INSTITUT DE FRANCE
Académie des sciences

Comptes Rendus
Mathématique

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Volume 361 (2023), p. 877-888

Published online: 18 July 2023

https://doi.org/10.5802/crmath.450

Les Comptes Rendus. Mathématique *sont membres du*
Centre Mersenne pour l'édition scientifique ouverte
www.centre-mersenne.org
e-ISSN : 1778-3569

Comptes Rendus
Mathématique
2023, Vol. 361, p. 877-888
https://doi.org/10.5802/crmath.450

Algebra, Algebraic geometry / *Algèbre, Géométrie algébrique*

Tensor weight structures and t-structures on the derived categories of schemes

Structures de poids tensorielles et t-structures sur les catégories dérivées des schémas

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Abstract. We give a condition which characterises those weight structures on a derived category which come from a Thomason filtration on the underlying scheme. Weight structures satisfying our condition will be called $\otimes^c$-weight structures. More precisely, for a Noetherian separated scheme X, we give a bijection between the set of compactly generated $\otimes^c$-weight structures on $\mathbf{D}(\mathrm{Qcoh}\,X)$ and the set of Thomason filtrations of X. We achieve this classification in two steps. First, we show that the bijection [12, Theorem 4.10] restricts to give a bijection between the set of compactly generated $\otimes^c$-weight structures and the set of compactly generated tensor t-structures. We then use our earlier classification of compactly generated tensor t-structures to obtain the desired result. We also study some immediate consequences of these classifications in the particular case of the projective line. We show that in contrast to the case of tensor t-structures, there are no non-trivial tensor weight structures on $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$.

Résumé. On dégage une condition qui caractérise les structures de poids sur une catégorie dérivée qui proviennent d'une filtration de Thomason sur le schéma sous-jacent. Les structures de poids satisfaisant notre condition s'appelleront des $\otimes^c$-structures de poids. Plus précisément, pour tout schéma séparé et noethérien X, nous construisons une bijection entre l'ensemble des $\otimes^c$-structures de poids à engendrement compact sur $\mathbf{D}(\mathrm{Qcoh}\,X)$ et l'ensemble des filtrations de Thomason sur X. La construction se fait en deux étapes. On montre d'abord que la bijection de [12, Theorem 4.10] donne par restriction une bijection entre l'ensemble des $\otimes^c$-structures de poids à engendrement compact et l'ensemble des t-structures tensorielles à engendrement compact. Nous utilisons ensuite notre classification précédente des $\otimes^c$-structures de poids à engendrement compact pour arriver au résultat. Nous étudions aussi quelques conséquences immédiates dans le cas particulier de la droite projective. Nous montrons que, contrairement au cas des t-structures tensorielles, il n'y a pas de structure de poids tensorielle non-triviale sur $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$.

2020 Mathematics Subject Classification. 14F08, 18G80.

Manuscript received 14 October 2022, revised and accepted 25 November 2022.

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1. Introduction

Weight structures on triangulated categories were introduced by Bondarko [4] as an important natural counterpart of t-structures with applications to Voevodsky's category of motives. Pauksztello independently came up with the same notion while trying to obtain a dual version of a result due to Hoshino, Kato and Miyachi; he termed it co-t-structures, see [11]. It has been observed by Bondarko that the two notions, t-structures and weight structures, are connected by interesting relations. In this vein, Šťovíček and Pospíšil have proved for a certain class of triangulated categories, the collection of compactly generated t-structures and compactly generated weight structures are in bijection [12, Theorem 4.10] with each other, where the bijection goes via a duality at the compact level. In particular, this bijection holds in the derived category of a Noetherian ring R and since in this case, we have the classification of compactly generated t-structures in terms of Thomason filtrations of $\operatorname{Spec} R$ [1, Theorem 3.11], they obtain a classification of compactly generated weight structures of $\mathbf{D}(R)$.

Our aim in this short article is twofold: first to generalize the theorem of Šťovíček and Pospíšil [12, Theorem 4.15] to the case of separated Noetherian schemes, and second to understand the two types of notions, in the simplest non-affine situation: the derived category of the projective line over a field k. Our interest in this special case arose partly from the work of Krause and Stevenson [10], where the authors study the localizing subcategories of $\mathbf{D}(\operatorname{Qcoh}\mathbb{P}^1_k)$, and partly from our desire to better understand the general results.

