot 1$:

$$1 \xrightarrow{B} \frac{1}{2} \xrightarrow{B} \frac{1}{3} \xrightarrow{A} \frac{4}{3} \xrightarrow{A} \frac{7}{3}.$$

According to Theorem 3, we can start with 1 and apply $A_q^2 B_q^2$ to compute $\left[\frac{7}{3}\right]_q$. Notice that the action of A_q and B_q are given by $A_q \cdot x = 1 + qx$ and $B_q \cdot x = \frac{qx}{1+qx}$.

$$1 \xrightarrow{B_q} \frac{q}{1+q} \xrightarrow{B_q} \frac{q^2}{1+q+q^2} \xrightarrow{A_q} \frac{1+q+q^2+q^3}{1+q+q^2} \xrightarrow{A_q} \frac{1+2q+2q^2+q^3+q^4}{1+q+q^2}.$$

2.1. *Combinatorial Interpretations*

Let $\left[\frac{r}{s}\right]_q = \frac{\mathscr{R}(q)}{\mathscr{S}(q)}$. One of the interpretations given in [9] says that the coefficient of q^k in $\mathscr{R}(q)$ is the number of "*k-vertex closures*" in a certain directed graph. Although worded differently, this is equivalent to the following interpretation in terms of posets (see [8]).

Definition 5. *Define a poset $F\left(\frac{r}{s}\right)$ on the set $\{x_1, x_2, \dots, x_{N-1}\}$, where $N = \sum_{i=1}^{2m} a_i$, with cover relations:*

$$x_1 < x_2 < \cdots < x_{a_1} > x_{a_1+1} > \cdots > x_{a_1+a_2} < \cdots.$$

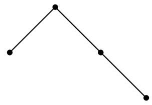
*In other words, the Hasse diagram is a "*fence*" which goes up $(a_1 - 1)$ times, then down a_2 times, then up a_3 times, ..., and finally up a_{2m-1} times, and down $(a_{2m} - 1)$ times.*

Let $L\left(\frac{r}{s}\right)$ be the lattice of order ideals in $F\left(\frac{r}{s}\right)$. In this language, what Morier-Genoud and Ovsienko call a "*k-vertex closure*" is the same as a lower order ideal of $F\left(\frac{r}{s}\right)$. In particular we have the following.

Theorem 6 ([9, Thm. 4]). *$\mathscr{R}(q)$ is the rank generating function of $L\left(\frac{r}{s}\right)$.*

Remark 7. The connection between these posets and F-polynomials in cluster algebras has been noted in [1,4,9,11,16]. These posets were called "*fence posets*" in [8] and "*piece-wise linear posets*" in [1]. It was shown in [9] that $\mathscr{R}(q)$ can be obtained from an F-polynomial by specialization of the variables.

Example 8. We continue with the running example of $\frac{r}{s} = \frac{7}{3}$ from Example 4. Recall that the continued fraction for $\frac{7}{3}$ is $[2,3]$. The corresponding fence $F\left(\frac{7}{3}\right)$ is



The Lattice $L\left(\frac{7}{3}\right)$ of order ideals in $F\left(\frac{7}{3}\right)$ is pictured in Figure 2. Notice the number of elements of rank k is the coefficient of q^k in $\mathscr{R}(q) = 1+2q+2q^2+q^3+q^4$.

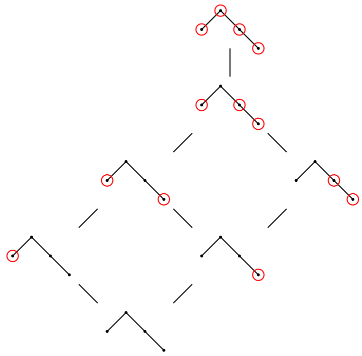


Figure 2. Poset structure of $L\left(\frac{7}{3}\right)$. Each order ideal is indicated by putting red circles around its elements.

