



INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus

Mathématique

Thomas Leistner and Thomas Munn

Completeness of certain compact Lorentzian locally symmetric spaces

Volume 361 (2023), p. 819-824

Published online: 11 May 2023

<https://doi.org/10.5802/crmath.449>



This article is licensed under the
CREATIVE COMMONS ATTRIBUTION 4.0 INTERNATIONAL LICENSE.
<http://creativecommons.org/licenses/by/4.0/>



Les Comptes Rendus. Mathématique sont membres du
Centre Mersenne pour l'édition scientifique ouverte

www.centre-mersenne.org

e-ISSN : 1778-3569



Geometry and Topology / *Géométrie et Topologie*

Completeness of certain compact Lorentzian locally symmetric spaces

Thomas Leistner^{*, a} and Thomas Munn^b

^a School of Mathematical Sciences, University of Adelaide, SA 5005, Australia

^b Lund University, Faculty of Science, Centre for Mathematical Sciences, Box 118, 22100 Lund, Sweden

E-mails: thomas.leistner@adelaide.edu.au, thomas.munn@math.lu.se

Abstract. We show that a compact Lorentzian locally symmetric space is geodesically complete if the Lorentzian factor in the local de Rham–Wu decomposition is of Cahen–Wallach type or if the maximal flat factor is one-dimensional and time-like. Our proof uses a recent result by Mehidi and Zeghib and an earlier result by Romero and Sánchez.

Keywords. Lorentzian manifolds, Lorentzian symmetric spaces, geodesic completeness.

Mathematical subject classification (2010). 53C50, 53C35.

Funding. This project has received funding from the Australian Research Council (DP190102360).

Manuscript received 15 November 2022, revised 22 November 2022, accepted 23 November 2022.

1. Introduction and statement of result

In contrast to Riemannian manifolds, a compact Lorentzian manifold is not necessarily geodesically complete — the fundamental example is the incomplete Clifton–Pohl torus, see [12, Example 7.16]. In fact, the set of incomplete Lorentzian metrics on the 2-torus is dense in the set of Lorentzian metrics [14]. Hence, the question whether or not a compact Lorentzian manifold is complete is fundamental in Lorentzian geometry, especially when addressing classification problems for compact manifolds.

One condition that ensures that a compact Lorentzian manifold (M, g) is complete is a sufficient amount of global symmetry: Marsden showed that any compact homogeneous semi-Riemannian manifold is complete [9]. This includes symmetric spaces, i.e. manifolds for which each point $p \in M$ is a fixed point of an isometry whose differential at p is $-\text{Id}$. In a striking generalisation of Marsden’s result in the Lorentzian context, Romero and Sánchez showed that (M, g) is complete if it admits a time-like conformal Killing vector field [15], so a rather small amount of global symmetry. In the case of only local homogeneity, the question is much harder to answer. Only when the dimension of M is 3, Dumitrescu and Zeghib [5] were able to show

* Corresponding author.

that if (M, g) is compact and locally homogeneous, then (M, g) is isometric to a quotient of a Lorentzian homogeneous space G/H by a discrete subgroup Γ of G acting properly and freely on G/H , and consequently is complete.

Other authors have imposed conditions on the curvature of compact Lorentzian manifold (M, g) : if (M, g) is flat [4] or, more generally, has constant sectional curvature [7], then it is complete. In a combination of global and curvature assumptions, in [8] it was shown that (M, g) is complete if it is a pp-wave, i.e. if (M, g) admits a parallel null vector field and satisfies a certain curvature condition. Very recently this result was generalised by Mehidi and Zeghib [10], who showed that the curvature condition can be dropped, i.e. that the existence of a parallel null vector field on a compact Lorentzian manifold implies that the manifold is complete. These results show that compactness and certain types of Lorentzian special holonomy imply completeness. Another result that links completeness to the holonomy group was given in [1].

Of course, assumptions on the homogeneity and on the curvature are not independent of each other. Examples are *locally symmetric spaces*, which are defined by the condition that the curvature tensor is parallel. A locally symmetric space (M, g) is locally isometric to a specific symmetric space, the model space for (M, g) . Since symmetric spaces are complete, by the de Rham–Wu decomposition theorem [13, 16], their universal cover decomposes into indecomposable symmetric spaces, i.e. those that cannot be further decomposed into a semi-Riemannian product. Hence, every locally symmetric space has a local product decomposition into locally symmetric spaces that are locally isometric to an indecomposable symmetric space. We refer to this as the *local de Rham–Wu decomposition* of a locally symmetric space, see Section 2 for more details. It has been shown by Cahen and Wallach [3] that there are only two types of indecomposable Lorentzian symmetric spaces: the spaces of nonzero constant sectional curvature — de Sitter and anti-de Sitter space — and the Cahen–Wallach spaces, see Section 2 for their definition.