In our earlier work [7], we have shown that a t-structure on $\mathbf{D}(\operatorname{Qcoh} X)$ supported on a Thomason filtration of a Noetherian scheme X satisfies a tensor condition. We call them tensor t-structures. In this article, we introduce the analogous notion of tensor weight structures, also a slightly weaker notion which we call $\otimes^c$-weight structures. We then show that the bijection [12, Theorem 4.10] restricts to a bijection between tensor t-structures and $\otimes^c$-weight structures; this can be seen as a consequence of our Lemma 10 and Theorem 15. Next, we specialize to the case of the derived categories of separated Noetherian schemes and classify compactly generated $\otimes^c$-weight structures in this case, see Theorem 21.

In the last section, we apply all the general theory and the classification results to the derived category of the projective line over a field k. By our Theorem 15 classifying compactly generated tensor t-structures of $\mathbf{D}(\operatorname{Qcoh}\mathbb{P}^1_k)$ is equivalent to classifying thick $\otimes$-preaisles of $\mathbf{D}^b(\operatorname{Coh}\mathbb{P}^1_k)$, so we restrict our attention to $\otimes$-preaisles of $\mathbf{D}^b(\operatorname{Coh}\mathbb{P}^1_k)$. We give a complete description of the $\otimes$-preaisles in Proposition 24, and in Proposition 28 we determine which of these are aisles or in other words give rise to t-structures on $\mathbf{D}^b(\operatorname{Coh}\mathbb{P}^1_k)$. The result of Proposition 28 is not new, it can possibly be deduced from [3]; also in [8], the authors describe the bounded t-structures on $\mathbf{D}^b(\operatorname{Coh}\mathbb{P}^1_k)$ by using the classification of t-stabilities on $\mathbf{D}^b(\operatorname{Coh}\mathbb{P}^1_k)$. Finally, we consider the same question for tensor weight structures, and to our surprise, we discovered that on $\mathbf{D}^b(\operatorname{Coh}\mathbb{P}^1_k)$ there are no non-trivial tensor weight structures.

2. Preliminaries

Let $\mathscr{T}$ be a triangulated category and $\mathscr{T}^c$ denote the full subcategory of compact objects. We recall the definition of t-structures which was introduced in [2].

Definition 1. *A* t-structure *on $\mathscr{T}$ is a pair of full subcategories $(\mathscr{U},\mathscr{V})$ satisfying the following properties:*

- (t1) $\Sigma\mathscr{U} \subset \mathscr{U}$ *and* $\Sigma^{-1}\mathscr{V} \subset \mathscr{V}$.
- (t2) $\mathscr{U} \perp \Sigma^{-1}\mathscr{V}$.
- (t3) *For any $T \in \mathscr{T}$ there is a distinguished triangle $U \to T \to V \to \Sigma U$ where $U \in \mathscr{U}$ and $V \in \Sigma^{-1}\mathscr{V}$. We call such a triangle* truncation decomposition *of T.*

Next, we quote the definition of weight structures from [5].

Definition 2. *A* weight structure *on $\mathcal{T}$ is a pair of full subcategories $(\mathcal{X},\mathcal{Y})$ satisfying the following properties:*

(w0) *$\mathcal{X}$ and $\mathcal{Y}$ are closed under direct summands.*
(w1) $\Sigma^{-1}\mathcal{X} \subset \mathcal{X}$ *and* $\Sigma\mathcal{Y} \subset \mathcal{Y}$.
(w2) $\mathcal{X} \perp \Sigma\mathcal{Y}$.
(w3) *For any object $T \in \mathcal{T}$ there is a distinguished triangle $X \to T \to Y \to \Sigma X$ where $X \in \mathcal{X}$ and $Y \in \Sigma\mathcal{Y}$. The above triangle is called a* weight decomposition *of T.*

Note that if $(\mathcal{U},\mathcal{V})$ is a t-structure on $\mathcal{T}$ then $(\mathcal{V},\mathcal{U})$ is a t-structure on $\mathcal{T}^{op}$. Similarly, If $(\mathcal{X},\mathcal{Y})$ is a weight structure on $\mathcal{T}$ then $(\mathcal{Y},\mathcal{X})$ is a weight structure on$\mathcal{T}^{op}$.

