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The connectedness of degeneracy loci in positive characteristic

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Abstract. A well-known result of Fulton–Lazarsfeld ensures the connectedness of degeneracy loci under an ampleness condition. We extend it to positive characteristic, along with the variants for degeneracy loci of symmetric and alternating maps of even rank, due to Tu in characteristic zero. The proof uses the explicit determination of the top étale cohomology group of an algebraic variety, a result communicated by Esnault.

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1. Introduction

It is classical that a hyperplane section of a projective variety of dimension greater than 1 is connected. This fundamental fact can be rephrased in terms of the connectedness of the zero scheme of a global section of an ample line bundle. This latter statement admits a generalisation to degeneracy loci, due to Fulton–Lazarsfeld [4] (see also [11, §7.2]). Recall that, for a map of vector bundles $\phi : \mathscr{E} \to \mathscr{F}$, the degeneracy locus $D_r(\phi)$ is the zero scheme of $\wedge^{r+1}\phi : \wedge^{r+1}\mathscr{E} \to \wedge^{r+1}\mathscr{F}$.

In this note we extend the Fulton–Lazarsfeld theorem to positive characteristic, namely

Theorem 1. *Let k be a field and X a proper k-scheme of dimension d. Let $\phi : \mathscr{E} \to \mathscr{F}$ be a morphism of vector bundles on X, of ranks e and f. If $\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}$ is ample, then $D_r(\phi)$ is*

(i) *nonempty if $d \geq (e-r)(f-r)$*
(ii) *geometrically connected if $d > (e-r)(f-r)$ and X is geometrically irreducible.*

This was shown in [4] for $k = \mathbb{C}$ and the proof given there also works in positive characteristic if X is smooth. For singular X in positive characteristic, Fulton–Lazarsfeld [5] deduced (i) from deep results in intersection cohomology. More recently, Flenner–Ulrich [3] showed that the conclusions of Theorem 1 hold in characteristic $p > 0$ under the stronger assumption that $\mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}$ is cohomologically p-ample.

As an immediate consequence of Theorem 1, we have the following result which, surprisingly, appears to have been open.

Corollary. *Let k be a field, X a proper, geometrically irreducible k-scheme, and $\mathscr{F}$ an ample vector bundle on X. If $\dim X > \operatorname{rk}\mathscr{F}$, then the zero scheme of a global section of $\mathscr{F}$ is geometrically connected.*

There are variants of Theorem 1 for symmetric and alternating degeneracy loci, conjectured in [5] and shown by Tu [14] in characteristic zero for $r \in 2\mathbb{Z}$, and by Flenner–Ulrich [3] for arbitrary r in characteristic $p > 0$ under their standing cohomologically p-ample assumption. Here we extend Tu's results to positive characteristic, obtaining

Theorem 2. *Let k be a field (resp. a field of characteristic other than 2) and X a proper k-scheme of dimension d. On X, let $\mathscr{E}$ be a vector bundle of rank e, $\mathscr{L}$ a line bundle, and $\varphi : \mathscr{E}^{\otimes 2} \to \mathscr{L}$ an alternating (resp. symmetric) map with induced map $\phi : \mathscr{E} \to \mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{L}$.*

If $\mathscr{L} \otimes_{\mathscr{O}_X} \wedge^2 \mathscr{E}^\vee$ is ample (resp. $\mathscr{L} \otimes_{\mathscr{O}_X} \operatorname{Sym}^2 \mathscr{E}^\vee$ is ample), then, for an even integer r, $D_r(\phi)$ is

(i) *nonempty if $d \geq \binom{e-r}{2}$ (resp. $d \geq \binom{e-r+1}{2}$)*

(ii) *geometrically connected if $d > \binom{e-r}{2}$ (resp. $d > \binom{e-r+1}{2}$) and X is geometrically irreducible.*

As in Theorem 1, case (i) has been known for a long time (without restriction on r or k) thanks to Fulton–Lazarsfeld [5]; we have only included it to emphasise that cases (i) and (ii) are proved here in parallel. Note that the hypothesis $r \in 2\mathbb{Z}$ is not restrictive in the alternating case. The restriction to characteristic $\neq 2$ in the symmetric case is necessary in order to interpret $\operatorname{Sym}^2 \mathscr{E}$ as a space of maps $\mathscr{E}^\vee \to \mathscr{E}$.