If one can show that all compact locally symmetric spaces are complete, then they are isometric quotients of their model spaces and their classification can be reduced to the classification of compact Clifford–Klein forms, and hence to the algebraic problem of classifying co-compact isometric group actions (see for example the results on compact quotients of Cahen–Wallach spaces and further references on compact Lorentzian Clifford–Klein forms in [6]). However, so far this has only been achieved in dimension 3 (in [5] as a special case of locally homogeneous spaces) and for the indecomposable spaces: the results in [4], [7] and [8] imply that a compact, indecomposable Lorentzian locally symmetric space is geodesically complete. In the second author’s thesis [11] this was generalised to locally symmetric spaces which are local products of Euclidean space and a Cahen–Wallach space. Using the same method as in [11], the new result in [10], as well as the result in [15], here we generalise this further.

Theorem 1. *Let (M, g) a compact Lorentzian locally symmetric space whose local de Rham–Wu decomposition satisfies one of the following conditions:*

- (i) *the Lorentzian factor is of Cahen–Wallach type, or*
- (ii) *the maximal flat factor is one-dimensional and time-like.*

Then the time-orientable cover of (M, g) admits a parallel vector field and hence (M, g) is geodesically complete.

This leaves open the cases when (M, g) does not have constant sectional curvature but the Lorentzian factor in the local de Rham–Wu decomposition of (M, g) does. The simplest examples of this situation are those when (M, g) is locally isometric to a product of Minkowski space and a sphere, or to a product of (anti-)de Sitter space with Euclidean space or a sphere. In [11] an attempt was made to apply the approach in [4] and [7] to such cases, but the methods are too specific to the constant curvature model spaces and do not easily transfer to products thereof.

2. The local de Rham–Wu decomposition of a Lorentzian locally symmetric space

In this section we will clarify what we mean by the *local de Rham–Wu decomposition* of a locally symmetric space and we will recall the notion of a Cahen–Wallach space.

We call a symmetric space (M, g) *indecomposable* if the connected component of its holonomy group does not admit a non-trivial invariant subspace of the tangent space on which the metric remains non-degenerate. Since symmetric spaces are complete, by the de Rham–Wu decomposition theorem [13, 16], this is equivalent to the condition that the universal cover $(\widetilde{M}, \widetilde{g})$ of (M, g) does not decompose into a semi-Riemannian product. Hence, the universal cover of a symmetric space is globally isometric to a semi-Riemannian product of simply connected indecomposable symmetric spaces. Note that a *Riemannian* symmetric space (M, g) is indecomposable if and only if the connected component of its holonomy group acts irreducibly, i.e. does not admit any non-trivial invariant subspace of the tangent space.

Cahen and Wallach [3] have shown that an *indecomposable* Lorentzian symmetric space (M, g) either has nonzero constant sectional curvature or is universally covered by $\widetilde{M} = \mathbb{R}^{n+2}$ with

$$\widetilde{g} = 2dt dv + x^a Q_{ab} x^b dt^2 + \delta_{ab} dx^a dx^b, \tag{1}$$

where Q_{ab} is a symmetric $n \times n$ -matrix with nonzero determinant and (t, v, x^1, \dots, x^n) are global coordinates on $\widetilde{M} = \mathbb{R}^{n+2}$. Then $(\widetilde{M}, \widetilde{g})$ is called a *Cahen–Wallach space*. A Cahen–Wallach space admits a parallel null vector field $\frac{\partial}{\partial v}$, and the only non-vanishing terms (up to symmetry) of its curvature tensor are

$$R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x^a}, \frac{\partial}{\partial t}, \frac{\partial}{\partial x^b}\right) = Q_{ab}, \tag{2}$$

for $a, b = 1, \dots, n$. The Ricci tensor ρ , as an endomorphism of $T\widetilde{M}$, is

$$\rho = q \frac{\partial}{\partial v} \otimes dt, \tag{3}$$

where q is the trace of the matrix Q_{ij} . In particular, $\rho^2 = 0$. Two Cahen–Wallach spaces are isometric if their Q_{ij} 's have the same spectrum with the same multiplicities.