For any subcategory $\mathcal{U}$ of $\mathcal{T}$, we denote $\mathcal{U}^{\perp}$ to be the full subcategory consisting of objects $B \in \mathcal{T}$ such that $\mathrm{Hom}(A,B) = 0$ for all $A \in \mathcal{U}$. Analogously we define ${}^{\perp}\mathcal{U}$ to be the full subcategory of objects $B \in \mathcal{T}$ such that $\mathrm{Hom}(B,A) = 0$ for all $A \in \mathcal{U}$.

Definition 3. *We say a t-structure $(\mathcal{U},\mathcal{V})$ is* compactly generated *if there is a set of compact objects $\mathcal{S}$ such that $\mathcal{U} = {}^{\perp}(\mathcal{S}^{\perp})$. A weight structure $(\mathcal{X},\mathcal{Y})$ is* compactly generated *if there is a set of compact objects $\mathcal{S}$ such that $\mathcal{X} = {}^{\perp}(\mathcal{S}^{\perp})$.*

Definition 4. *A subcategory $\mathcal{U}$ of $\mathcal{T}$ is a* preaisle *if it is closed under positive shifts and extensions. Dually, we say $\mathcal{U}$ is a* copreaisle *of $\mathcal{T}$, if $\mathcal{U}$ is a preaisle of $\mathcal{T}^{op}$.*

A preaisle is called thick *if it is closed under direct summands. We say a preaisle is* cocomplete *if it is closed under coproducts in $\mathcal{T}$, and* complete *if it is closed under products. Similarly, we define* thick, cocomplete *and* complete *copreaisles.*

For a t-structure $(\mathcal{U},\mathcal{V})$ the subcategory $\mathcal{U}$ is a cocomplete preaisle of $\mathcal{T}$, and for a weight structure $(\mathcal{X},\mathcal{Y})$ the subcategory $\mathcal{X}$ is a cocomplete copreaisle of $\mathcal{T}$.

We need the notion of stable derivators to formulate the next theorem but our requirement of the theory of derivators is the bare minimum. We will not go into the precise lengthy definition here, instead, we refer the reader to [12, Section 2.1] and references therein.

Theorem 5 ([12, Theorem 4.5]). *Let $\mathcal{T} = \mathbb{D}(e)$, where $\mathbb{D}$ is a stable derivator such that for each small category I, $\mathbb{D}(I)$ has all small coproducts. Then,*

(i) *There is a bijection between the set of compactly generated t-structures of $\mathcal{T}$ and the set of thick preaisles of $\mathcal{T}^c$ given by*

$$(\mathcal{U},\mathcal{V}) \mapsto \mathcal{U} \cap \mathcal{T}^c \qquad \mathcal{P} \mapsto ({}^{\perp}(\mathcal{P}^{\perp}), \Sigma\mathcal{P}^{\perp}).$$

(ii) *There is a bijection between the set of compactly generated weight structures of $\mathcal{T}$ and the set of thick copreaisles of $\mathcal{T}^c$ given by*

$$(\mathcal{X},\mathcal{Y}) \mapsto \mathcal{X} \cap \mathcal{T}^c \qquad \mathcal{P} \mapsto ({}^{\perp}(\mathcal{P}^{\perp}), \Sigma^{-1}\mathcal{P}^{\perp}).$$

3. Tensor weight and t-structures

We recall the definition of tensor triangulated category from [9, Definition A.2.1].

Definition 6. *A* tensor triangulated category $(\mathcal{T},\otimes,\mathbf{1})$ *is a triangulated category with a compatible closed symmetric monoidal structure. This means there is a functor $-\otimes-\colon \mathcal{T}\times\mathcal{T} \to \mathcal{T}$ which is triangulated in both the variables and satisfies certain compatibility conditions. Moreover, for each $B \in \mathcal{T}$ the functor $-\otimes B$ has a right adjoint which we denote by $\mathcal{H}\mathrm{om}(B,-)$. The functor $\mathcal{H}\mathrm{om}(-,-)$ is triangulated in both the variables, and for any A, B, and C in $\mathcal{T}$ we have natural isomorphisms* $\mathrm{Hom}(A\otimes B, C) \to \mathrm{Hom}(A, \mathcal{H}\mathrm{om}(B,C))$.