Apart from a single topological lemma, the proofs given in [4] and [14] can be made characteristic free. So what remains is to find a suitable substitute for that lemma, and this is the original content of this note. The required result in étale cohomology (Corollary 7) turns out to be a straightforward consequence of de Jong's theorem on alterations. In fact, as pointed out to us by Esnault, one can use Gabber's refinement of de Jong's theorem to precisely determine the top étale cohomology group, which easily implies the required result; see Proposition 3.

There are further generalisations of Theorems 1 and 2 that may be deduced from the results of this paper, for instance the generalisation to n-ample vector bundles of [15] and [13]. Broadly speaking, any generalisation based on the approach of [4] should be accessible.

Conventions

If $\mathscr{E}$ is a vector bundle (i.e., a locally free coherent sheaf) on X we write $\mathscr{E}^\vee = \underline{\operatorname{Hom}}_{\mathscr{O}_X}(\mathscr{E}, \mathscr{O}_X)$ for the dual bundle. If $s : \mathscr{O}_X \to \mathscr{E}$ a section, then the *zero scheme* of s is the closed subscheme $Z(s) \subset X$ whose defining ideal sheaf is the image of the dual map $\check{s} : \mathscr{E}^\vee \to \mathscr{O}_X$. If $\phi : \mathscr{E} \to \mathscr{F}$ is a map of vector bundles, then the zero scheme of ϕ is defined to be the zero scheme of the induced section $\mathscr{O}_X \to \mathscr{E}^\vee \otimes_{\mathscr{O}_X} \mathscr{F}$.

Our convention for projective bundles and Grassmannians is [7], the opposite of [4, 14]. With this convention, a vector bundle $\mathscr{E}$ is *ample* if and only if the tautological quotient line bundle $\mathscr{O}_{\mathbb{P}(\mathscr{E})}(1)$ on $\mathbb{P}(\mathscr{E})$ is ample [8, 3.2].

To simplify the notation, when k is separably closed and $(n, \operatorname{char}(k)) = 1$ we will fix an isomorphism $\mathbb{Z}/n(1) \simeq \mathbb{Z}/n$, and omit to write Tate twists.

All cohomology is étale. Cohomology with compact support is denoted $H^i_c(X, -)$.

2. A result in étale cohomology

In this section we determine the top étale cohomology group of algebraic varieties and derive some useful consequences. The main result, communicated by Esnault, is of independent interest. Its proof uses Gabber's refinement of de Jong's alterations.

Let k be an algebraically closed field, X be a separated k-scheme of finite type, $d := \dim X$, and n an integer prime to $\mathrm{char}(k)$.

Proposition 3 (Esnault). *We have*

$$H^{2d}(X, \mathbb{Z}/n) = \oplus_Y \mathbb{Z}/n$$

where the direct sum is taken over the irreducible components Y of X of dimension d which are proper over k. Moreover, the trace map is an isomorphism on each direct summand.

We first prove a lemma.

Lemma 4.

(a) *If X is irreducible but not proper, then $H^{2d}(X, \mathbb{Z}/n) = 0$.*

(b) *If $\widetilde{X}$ is the disjoint union of the proper irreducible components of dimension d of X, then the canonical map $H^{2d}(X, \mathbb{Z}/n) \to H^{2d}(\widetilde{X}, \mathbb{Z}/n)$ is an isomorphism.*

Proof. We may assume X is reduced.

(a). First assume X is smooth. Since X is not proper we have $H^0_c(X, \mathbb{Z}/n) = 0$, hence the result follows by Poincaré duality.

We reduce the general case to the smooth case as follows. Firstly, we may assume $n = l^m$ for a prime $l \neq \mathrm{char}(k)$. By Gabber's theorem on alterations [9, 2.1], there exists an alteration $\pi : X' \to X$ with X' smooth, connected and quasi-projective, with π of degree prime to n. Note that X' cannot be proper, so $H^{2d}(X', \mathbb{Z}/n) = 0$.