Applying the de Rham–Wu decomposition to the universal cover of a Lorentzian symmetric space (M, g) , we obtain that $(\widetilde{M}, \widetilde{g})$ is globally isometric to a product of

- (a) an indecomposable Lorentzian symmetric space, i.e. of nonzero constant sectional curvature or a Cahen–Wallach space, or $(\mathbb{R}, -dt^2)$,
- (b) irreducible Riemannian symmetric spaces, and
- (c) Euclidean space.

Of course, (b) or (c) may not be present in the decomposition. Moreover, any such a decomposition has a maximal flat factor, which is either empty or equal to $(\mathbb{R}, -dt^2)$, Minkowski space, or Euclidean space.

A locally symmetric space (M, g) is a semi-Riemannian manifold with parallel curvature tensor, $\nabla R = 0$. Every locally symmetric space is locally isometric to a globally symmetric space, and hence locally isometric to a product of indecomposable symmetric spaces. We refer to this local product decomposition as the *local de Rham–Wu decomposition* of (M, g) . We call a locally symmetric space *indecomposable* if it is locally isometric to an indecomposable symmetric space. Consequently, an indecomposable Lorentzian locally symmetric space either has constant sectional curvature or is locally isometric to a Cahen–Wallach space. The latter means that there are local coordinates such that the metric is of the form (1). Indecomposable compact Lorentzian locally symmetric spaces have been shown to be complete, the ones with constant sectional curvature in [4, 7] and the ones of Cahen–Wallach type in [8].

Since a locally symmetric Lorentzian symmetric space (M, g) is locally isometric to a product of symmetric spaces as in (a), (b), and (c), the tangent space at a point decomposes into

orthogonal subspaces that correspond to the tangent spaces of the local factors. Moreover, the curvature tensor and the Ricci tensor of (M, g) decompose accordingly into the sum of the curvature tensors and Ricci tensors of the factors in the local decomposition. Since irreducible Riemannian symmetric spaces of dimension > 1 are Einstein with nonzero Einstein constant, this implies the following: if the local de Rham–Wu decomposition of (M, g) does not contain a Cahen–Wallach factor, then the Ricci tensor of (M, g) is given by the sum of multiples of the identity, and if it does contain a Cahen–Wallach factor, then the Ricci tensor is a sum of ρ as in (3) and multiples of the identity.

3. Proof of Theorem 1

Let (M, g) be a locally symmetric Lorentzian space of dimension m and let Hol_p be its holonomy group at a point $p \in M$. Let R be the curvature tensor of (M, g) and ρ the Ricci tensor considered as an endomorphism of the tangent bundle. Since $\nabla R = 0$, $\nabla \rho = 0$, so both R and ρ are invariant under Hol_p . We define two vector distributions, the *nullity of R* ,

$$\mathcal{N} := \{X \in TM \mid X \lrcorner R = 0\},$$

and the *generalised kernel of ρ* ,

$$\mathcal{K} := \{X \in TM \mid \exists m \in \mathbb{N} : \rho^m(X) = 0\}.$$

Clearly, both \mathcal{N} and \mathcal{K} are invariant under Hol_p , since for $H \in \text{Hol}_p$ we have

$$R(H\mathcal{N}|_p, X, Y, Z) = R(\mathcal{N}|_p, H^{-1}X, H^{-1}Y, H^{-1}Z) = 0$$

and

$$\rho(HX) = H^{-1}\rho(X),$$

for all $X, Y, Z \in T_pM$. This implies that also the orthogonal distributions \mathcal{N}^\perp and \mathcal{K}^\perp are invariant under Hol_p . We clearly have that $\mathcal{N} \subset \mathcal{K}$.

The remarks at the end of the previous section imply that in the case when the local de Rham–Wu decomposition of (M, g) does not contain a Cahen–Wallach factor, we have that $\mathcal{N} = \mathcal{K}$, $TM = \mathcal{N} \oplus \mathcal{N}^\perp$, and $\mathcal{N} \neq 0$ only if the local de Rham–Wu decomposition contains a flat factor. If the flat factor is one-dimensional and time-like, we have that \mathcal{N} is a time-like line bundle and an arbitrary element $H \in \text{Hol}_p$ in the decomposition $T_pM = \mathcal{N}|_p \oplus \mathcal{N}^\perp|_p$ is of the form

$$H = \begin{pmatrix} \pm 1 & 0 \\ 0 & A \end{pmatrix},$$

with $A \in \mathbf{O}(m - 1)$. Hence, the time-orientable cover of (M, g) admits a time-like parallel vector field and is compact if M is compact. A parallel vector field is of course conformally Killing, so we can apply the result in [15] to conclude that (M, g) is complete. This proves case (ii) in Theorem 1.