Definition 7. *Let $\mathcal{T}$ be a tensor triangulated category given with a preaisle $\mathcal{T}^{\leq 0}$ satisfying*

$$\mathcal{T}^{\leq 0} \otimes \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 0} \text{ and } \mathbf{1} \in \mathcal{T}^{\leq 0}.$$

A preaisle $\mathcal{U}$ of $\mathcal{T}$ is a $\otimes$-preaisle (with respect to $\mathcal{T}^{\leq 0}$) *if $\mathcal{T}^{\leq 0} \otimes \mathcal{U} \subset \mathcal{U}$. We say a copreaisle $\mathcal{X}$ of $\mathcal{T}$ is a* $\otimes$-copreaisle (with respect to $\mathcal{T}^{\leq 0}$) *if $\mathcal{H}\mathrm{om}(\mathcal{T}^{\leq 0}, \mathcal{X}) \subset \mathcal{X}$. A t-structure $(\mathcal{U}, \mathcal{V})$ is a called a* tensor t-structure *if $\mathcal{U}$ is a $\otimes$-preaisle, and a weight structure $(\mathcal{X}, \mathcal{Y})$ is a* tensor weight structure *if $\mathcal{X}$ is a $\otimes$-copreaisle.*

Let $\mathcal{S} \subset \mathcal{T}$ be a class of objects. We denote the smallest cocomplete preaisle containing $\mathcal{S}$ by $\langle \mathcal{S} \rangle^{\leq 0}$ and call it the *cocomplete preaisle generated by $\mathcal{S}$*. If $\mathcal{T}$ does not have coproducts we denote $\langle \mathcal{S} \rangle^{\leq 0}$ to be the smallest preaisle containing $\mathcal{S}$.

Lemma 8. *Let $\mathcal{T}^{\leq 0}$ be generated by a set of objects $\mathcal{K}$, that is, $\mathcal{T}^{\leq 0} = \langle \mathcal{K} \rangle^{\leq 0}$. Then*

(i) *a cocomplete preaisle $\mathcal{U}$ of $\mathcal{T}$ is a $\otimes$-preaisle if and only if $\mathcal{K} \otimes \mathcal{U} \subset \mathcal{U}$.*
(ii) *a complete copreaisle $\mathcal{X}$ of $\mathcal{T}$ is a $\otimes$-copreaisle if and only if $\mathcal{H}\mathrm{om}(\mathcal{K}, \mathcal{X}) \subset \mathcal{X}$.*

Proof. Part (i). Suppose $\mathcal{K} \otimes \mathcal{U} \subset \mathcal{U}$. We define $\mathcal{B} = \{X \in \mathcal{T}^{\leq 0} \mid X \otimes \mathcal{U} \subset \mathcal{U}\}$. Since $\mathcal{U}$ is a cocomplete preaisle we can observe that $\mathcal{B}$ is also a cocomplete preaisle. Now, by our assumption $\mathcal{K} \subset \mathcal{B}$ so we get $\mathcal{T}^{\leq 0} \subset \mathcal{B}$ which proves $\mathcal{U}$ is $\otimes$-preaisle. The converse is immediate.

Part (ii). Let $\mathcal{B} = \{X \in \mathcal{T}^{\leq 0} \mid \mathcal{H}\mathrm{om}(X, \mathcal{X}) \subset \mathcal{X}\}$. Since $\mathcal{X}$ is a copreaisle we can see that $\mathcal{B}$ is a preaisle. Completeness of $\mathcal{X}$ implies $\mathcal{B}$ is cocomplete. Now, by following a similar argument as in (i) we get $\mathcal{X}$ is $\otimes$-copreaisle. □

An immediate consequence of the above lemma is if $\mathcal{T}^{\leq 0} = \langle \mathbf{1} \rangle^{\leq 0}$ then every cocomplete preaisle of $\mathcal{T}$ is a $\otimes$-preaisle and every complete copreaisle of $\mathcal{T}$ is a $\otimes$-copreaisle. In particular, for a commutative ring R, all cocomplete preaisle and complete copreaisle of $\mathbf{D}(R)$ satisfy the tensor condition.