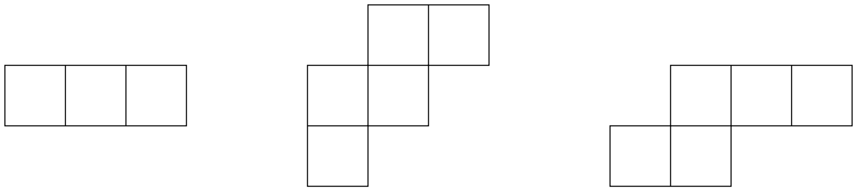
Remark 9. Two properties that are common for q-analogues are unimodality and palindromicity. A polynomial is "*unimodal*" if its coefficients form a unimodal sequence. In [8], it was shown in some special cases that $\mathscr{R}(q)$ is unimodal, using the language of fence posets. Unimodality of $\mathscr{R}(q)$ was proven in full generality in [13], again using the language of fence posets. A polynomial is "*palindromic*" if its sequence of coefficients is a palindrome. The polynomials $\mathscr{R}(q)$ are not palindromic in general (as seen in earlier examples). However, it was shown in [7] that traces of the $\mathrm{PSL}_2^q(\mathbb{Z})$-matrices used to compute the q-rationals (as in Example 4) are palindromic.

2.2. *Snake Graphs*

Yet another equivalent description can be given in terms of certain planar graphs, which are called "*snake graphs*" in the cluster algebra literature (e.g. [2, 10, 15]). These graphs are built out of square tiles, such that each tile is either above or to the right of the previous one. Equivalently, they are skew Young diagrams containing no 2×2 blocks. To each snake graph G we can naturally associate a *word* $W(G)$ in the alphabet $\{R,U\}$, indicating whether going from one tile to the next is "*right*" or "*up*". Some examples are shown in Figure 3.

Definition 10. *Let $\overline{R} = U$ and $\overline{U} = R$. If $w = w_1w_2\cdots w_k$ is a word in the alphabet $\{R,U\}$, define the* dual word *to be $w^* = \overline{w}_1 w_2 \overline{w}_3 w_4 \cdots \overline{w}_{2n-1} w_{2n}$ or $w^* = \overline{w}_1 w_2 \overline{w}_3 w_4 \cdots w_{2n}\overline{w}_{2n+1}$ (depending on parity). In other words, w^* is obtained by toggling the odd-indexed letters. Extend this definition to snake graphs by defining G^* to be the snake graph so that $W(G^*) = W(G)^*$.*

Remark 11. Snake graphs have appeared numerous times in the cluster algebra literature (see for example [2, 4, 10, 11, 15, 16]). The dual construction (G^* rather than G) seems to be more popular in the literature (in [15], Propp first constructs G^* and refers to G as the "dual" snake). As explained in [4] and [15], there is a bijection between perfect matchings of G^* and north-east lattice paths on G. In what follows, results from the literature are paraphrased in terms of G rather than G^*, so as to use lattice paths instead of perfect matchings.



$W(G) = RR$ $W(G) = URUR$ $W(G) = RURR$

$G = G(4/1)$ $G = G(13/8)$ $G = G(11/4)$

$4/1 = [3,1]$ $13/8 = [1,1,1,1,1,1]$ $11/4 = [2,1,2,1]$

Figure 3. Examples of snake graphs. For each example, we indicate the word $W(G)$, the corresponding rational number, and its continued fraction expansion.

Definition 12 (The "dual" version of [2, Def. 3.1]). *Given a continued fraction $\frac{r}{s} = [a_1, \ldots, a_{2m}]$, define a snake graph $G\left(\frac{r}{s}\right)$ such that its word is $W(G) = R^{a_1-1}U^{a_2}R^{a_3}U^{a_4}\cdots R^{a_{2m-1}}U^{a_{2m}-1}$. See Figure 3 for examples.*

Definition 13. *Given a snake graph G, let $P(G)$ be the set of lattice paths on G, going from the bottom-left corner to the top-right corner, which only take steps right or up. We also use the notation $P\left(\frac{r}{s}\right) = P\left(G\left(\frac{r}{s}\right)\right)$.*

The following result was mentioned in [15, §4], and the "dual" version (in terms of perfect matchings of G^*) also appears in [2, Thm. 3.4].

Theorem 14. *The number of lattice paths on $G\left(\frac{r}{s}\right)$ is r. In other words, $\left|P\left(\frac{r}{s}\right)\right| = r$.*

We now give a q-analogue of Theorem 14, where r is replaced by $\mathscr{R}(q)$, the numerator of the corresponding q-rational. Note that there is a natural partial order on $P(G)$ so that $a \leq b$ if the set of boxes above a is a subset of the boxes above b. Let $|a|$ denote the number of boxes above the path a. The following is an equivalent reformulation of Theorem 6.