From now on assume that the local de Rham–Wu decomposition of (M, g) does contain a Cahen–Wallach factor. Since $\frac{\partial}{\partial v}$ is a local parallel null vector field, and with (2), we get that

$$\mathcal{L} := \mathcal{N} \cap \mathcal{N}^\perp \subset TM$$

is not trivial, and hence a bundle of null lines that is invariant under the holonomy group of (M, g) . In particular, we have the following holonomy invariant filtrations

$$\mathcal{L} \subset \mathcal{N} \subset \mathcal{K}, \quad \mathcal{L} \subset \mathcal{N}^\perp \cap \mathcal{K} \subset \mathcal{K}.$$

Moreover, since the Ricci-tensor of a Cahen–Wallach space is nilpotent and the irreducible Riemannian factors are Einstein with non-zero Einstein constant, \mathcal{K}^\perp is non-degenerate, so that

$$TM = \mathcal{K} \oplus \mathcal{K}^\perp$$

is a holonomy invariant decomposition of TM . At $p \in M$, we have the Hol_p -invariant decomposition

$$T_pM = \mathcal{K}|_p \oplus \mathcal{K}^\perp|_p,$$

with $\mathcal{K}|_p = T \oplus \bar{T}$, where $T \neq \{0\}$ is the tangent space at p of the local Cahen–Wallach factor and \bar{T} the (possibly trivial) tangent space of the Euclidean (i.e. flat and Riemannian) factor. Note that the decomposition $\mathcal{K}|_p = T \oplus \bar{T}$ may not be Hol_p -invariant. In order to describe the action of Hol_p on T_pM , we denote by (t, v, x^1, \dots, x^n) the coordinates on the local Cahen–Wallach factor as in (1) and fix the following basis of $\mathcal{K}|_p$,

$$e_- = \frac{\partial}{\partial v}, \quad e_1, \dots, e_n, \quad e_{n+1}, \dots, e_{n+k}, \quad e_+ = \frac{\partial}{\partial v} - x^a Q_{ab} x^b \frac{\partial}{\partial t},$$

where $e_a = \frac{\partial}{\partial x^a}$ for $a = 1, \dots, n$ and e_{n+b} for $b = 1, \dots, k$ an orthonormal basis for the Euclidean metric on \bar{T} . In such a basis, the metric restricted to $\mathcal{K}|_p$ is of the form

$$g|_p = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

In particular we have

$$T = \text{span}\{e_-, e_1, \dots, e_n, e_+\}, \quad \bar{T} = \text{span}\{e_{n+1}, \dots, e_{n+k}\},$$

and that

$$\mathcal{L}|_p = \mathbb{R} \cdot e_-, \quad (\mathcal{N}^\perp \cap \mathcal{K})|_p = \text{span}\{e_-, e_1, \dots, e_n\}, \quad \mathcal{N}|_p = \text{span}\{e_-, e_{n+1}, \dots, e_{n+k}\},$$

are Hol_p -invariant. Then, in the Hol_p -invariant decomposition $T_pM = \mathcal{K}|_p \oplus \mathcal{K}^\perp|_p$, an arbitrary element $H \in \text{Hol}_p$ is of the form

$$H = \begin{pmatrix} F & 0 \\ 0 & G \end{pmatrix},$$

where F , in the above basis, writes as

$$F = \begin{pmatrix} \lambda & * & * & * \\ 0 & A & 0 & u \\ 0 & 0 & B & v \\ 0 & 0 & 0 & \lambda^{-1} \end{pmatrix},$$

with $A \in \mathbf{O}(n)$, $B \in \mathbf{O}(k)$, $u \in \mathbb{R}^n$, $v \in \mathbb{R}^k$, $0 \neq \lambda \in \mathbb{R}$, and the $*$ -terms are determined by the other terms. In particular, we have that

$$H(e_a) = F(e_a) = A_a^b e_b \pmod{e_-}, \quad H(e_+) = F(e_+) = \lambda e_+ + u^a e_a + v^{\bar{c}} e_{\bar{c}} \pmod{e_-},$$

where the unbarred indices $a, b, c \dots$ run from 1 to n and the barred indices $\bar{a}, \bar{b}, \bar{c}, \dots$ from $n+1$ to $n+k$. Then, with (2) and since $e_- \lrcorner R = 0$ and $e_{\bar{c}} \lrcorner R = 0$, we have

$$(H^{-1} \cdot R)(e_a, e_+, e_b, e_+) = R(He_a, He_+, He_b, He_+) = R(Fe_a, Fe_+, Fe_b, Fe_+) = \frac{A_a^c Q_{cd} A_b^d}{\lambda^2}.$$