Definition 9. *We say an object $X \in \mathcal{T}$ is* rigid *or* strongly dualizable *if for each $Y \in \mathcal{T}$ the natural map $\mu : \mathcal{H}\mathrm{om}(X, \mathbf{1}) \otimes Y \to \mathcal{H}\mathrm{om}(X, Y)$ is an isomorphism. A tensor triangulated category $(\mathcal{T}, \otimes, \mathbf{1})$ is* rigidly compactly generated *if the following conditions hold:*

(i) *$\mathcal{T}$ is compactly generated;*
(ii) *$\mathbf{1}$ is compact;*
(iii) *every compact object is rigid.*

Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category. For such a triangulated category the tensor product on $\mathcal{T}$ restricts to $\mathcal{T}^c$ therefore $(\mathcal{T}^c, \otimes, \mathbf{1})$ is also a tensor triangulated category. Suppose $\mathcal{T}$ is given with a preaisle $\mathcal{T}^{\leq 0}$ satisfying the condition of Definition 7 then so does the preaisle $\mathcal{T}^c \cap \mathcal{T}^{\leq 0}$ of $\mathcal{T}^c$. So we can define $\otimes$-preaisles and $\otimes$-copreaisles of $\mathcal{T}^c$ with respect to $\mathcal{T}^c \cap \mathcal{T}^{\leq 0}$.

Lemma 10. *Let $\mathcal{T}$ be a rigidly compactly generated tensor triangulated category given with a preaisle $\mathcal{T}^{\leq 0}$ satisfying the condition of Definition 7.*

Then, there is a one-to-one correspondence between the set of $\otimes$-preaisles and the set of $\otimes$-copreaisles of $\mathcal{T}^c$.

Proof. Let $\mathcal{U}$ be a full subcategory of $\mathcal{T}^c$. We denote $\mathcal{U}^*$ the full subcategory given by

$$\mathcal{U}^* = \left\{ X \in \mathcal{T}^c \,\middle|\, X \cong \mathcal{H}\mathrm{om}(Y, \mathbf{1}) \text{ for some } Y \in \mathcal{U} \right\}.$$

The assignment $\mathcal{U} \mapsto \mathcal{U}^*$ induces an equivalence between the preaisles and copreaisles of $\mathcal{T}^c$; see [12, Lemma 4.9]. We only need to show that the above assignment preserves the tensor condition.

Let $\mathcal{U}$ be a preaisle of $\mathcal{T}^c$, $X \in \mathcal{U}^*$ and $T \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$. We have

$$\mathscr{H}\mathrm{om}(T, X) \cong \mathscr{H}\mathrm{om}(T, \mathscr{H}\mathrm{om}(Y, \mathbf{1})) \cong \mathscr{H}\mathrm{om}(T \otimes Y, \mathbf{1}).$$

If we assume $\mathcal{U}$ is a $\otimes$-preaisle then $T \otimes Y \in \mathcal{U}$. Hence $\mathscr{H}\mathrm{om}(T, X) \in \mathcal{U}^*$, this proves $\mathcal{U}^*$ is a $\otimes$-copreaisle.

Now, suppose $\mathcal{U}$ is a copreaisle. Let $X \in \mathcal{U}^*$ and $T \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$. Then,

$$\begin{aligned} T \otimes X &\cong T \otimes \mathscr{H}\mathrm{om}(Y, \mathbf{1}) \\ &\cong \mathscr{H}\mathrm{om}(Y, T) \\ &\cong \mathscr{H}\mathrm{om}(Y \otimes \mathscr{H}\mathrm{om}(T, \mathbf{1}), \mathbf{1}) \\ &\cong \mathscr{H}\mathrm{om}(\mathscr{H}\mathrm{om}(T, Y), \mathbf{1}). \end{aligned}$$

If we assume $\mathcal{U}$ is a $\otimes$-copreaisle then $\mathscr{H}\mathrm{om}(T, Y) \in \mathcal{U}$. We get, $T \otimes X \in \mathcal{U}^*$, which proves $\mathcal{U}^*$ is a $\otimes$-preaisle. □