Theorem 15. *$\mathscr{R}(q)$ is the rank-generating function of the poset $P\left(\frac{r}{s}\right)$. In other words,*

$$\mathscr{R}(q) = \sum_{p \in P\left(\frac{r}{s}\right)} q^{|p|}.$$

Proof. It is easy to see that $P\left(\frac{r}{s}\right)$ is isomorphic to the poset $L\left(\frac{r}{s}\right)$ described earlier. The points of $F\left(\frac{r}{s}\right)$ correspond to the boxes of the snake graph, and the order ideals of $F\left(\frac{r}{s}\right)$ are the sets of boxes above the lattice paths.

More specifically, we can construct $G\left(\frac{r}{s}\right)$ from $F\left(\frac{r}{s}\right)$ as follows (see Figure 4 for an illustration). Start with the Hasse diagram of $F\left(\frac{r}{s}\right)$, and reflect it over a horizontal line. Next, rotate it 45° counter-clockwise. Then draw a square around each vertex. This will be $G\left(\frac{r}{s}\right)$. □

Example 16. The poset $P\left(\frac{7}{3}\right)$ is shown in Figure 5. Compare with $L\left(\frac{7}{3}\right)$ from Example 8 and Figure 2.

The snake graph interpretation allows to make a nice analogy with the situation for q-binomial coefficients. In particular, $P\left(\frac{r}{s}\right)$ is also isomorphic to an interval in Young's lattice. This follows from the observation that any snake graph is a skew Young diagram of shape λ/μ where λ is determined by the bottom boundary of G, and μ is determined by the top boundary of G.

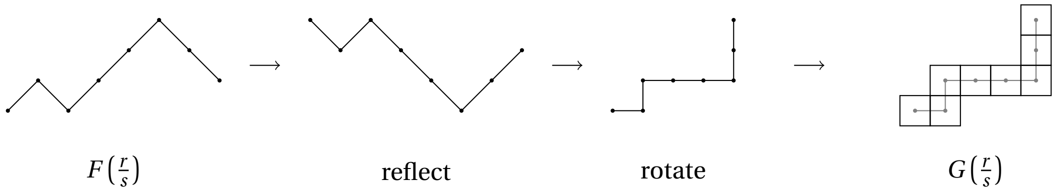


Figure 4. Illustration of the equivalence of Theorem 6 and Theorem 15

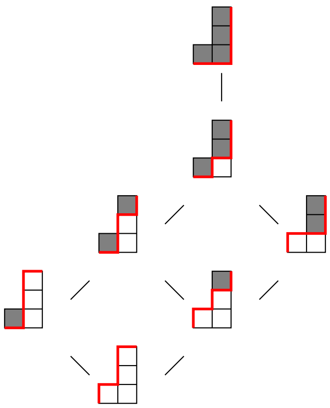
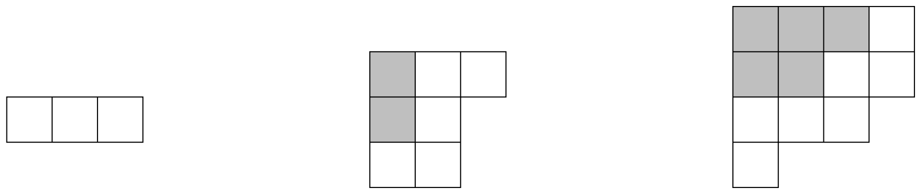


Figure 5. Poset structure of $P\left(\frac{7}{3}\right)$

Definition 17. *If G is a snake graph, let $\lambda(G)$ be the partition such that G is the subset of boxes adjacent to the bottom boundary of λ. Also, let $\mu(G) < \lambda(G)$ be the partition such that G is the skew Young diagram of shape $\lambda(G)/\mu(G)$. If $G = G\left(\frac{r}{s}\right)$, then we use the notations $\lambda\left(\frac{r}{s}\right)$ and $\mu\left(\frac{r}{s}\right)$.*

Example 18. Some examples of λ and μ, along with their corresponding fractions $\frac{r}{s}$, are shown in Figure 6.