Let $j : U \hookrightarrow X$ be a dense open such that $\pi_U := \pi|_U$ is finite flat, $U' = U \times_X X'$, and $j' : U' \to X'$ be the inclusion. Since π_U is finite flat, composition of the adjunction map

$$\mathbb{Z}/n \longrightarrow \pi_{U,*}\pi_U^*\mathbb{Z}/n = \pi_{U,*}\mathbb{Z}/n \tag{1}$$

with the trace map for π_U is multiplication by the degree on $\mathbb{Z}/n$ [2, 2.9]. On the other hand, note that $j \circ \pi_U = \pi \circ j'$, so $j_!\pi_{U,*}\mathbb{Z}/n = R\pi_* j'_!\mathbb{Z}/n$ and the global sections of this complex compute $H^*(X', j'_!\mathbb{Z}/n)$. Therefore, for all i the canonical map

$$H^i(X, j_!\mathbb{Z}/n) \longrightarrow H^i(X', j'_!\mathbb{Z}/n)$$

obtained by applying $j_!$ to (1) has kernel annihilated by the degree of π, hence is injective by choice of π and n. So it suffices to show that $H^{2d}(X, j_!\mathbb{Z}/n) = H^{2d}(X, \mathbb{Z}/n)$ and similarly for X'. This follows from the exact sequence

$$H^{2d-1}(X \setminus U, \mathbb{Z}/n) \longrightarrow H^{2d}(X, j_!\mathbb{Z}/n) \longrightarrow H^{2d}(X, \mathbb{Z}/n) \longrightarrow H^{2d}(X \setminus U, \mathbb{Z}/n)$$

because U is dense in X, hence $\dim(X \setminus U) < d$. Similarly for X'.

(b). Let $\widetilde{X}'$ be the disjoint union of all of the irreducible components of X and $\mu : \widetilde{X}' \to X$ the canonical finite map. Consider the adjunction map

$$\mathbb{Z}/n \longrightarrow \mu_*\mu^*\mathbb{Z}/n = R\mu_*\mathbb{Z}/n$$

Since the kernel and cokernel of this map are torsion sheaves supported on subschemes of dimension at most $d-1$, it follows that the canonical map

$$H^{2d}(X, \mathbb{Z}/n) \longrightarrow H^{2d}(\widetilde{X}', \mathbb{Z}/n)$$

is an isomorphism. Now apply (a) to $\widetilde{X}'$. □

Proof of Proposition 3. By Lemma 4 we may assume X is proper and irreducible. Furthermore, we may assume X to be reduced. If $U \subset X$ is a smooth dense open, then $\dim(X \setminus U) < d$, so $H^{2d}(X, \mathbb{Z}/n) = H^{2d}_c(U, \mathbb{Z}/n)$, with the latter isomorphic to $\mathbb{Z}/n$ via the trace by Poincaré duality [2, 3.2.6]. □

Corollary 5. *If X is proper and $Z \subset X$ is a closed subset intersecting an irreducible component of dimension d, the trace map $H^{2d}_Z(X, \mathbb{Z}/n) \to \mathbb{Z}/n$ is surjective.*

Proof. Let $X' \subset X$ be the union of the irreducible components meeting Z. Then $\dim X' = d$ by assumption, and $H^{2d}_Z(X', \mathbb{Z}/n) = H^{2d}_Z(X, \mathbb{Z}/n)$ by excision. So we may assume $X = X'$, i.e. Z meets every irreducible component Y of X. In that case, no irreducible component of $X \setminus Z$ is proper, hence $H^{2d}(X \setminus Z, \mathbb{Z}/n) = 0$ by Proposition 3. Now we have the exact sequence

$$H^{2d}_Z(X, \mathbb{Z}/n) \longrightarrow H^{2d}(X, \mathbb{Z}/n) \longrightarrow H^{2d}(X \setminus Z, \mathbb{Z}/n)$$

and the trace map $H^{2d}(X, \mathbb{Z}/n) \to \mathbb{Z}/n$ is surjective by Proposition 3. □

Corollary 6. *If $x \in X$ is a closed point, then the trace map $H^{2d}_x(X, \mathbb{Z}/n) \to \mathbb{Z}/n$ is surjective, where $d = \dim_x X$.*

Proof. Use excision to reduce to the case X is projective and apply Corollary 5. □

In the sequel we will make use of the following.

Corollary 7. *Fix a prime $l \neq \mathrm{char}(k)$. Assume X is proper and let $Z \subset X$ be closed.*

(a) *If $H^{2d}(X \setminus Z, \mathbb{Q}_l) = 0$, then $Z \neq \emptyset$.*
(b) *If X is irreducible and $H^{2d-1}(X \setminus Z, \mathbb{Q}_l) = 0 = H^{2d}(X \setminus Z, \mathbb{Q}_l)$, then Z is connected.*