Definition 11. *A preaisle $\mathcal{U}$ is* compactly generated *if $\mathcal{U} = \langle \mathcal{S} \rangle^{\leq 0}$ for a set of compact objects $\mathcal{S}$.*

Definition 12. *We say a triangulated category $\mathcal{T}$ has the property $(*)$ if:*

(i) *$\mathcal{T}$ is rigidly compactly generated;*
(ii) *$\mathcal{T}$ has a preaisle $\mathcal{T}^{\leq 0}$ satisfying $\mathcal{T}^{\leq 0} \otimes \mathcal{T}^{\leq 0} \subset \mathcal{T}^{\leq 0}$ and $\mathbf{1} \in \mathcal{T}^{\leq 0}$;*
(iii) *$\mathcal{T}^{\leq 0}$ is compactly generated, that is, $\mathcal{T}^{\leq 0} = \langle \mathcal{T}^c \cap \mathcal{T}^{\leq 0} \rangle^{\leq 0}$.*

For a triangulated category $\mathcal{T}$ having the property $(*)$, we define a weaker notion than $\otimes$-(co)preaisle.

Definition 13. *Let $\mathcal{T}$ have the property $(*)$.*

A preaisle $\mathcal{U}$ of $\mathcal{T}$ is a $\otimes^c$-preaisle *if for any $T \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$ and $U \in \mathcal{U}$ we have $T \otimes U \in \mathcal{U}$. Similarly a copreaisle $\mathcal{X}$ of $\mathcal{T}$ is a* $\otimes^c$-copreaisle *if for any $T \in \mathcal{T}^c \cap \mathcal{T}^{\leq 0}$ and $X \in \mathcal{X}$ we have $\mathscr{H}\mathrm{om}(T, X) \in \mathcal{X}$.*

A t-structure $(\mathcal{U}, \mathcal{V})$ on $\mathcal{T}$ is a $\otimes^c$-t-structure *if $\mathcal{U}$ is a $\otimes^c$-preaisle and a weight structure $(\mathcal{X}, \mathcal{Y})$ on $\mathcal{T}$ is a* $\otimes^c$-weight structure *if $\mathcal{X}$ is a $\otimes^c$-copreaisle.*

Remark 14. This weaker notion gives something new only for preaisles(resp. copreaisles) which are not cocomplete(resp. complete) since by Lemma 8 it can easily be observed that for $\mathcal{T}$ having the property $(*)$: (i) every cocomplete $\otimes^c$-preaisles of $\mathcal{T}$ is a $\otimes$-preaisle of $\mathcal{T}$, and (ii) every complete $\otimes^c$-copreaisle of $\mathcal{T}$ is a $\otimes$-copreaisle of $\mathcal{T}$.

With this weaker notion, we now prove the tensor analogue of Theorem 5.

Theorem 15. *Let $\mathcal{T}$ have the property $(*)$ (see Definition 12) and $\mathcal{T} = \mathbb{D}(e)$, where $\mathbb{D}$ is a stable derivator such that for each small category I, $\mathbb{D}(I)$ has all small coproducts. Then,*

(i) *There is a bijective correspondence between the set of compactly generated tensor t-structures of $\mathcal{T}$ and the set of thick $\otimes$-preaisles of $\mathcal{T}^c$ given by*

$$(\mathcal{U}, \mathcal{V}) \mapsto \mathcal{U} \cap \mathcal{T}^c \qquad \mathcal{P} \mapsto ({}^{\perp}(\mathcal{P}^{\perp}), \Sigma \mathcal{P}^{\perp}).$$

(ii) *There is a bijective correspondence between the set of compactly generated $\otimes^c$-weight structures of $\mathcal{T}$ and the set of thick $\otimes$-copreaisles of $\mathcal{T}^c$ given by*

$$(\mathcal{X}, \mathcal{Y}) \mapsto \mathcal{X} \cap \mathcal{T}^c \qquad \mathcal{P} \mapsto ({}^{\perp}(\mathcal{P}^{\perp}), \Sigma^{-1} \mathcal{P}^{\perp}).$$

Before proving the theorem, we will make some comments about the lack of symmetry in the above statement:

Remark 16. It is easy to observe that given a subcategory $\mathscr{P}$ the subcategory ${}^{\perp}(\mathscr{P}^{\perp})$ is always cocomplete, that is, closed under coproducts. Therefore, by Remark 14 saying ${}^{\perp}(\mathscr{P}^{\perp})$ is a $\otimes^{c}$-preaisle of $\mathscr{T}$ is equivalent to saying it is a $\otimes$-preaisle. However, compactly generated copreaisles are not closed under products in general, so Remark 14 is not applicable here.