$\frac{r}{s} = \frac{4}{1} = [3,1]$	$\frac{r}{s} = \frac{12}{5} = [2,2,1,1]$	$\frac{r}{s} = \frac{31}{18} = [1,1,2,1,1,2]$
$W(4/1) = RR$	$W(12/5) = RUUR$	$W(31/18) = URRURU$
$\lambda = (3)$, $\mu = \emptyset$	$\lambda = (3,2,2)$, $\mu = (1,1)$	$\lambda = (4,4,3,1)$, $\mu = (3,2)$

Figure 6. Examples of $\lambda\left(\frac{r}{s}\right)$ and $\mu\left(\frac{r}{s}\right)$. The partition μ is shaded in gray.

In order to state the next result, we introduce the following notation. We will write partitions as $\lambda = (\lambda_1^{b_1}, \lambda_2^{b_2}, \ldots)$, where $\lambda_i^{b_i}$ means that λ_i is repeated b_i times. For example, $(3^4, 2, 1^2) = (3,3,3,3,2,1,1)$.

Proposition 19. *Suppose $\frac{r}{s} = [a_1, b_1, \dots, a_m, b_m]$. The partitions $\lambda\left(\frac{r}{s}\right)$ and $\mu\left(\frac{r}{s}\right)$ are given explicitly as follows.*

(a) $\lambda = (\lambda_1^{b_m}, \lambda_2^{b_{m-1}}, \dots, \lambda_m^{b_1})$, *where* $\lambda_k = \sum_{i=1}^{m+1-k} a_i$.
(b) $\mu = (\mu_1^{c_1}, \dots, \mu_m^{c_m})$, *where* $\mu_k = \lambda_k - 1$, *and* $c_k = b_{m+1-k} - \delta_{1,k}$.

Proof. **(a).** We will induct on m. Recall the definition of the snake graph $G\left(\frac{r}{s}\right)$. It is defined by the word $W\left(\frac{r}{s}\right) = R^{a_1-1}U^{b_1}R^{a_2}U^{b_2}\cdots R^{a_m}U^{b_m-1}$. By definition of $\lambda\left(\frac{r}{s}\right)$, the partition λ will have the same bottom boundary as $G\left(\frac{r}{s}\right)$. Suppose $m=1$. Then $W(r/s) = R^{a_1-1}U^{b_1-1}$. The young diagram which has this as its bottom boundary is the rectangle with width a_1 and height b_1 (i.e. $\lambda = (a_1^{b_1})$).

Now suppose the result is true for all continued fractions $\frac{r'}{s'} = [a_1, b_1, \dots, a_{m-1}, b_{m-1}]$. Then by induction $\lambda\left(\frac{r'}{s'}\right) = \lambda' = (\lambda_1^{b_{m-1}}, \lambda_2^{b_{m-2}}, \dots, \lambda_{m-1}^{b_1})$, where $\lambda_k = \sum_{i=1}^{m-k} a_i$. The snake graph for $\frac{r}{s} = [a_1, b_1, \dots, a_m, b_m]$ is built from the previous one by adding some more boxes to the end. This means the parition λ will be obtained from λ' by adding some more rows to the top.

First let us point out a subtle point. The word $W\left(\frac{r'}{s'}\right)$ ends with $U^{b_{m-1}-1}$, but $W\left(\frac{r}{s}\right)$ contains $U^{b_{m-1}}$. This means we must add one more box going up at the end of $G\left(\frac{r'}{s'}\right)$ before continuing to build $G\left(\frac{r}{s}\right)$. This simply means the rows of λ' will not be changed. Now that we have added an extra U, the next part of $W\left(\frac{r}{s}\right)$ is R^{a_m}. This means we add another row on top of λ' which has a_m more boxes than the previous row. By induction, the previous row has $\sum_{i=1}^{m-1} a_i$ boxes, and so this new row has $\sum_{i=1}^{m} a_i$ boxes. Finally, $W\left(\frac{r}{s}\right)$ ends with U^{b_m-1}, and so there are a total of b_m rows at the top, all of this length.

(b). The statement says μ is obtained from λ by subtracting one from all parts (i.e. $\mu_i = \lambda_i - 1$), and then the multiplicities are the same for all but the first (i.e. the greatest) part, which is one less. To see this, consider a part of the word $W\left(\frac{r}{s}\right)$ of the form R^aU^b. When the snake graph goes right a times, this corresponds to the bottom-most row of the Young diagram of some length λ_i. After this, every time it goes up (except the last), we get another row of length λ_i. The exception is the last time it goes up, since if the snake graph continues to go right again after this, the final "up" will be part of a longer row.