Proof. (a) follows at once from Proposition 3. In case (b) we find

$$H^{2d}_Z(X, \mathbb{Q}_l) = H^{2d}(X, \mathbb{Q}_l)$$

so by Proposition 3 we get $\dim H^{2d}_Z(X, \mathbb{Q}_l) = 1$. If $Z = Z_1 \sqcup Z_2$ is the disjoint union of two closed subspaces, then $H^{2d}_Z(X, \mathbb{Q}_l) = H^{2d}_{Z_1}(X, \mathbb{Q}_l) \oplus H^{2d}_{Z_2}(X, \mathbb{Q}_l)$, so $H^{2d}_{Z_i}(X, \mathbb{Q}_l) = 0$ for some i, say $i = 2$. Then it follows from Corollary 5 that $Z_2 = \emptyset$, whence $Z = Z_1$. Thus, Z is connected. □

Remark. One can prove Corollary 7 using only de Jong's alteration theorem [10], avoiding Gabber's more delicate result. This is what we had originally done, and it sufficed for Theorems 1 and 2.

3. Proofs of the theorems

We first make some reductions. Clearly, we may assume k algebraically closed. If $s : \mathcal{O}_X \to \mathcal{G}$ is a global section of a vector bundle, then from the exact sequence

$$\mathcal{G}^\vee \xrightarrow{\check{s}} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z(s)} \longrightarrow 0$$

we see that for a morphism $Y \to X$ the zero scheme satisfies $Z(s) \times_X Y = Z(s|_Y)$. In particular, for $Y = X^{\mathrm{red}}$ we find that $Z(s|_{X^{\mathrm{red}}}) \to Z(s)$ is a homeomorphism. Since the pullback of an ample vector bundle by a finite morphism is ample, to show a topological property of $D_r(\phi)$ we may assume X is reduced. And in case (i), taking Y to be an irreducible component of X of dimension d, we may also assume X is irreducible.

3.1. *The codimension of simple degeneracy loci*

A key input to the arguments of [4] and [14] is the computation of the codimension of some simple degeneracy loci (cf. [4, p. 276] and [14, p. 388]), used almost implicitly. We provide a proof here, which also demonstrates how one arrives at the expected codimensions.

Let E be a vector space of dimension e over a field k of characteristic different from 2. Let $P := \mathbb{P}((\mathrm{Sym}^2 E)^\vee)$ and $\mathcal{E} := E \otimes_k \mathcal{O}_P$. Since $\mathrm{char}(k) \neq 2$, we have $(\mathrm{Sym}^2 E)^\vee = \mathrm{Sym}^2(E^\vee)$, hence on P there is a canonical map $\varphi : \mathcal{E}^\vee \to \mathcal{E}(1) := \mathcal{E} \otimes_{\mathcal{O}_P} \mathcal{O}_P(1)$.

Proposition 8. *For* $0 \le r \le e$, $D_r := D_r(\varphi) \subset P$ *is an irreducible subscheme of codimension* $\binom{e-r+1}{2}$.

Proof. Let $G = \mathrm{Grass}_{e-r}(E)$ and $\mathscr{Q}$ the tautological quotient on G. On $P \times G$ we have the vector bundle $\overline{\mathscr{E}} := \ker(\mathscr{E} \to \mathscr{Q})$.

Consider the closed subscheme $\overline{P} := \mathbb{P}(\mathrm{Sym}^2 \overline{\mathscr{E}}^\vee) \subset P \times G$. A simple dimension count, using the well-known dimensions of projective bundles and Grassmannians, yields $\dim \overline{P} = \dim P - \binom{e-r+1}{2}$.

Lemma 9. *The canonical map* $\overline{P} \to P$ *induced by the projection* $P \times G \to P$ *factors over* D_r. *Moreover, the resulting map* $\overline{P} \to D_r$ *is surjective and an isomorphism over a dense open subset of* $(D_r)^{\mathrm{red}}$.

Proof. The key observation is that $\overline{P}$ is the zero subscheme of the composition $\mathscr{E}^\vee \to \mathscr{E}(1) \to \mathscr{Q}(1)$. This follows easily from the exact sequence

$$\mathscr{Q}^\vee \otimes \mathscr{E}^\vee \longrightarrow \mathrm{Sym}^2 \mathscr{E}^\vee \longrightarrow \mathrm{Sym}^2 \overline{\mathscr{E}}^\vee \longrightarrow 0$$

In particular, on $\overline{P}$ the canonical map $\mathscr{E}^\vee \to \mathscr{E}(1)$ has rank at most r, hence the first assertion.