Hence, the holonomy invariance of R ,

$$(H^{-1} \cdot R)(e_a, e_+, e_b, e_+) = R(e_a, e_+, e_b, e_+) = Q_{ab},$$

yields the matrix equation for $Q = (Q_{ab})$,

$$\lambda^2 Q = A^\top Q A.$$

Since $A \in \mathbf{O}(n)$ and $\det(Q) \neq 0$, this implies that $\lambda = \pm 1$. This however shows that the time-orientable cover of (M, g) admits a global parallel null vector field, see [2, Proposition 2] for details.

If we now assume that M is compact, the time-orientable cover is compact and admits a global parallel null vector field, and hence, by the result in [10] is geodesically complete. This implies that (M, g) is geodesically complete. \square

Acknowledgements

A special case of Theorem 1 was obtained in the second author's thesis [11], written under supervision of the first author. We would like to thank Michael Eastwood for taking the role of co-supervisor and for helpful discussions and comments.

References

- [1] L. A. Aké Hau, M. Sánchez, "Compact affine manifolds with precompact holonomy are geodesically complete", *J. Math. Anal. Appl.* **436** (2016), no. 2, p. 1369-1371.
- [2] H. Baum, K. Lärz, T. Leistner, "On the full holonomy group of Lorentzian manifolds", *Math. Z.* **277** (2014), no. 3-4, p. 797-828.
- [3] M. Cahen, N. Wallach, "Lorentzian symmetric spaces", *Bull. Am. Math. Soc.* **76** (1970), p. 585-591.
- [4] Y. Carrière, "Autour de la conjecture de L. Markus sur les variétés affines", *Invent. Math.* **95** (1989), no. 3, p. 615-628.
- [5] S. Dumitrescu, A. Zeghib, "Géométries lorentziennes de dimension 3: classification et complétude", *Geom. Dedicata* **149** (2010), p. 243-273.
- [6] I. Kath, M. Olbrich, *Compact quotients of Cahen-Wallach spaces*, Memoirs of the American Mathematical Society, vol. 1264, American Mathematical Society, 2019, v+84 pages.
- [7] B. Klingler, "Complétude des variétés lorentziennes à courbure constante", *Math. Ann.* **306** (1996), no. 2, p. 353-370.
- [8] T. Leistner, D. Schliebner, "Completeness of compact Lorentzian manifolds with abelian holonomy", *Math. Ann.* **364** (2016), no. 3-4, p. 1469-1503.
- [9] J. Marsden, "On completeness of homogeneous pseudo-riemannian manifolds", *Math. J. Indiana Univ.* **22** (1973), p. 1065-1066.
- [10] L. Mehidi, A. Zeghib, "On completeness and dynamics of compact Brinkmann spacetimes", <https://arxiv.org/abs/2205.07243>, 2022.
- [11] T. J. Munn, "Symmetric spaces, geometric manifolds and geodesic completeness", 2021, Master's thesis, University of Adelaide, School of Mathematical Sciences, <https://hdl.handle.net/2440/134331>.
- [12] B. O'Neill, *Semi-Riemannian Geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc., 1983, xiii+468 pages.
- [13] G. de Rham, "Sur la reductibilité d'un espace de Riemann", *Comment. Math. Helv.* **26** (1952), p. 328-344.
- [14] A. Romero, M. Sánchez, "New properties and examples of incomplete Lorentzian tori", *J. Math. Phys.* **35** (1992), no. 4, p. 1992-1997.
- [15] ———, "Completeness of compact Lorentz manifolds admitting a timelike conformal Killing vector field", *Proc. Am. Math. Soc.* **123** (1995), no. 9, p. 2831-2833.
- [16] H.-H. Wu, "On the de Rham decomposition theorem", *Ill. J. Math.* **8** (1964), p. 291-311.