At each of these "up" steps, the corresponding part of μ will have length $(\lambda_i - 1)$. If this part R^aU^b is not the end of the snake graph, then by part (a), b is the number of parts of λ of length λ_i. However, in the special case that U^{b_m-1} is the end of the snake graph, we only get $b_m - 1$ parts of length μ_m (but there are b_m parts of length λ_m). $\square$

The following is yet another equivalent restatement of Theorem 6 and Theorem 15.

Proposition 20. *Let $\lambda = \lambda\left(\frac{r}{s}\right)$ and $\mu = \mu\left(\frac{r}{s}\right)$. The poset $P\left(\frac{r}{s}\right)$ is isomorphic to the interval $[\mu, \lambda]$ in Young's lattice, and*

$$\mathscr{R}(q) = \frac{1}{q^{|\mu|}} \sum_{\mu \le \nu \le \lambda} q^{|\nu|}.$$

Proof. There is the natural bijection between $P\left(\frac{r}{s}\right)$ and the interval $[\mu, \lambda]$ in Young's lattice, where a lattice path corresponds to the Young diagram (inside λ) whose boxes lie above the path. Under this bijection, the rank of a path corresponding to a partition ν is $|\nu| - |\mu|$, which is the number of boxes in the snake graph above the path. The formula for $\mathscr{R}(q)$ follows from this bijection together with Theorem 15. $\square$

3. Schubert Varieties

3.1. *Definitions*

Let $\mathbb{K}$ be a field, and consider the Grassmannian $\mathrm{Gr}_k(\mathbb{K}^n)$ of k-planes in $\mathbb{K}^n$. We will identify $\mathrm{Gr}_k(\mathbb{K}^n)$ with the set of $k \times n$ matrices of rank k, modulo left multiplication by elements of GL_k.

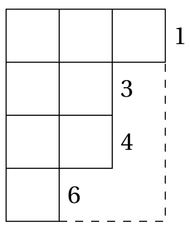
The following definitions and results are all well-known (see [5] for a standard reference, although here we use notational conventions as in [14]).

There is a simple bijection between the interval $[\emptyset,(n-k)^k]$ in Young's lattice (i.e. partitions which fit inside a $k\times(n-k)$ rectangle) and the set $\binom{[n]}{k}$ of k-element subsets of $[n]=\{1,2,\dots,n\}$, given as follows. Recall from Section 1.2 the bijection $Y_{n,k}\to P_{n,k}$ that associates to a Young diagram λ the lattice path from $(0,0)$ to $(n-k,k)$ which is formed by the bottom boundary of λ. If the path is traversed backwards, and the steps are labelled $1,2,\dots,n$, then taking the subset of vertical steps gives an element $I_\lambda\in\binom{[n]}{k}$.

Definition 21. *The* open Schubert cell $\Omega_\lambda^\circ\subseteq \mathrm{Gr}_k(\mathbb{K}^n)$ *is the subset whose representatives, when written in echelon form, have the identity matrix in the columns indexed by I_λ. The remaining non-zero entries (to the right of the pivots) form the shape of λ (but backwards).*

From this definition, it is easy to see that $\dim(\Omega_\lambda^\circ)=|\lambda|$. Once the matrices are in echelon form, there is a clear parameterization by $\mathbb{K}^{|\lambda|}$.

Example 22. The partition $\lambda=(3,2,2,1)$ fits inside a 3×4 rectangle (corresponding to $\mathrm{Gr}_4(\mathbb{K}^7)$). The corresponding column set is $I_\lambda=\{1,3,4,6\}$. The open Schubert cell Ω_λ° contains all matrices whose echelon form has the shape shown in Figure 7. Notice that the $*$'s are in the shape of λ, reflected horizontally.



$$\begin{pmatrix} 1 & * & 0 & 0 & * & 0 & * \\ 0 & 0 & 1 & 0 & * & 0 & * \\ 0 & 0 & 0 & 1 & * & 0 & * \\ 0 & 0 & 0 & 0 & 0 & 1 & * \end{pmatrix}$$