The surjectivity of this map is readily checked on fibres. In particular, D_r is irreducible. Consider the open subset $U := D_r \setminus D_{r-1}$ of D_r. It is nonempty because symmetric $e \times e$ matrices of rank r exist (take the $r \times r$ identity matrix and add zeros). We claim that $\overline{P}_U \to U^{\mathrm{red}}$ is an isomorphism. Indeed, it is easily checked that its fibres are trivial, so the map is unramified, and by [6, 17.4.1] it suffices find a section. But over U^{red} the cokernel of the map $\mathscr{E}^\vee \to \mathscr{E}(1)$ is a vector bundle of rank $e - r$, which defines a section. □

Thus, $\dim D_r = \dim \overline{P}$ and, since $\dim \overline{P} = \dim P - \binom{e-r+1}{2}$, this concludes the proof of Proposition 8. □

The alternating case is analogous, replacing Sym by $\wedge$ throughout, and also works in characteristic 2.

For the (easier) variant of Proposition 8 required for the proof of Theorem 1 we consider another k-vector space F of dimension f and let $P = \mathbb{P}(E^\vee \otimes F)$. On P there is a canonical map $\varphi : \mathscr{F} \to \mathscr{E}(1)$, where $\mathscr{F} := F \otimes_k \mathscr{O}_P$.

Proposition 10. *For* $0 \le r \le \min\{e, f\}$, $D_r(\varphi) \subset P$ *is an irreducible subscheme of codimension* $(e-r)(f-r)$.

Proof. Define G and $\overline{\mathscr{E}}$ as above, and let $\overline{P} = \mathbb{P}(\overline{\mathscr{E}}^\vee \otimes F)$. Then $\overline{P} \subset P \times G$ is the zero scheme of the composition $\mathscr{F} \xrightarrow{\varphi} \mathscr{E}(1) \to \mathscr{Q}(1)$ and we can argue as before. The details are left to the reader. □

3.2. *Proofs of Theorems 1 and 2*

Fix a prime $l \ne \mathrm{char}(k)$. The basic strategy of [4] and [14] for proving Theorems 1 and Theorem 2 is as follows. A Grassmannian $\mathbb{G}$ over X is constructed with a closed subset $Y \subset \mathbb{G}$ (denoted $Z(t)$ in [14]) mapping surjectively onto $D_r(\phi)$, and it suffices to show that Y is nonempty (resp. connected) in case (i) (resp. (ii)). By Corollary 7, this reduces to showing $H^j(\mathbb{G} \setminus Y, \mathbb{Q}_l) = 0$ for $j = 2\dim \mathbb{G}$ (resp. $j \ge 2\dim \mathbb{G} - 1$).

Proof of Theorem 1. The point is that the analogue of [4, Prop. 1.2] in l-adic étale cohomology holds over an arbitrary algebraically closed field, cf. [4, Rem. 1.7]. Indeed, the constructions are geometric and, besides the dimension count of Proposition 10, the proof only relies on the classical finiteness and vanishing results of SGA 4. In fact, the only place in the proof of [4, Thm. 1.1] where the hypothesis $k = \mathbb{C}$ is necessary is [4, Lem. 1.3], which is not used in the proof of [4, Prop. 1.2].

From [4, Prop. 1.2] we obtain $H^j(\mathbb{G} \setminus Y, \mathbb{Q}_l) = 0$ for $j = 2\dim \mathbb{G}$ in case (i) (resp. for $j \ge 2\dim \mathbb{G} - 1$ in case (ii)), proving Theorem 1. □

Proof of Theorem 2. Again most of the arguments of [14] hold in our setting with singular cohomology replaced by l-adic étale cohomology.

In [14, §1], the assumption $k = \mathbb{C}$ can be replaced by k algebraically closed and $\mathrm{char}(k) \neq 2$ (cf. [1, III, §3], [12, Ch. 2]). However, the stated results of [14, §1] also hold for $\mathrm{char}(k) = 2$ in the alternating case by [1, III, §3].

Then, in [14, §2], the conclusion of [14, Prop. 2.1] should be replaced by: $\mathbb{P}(E^*) - Z(s^*)$ is an affine space bundle on $X - Z(s)$.

Now, after replacing [14, Lem. 3.3] by our Corollary 7, the results of [14, §3–4] hold with étale cohomology (here one needs Proposition 8), without any smoothness hypothesis.

This proves Theorem 2 in the symmetric case when $\mathscr{L}$ is trivial. To remove this latter hypothesis, follow the argument of [14, §5] from [14, Lem. 5.1] onwards. For the alternating case, apply the argument of [14, §6]. □

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