$I_\lambda=\{1,3,4,6\}$

Figure 7. (left) The subset $I_\lambda\in\binom{[n]}{k}$ corresponding to the partition $\lambda=(3,2,2,1)$. (right) The echelon form of a matrix representative in Ω_λ°

Definition 23. *The* Schubert variety *(or* closed Schubert cell*) is the union of all open Schubert cells of $\mu\le\lambda$:*

$$\Omega_\lambda := \bigcup_{\mu\le\lambda}\Omega_\mu^\circ.$$

In particular, the Grassmannian is the "biggest" closed Schubert cell Ω_λ°, where $\lambda=(n-k)^k$ is the entire $k\times(n-k)$ rectangle, and thus $\mathrm{Gr}_k(\mathbb{K}^n)$ is the disjoint union of all the open Schubert cells. This gives a connection between the geometric and combinatorial interpretations of the q-binomial coefficients given earlier in Sections 1.1 and 1.2. If we take $\mathbb{K}=\mathbb{F}_q$, then clearly $\left|\Omega_\lambda^\circ\right|=q^{|\lambda|}$. The fact that the Grassmannian is the disjoint union of all open Schubert cells implies that

$$\left|\mathrm{Gr}_k(\mathbb{F}_q^n)\right| = \sum_{\lambda\le(n-k)^k}\left|\Omega_\lambda^\circ\right| = \sum_{\lambda\in Y_{n,k}} q^{|\lambda|} = \binom{n}{k}_q.$$

3.2. *Numerators of q-Rationals*

Finally, we return to the discussion of $\mathscr{R}(q)$, then numerator of the q-rational $\left[\frac{r}{s}\right]_q$, and we will show that it is the size of a certain subvariety of $\mathrm{Gr}_k(\mathbb{F}_q^n)$, where k and n can be determined by the continued fraction $\frac{r}{s}=[a_1,a_2,\dots,a_{2m}]$.

Definition 24. *For two partitions $\mu \leq \lambda$ which fit inside the $k \times (n-k)$ rectangle, let $\Omega_{[\mu,\lambda]}$ be the union of open Schubert cells in the interval $[\mu, \lambda]$:*

$$\Omega_{[\mu,\lambda]} := \bigcup_{\mu \leq \nu \leq \lambda} \Omega_\nu^\circ.$$

Theorem 25. *Let $\frac{r}{s} = [a_1, \dots, a_{2m}] \in \mathbb{Q}$, with $\frac{r}{s} > 1$, with corresponding q-rational $\left[\frac{r}{s}\right]_q = \frac{\mathscr{R}(q)}{\mathscr{S}(q)}$. Let $\mu = \mu\left(\frac{r}{s}\right)$ and $\lambda = \lambda\left(\frac{r}{s}\right)$. Then up to a factor of $q^{|\mu|}$, the numerator $\mathscr{R}(q)$ is the number of $\mathbb{F}_q$-points of $\Omega_{[\mu,\lambda]}$:*

$$\mathscr{R}(q) = q^{-|\mu|} \left|\Omega_{[\mu,\lambda]}\right|.$$

Moreover, this is a subvariety of the Grassmannian $\mathrm{Gr}_k(\mathbb{F}_q^n)$, with $n = \sum_{i=1}^{2m} a_i$ and $k = \sum_{i=1}^m a_{2i}$.

Proof. Recall from Proposition 20 that the numerator $\mathscr{R}(q)$ is the rank generating function of the interval $[\mu, \lambda]$ in Young's lattice (scaled by $q^{|\mu|}$). The width of the snake graph (also the width of λ) is $\sum_{i=1}^m a_{2i-1}$, and the height is $\sum_{i=1}^m a_{2i}$. Therefore the dimensions of the $k \times (n-k)$ rectangle which surrounds λ are given by $k = \sum_{i=1}^m a_{2i}$ and $n - k = \sum_{i=1}^m a_{2i-1}$.

Since $q^{|\nu|}$ is the size of the open Schubert cell Ω_ν°, the result follows. □

Example 26. If $\frac{r}{s} = \frac{n}{1}$ or $\frac{r}{s} = \frac{n+1}{n}$, then the snake graph $G\left(\frac{r}{s}\right)$ is a straight row or column of boxes. In this case, the partition μ is empty, and λ is the full $n \times 1$ or $1 \times n$ rectangle. In this case Theorem 25 reduces to a special case described in Example 1, which simply says that $|\mathbb{P}^{n-1}| = [n]_q$. The interpretation given in Theorem 25 in terms of Schubert cells just states that $\mathbb{P}^{n-1} = \bigcup_{k=0}^{n-1} X_k$, where X_k consists of those points whose homogeneous coordinates have the form

$$[x_0 : x_1 : \cdots : x_{n-1}] = [0 : 0 : \cdots : 0 : 1 : * : \cdots : *]$$

where $x_k = 1$ and $x_i = 0$ for $i < k$. Clearly $|X_k| = q^{n-1-k}$, and so $\sum_k |X_k| = 1 + q + \cdots + q^{n-1} = [n]_q$.

Example 27. Continuing with Example 8 and Example 16, let $\frac{r}{s} = \frac{7}{3} = [2, 3]$. The corresponding q-rational is

$$\left[\frac{7}{3}\right]_q = \frac{\mathscr{R}(q)}{\mathscr{S}(q)} = \frac{1 + 2q + 2q^2 + q^3 + q^4}{1 + q + q^2}.$$

The snake graph $G\left(\frac{7}{3}\right)$ has word $W(G) = RUU$, and is pictured (along with the poset $P\left(\frac{7}{3}\right)$) in Figure 5. The snake graph has width 2 and height 3, which means the corresponding $\Omega_{[\mu,\lambda]}$ lives in $\mathrm{Gr}_3(\mathbb{F}_q^5)$ (since $k = 3$ and $n - k = 2$).

In the notation of Theorem 25, we have $\lambda = (2, 2, 2)$ and $\mu = (1, 1)$. In this case Theorem 25 says that $q^2 \mathscr{R}(q)$ counts the number of points in $\Omega_{[\mu,\lambda]}$, which consists of all 3-dimensional subspaces of $\mathbb{F}_q^5$ which have a matrix representative of one of the following seven forms:

$$\begin{pmatrix} 1&0&0&*&* \\ 0&1&0&*&* \\ 0&0&1&*&* \end{pmatrix}, \quad \begin{pmatrix} 1&0&*&0&* \\ 0&1&*&0&* \\ 0&0&0&1&* \end{pmatrix}, \quad \begin{pmatrix} 1&0&*&*&0 \\ 0&1&*&*&0 \\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 1&*&0&0&* \\ 0&0&1&0&* \\ 0&0&0&1&* \end{pmatrix},$$

$$\begin{pmatrix} 0&1&0&0&* \\ 0&0&1&0&* \\ 0&0&0&1&* \end{pmatrix}, \quad \begin{pmatrix} 1&*&0&*&0 \\ 0&0&1&*&0 \\ 0&0&0&0&1 \end{pmatrix}, \quad \begin{pmatrix} 0&1&0&*&0 \\ 0&0&1&*&0 \\ 0&0&0&0&1 \end{pmatrix}.$$

Further Directions

Lastly, we mention some remaining open questions and interesting directions for further study.

As mentioned in Remark 9, the authors of [6] and [7] studied the traces of q-deformed matrices, showed that they are palindromic, and gave an enumerative interpretation in terms of triangulated annuli. It would be interesting to see if these traces have interpretations similar to the ones discussed in this paper.

For instance, is there some version of Theorem 25 for traces of q-deformed matrices? Is there some variety over $\mathbb{F}_q$ whose size is counted by the trace of the corresponding q-deformed matrix?

In [7], the authors mention (in section 5.2) that their combinatorial generating function for these q-deformed traces can be phrased in terms of closures of a directed graph. They also mention, as in Section 2.1 of the present paper, that this can be re-stated in terms of order ideals of the corresponding "*circular fence poset*". It might be interesting to see the equivalent statement in terms of snake graphs (or the circular versions, sometimes called "*band graphs*", which appeared in [3, 10, 11]).

Also, in [12] it was shown that the traces of certain products of matrices in $\mathrm{PSL}_2(\mathbb{R})$ corresponding to closed loops on a surface are given by a generating function for perfect matchings of a band graph. There should be some sense in which the combinatorial formulation of [7] is a q-specialization of the formulas in [12], in the same way that [9] (Appendix B) explains how the numerators $\mathscr{R}(q)$ are certain q-specializations of cluster variables. It would be interesting to make this formulation and connection more precise.

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