Example 17.[1] Consider the derived category $\mathbf{D}(\mathbb{Z})$, it is equivalent to the homotopy category of K-projectives or dg-projectives $\mathbf{K}(dg\text{-proj}\,\mathbb{Z})$. The copreaisle $\mathbf{K}^{\geq 0}(dg\text{-proj}\,\mathbb{Z})$ is compactly generated but it is not closed under products. Indeed, if $\mathbf{K}^{\geq 0}(dg\text{-proj}\,\mathbb{Z})$ is closed under product then the countable product of $\mathbb{Z}$, that is, $\prod_i \mathbb{Z} \in \mathbf{K}^{\geq 0}(dg\text{-proj}\,\mathbb{Z})$. Let $P^{\cdot}$ be a projective (hence K-projective) resolution of $\prod_i \mathbb{Z}$. Now, take the brutal truncations of $P^{\cdot}$ at degree 0, $\sigma^{\geq 0}(P^{\cdot})$ and $\sigma^{<0}(P^{\cdot})$. As $\sigma^{<0}(P^{\cdot}) \in \mathbf{K}^{<0}(dg\text{-proj}\,\mathbb{Z})$, we have $\mathrm{Hom}(\prod_i \mathbb{Z}, \sigma^{<0}(P^{\cdot})) = 0$ which implies $\sigma^{\geq 0}(P^{\cdot})$ is isomorphic to $\prod_i \mathbb{Z}$. Since $\sigma^{\geq 0}(P^{\cdot})$ is projective we get $\prod_i \mathbb{Z}$ is projective, which is a contradiction; see for instance [6, Corollary to Theorem 3.1, p. 466].

Proof of Theorem 15. Part (i). It has already been shown in [12, Theorem 4.5 (i)] that the above assignments are bijections between the set of compactly generated t-structures of $\mathscr{T}$ and the set of thick preaisles of $\mathscr{T}^{c}$. We only need to show that the assignments preserve the tensor conditions.

From the definition of $\otimes$-preaisle, it is easy to observe that if $\mathscr{U}$ is a $\otimes$-preaisle of $\mathscr{T}$ then $\mathscr{U} \cap \mathscr{T}^{c}$ is a $\otimes$-preaisle of $\mathscr{T}^{c}$. Suppose $\mathscr{P}$ is a $\otimes$-preaisle of $\mathscr{T}^{c}$. Since ${}^{\perp}(\mathscr{P}^{\perp})$ is a cocomplete preaisle of $\mathscr{T}$ by Remark 14 it is enough to show ${}^{\perp}(\mathscr{P}^{\perp})$ is a $\otimes^{c}$-preaisle of $\mathscr{T}$. Let $\mathscr{B} = \{X \in {}^{\perp}(\mathscr{P}^{\perp}) \mid (\mathscr{T}^{c} \cap \mathscr{T}^{\leq 0}) \otimes X \subset {}^{\perp}(\mathscr{P}^{\perp})\}$. We note that $\mathscr{B}$ is a cocomplete preaisle containing $\mathscr{P}$. Since ${}^{\perp}(\mathscr{P}^{\perp})$ is the smallest cocomplete preaisle containing $\mathscr{P}$ by [7, Lemma 1.9] we get $\mathscr{B} = {}^{\perp}(\mathscr{P}^{\perp})$.

Part (ii). In view of [12, Theorem 4.5 (ii)], again we only need to show that the assignments preserve the appropriate tensor conditions. If $\mathscr{X}$ is a $\otimes^{c}$-copreaisle of $\mathscr{T}$ then it is easy to observe that $\mathscr{X} \cap \mathscr{T}^{c}$ is a $\otimes$-copreaisle of $\mathscr{T}^{c}$. Suppose $\mathscr{P}$ is a $\otimes$-copreaisle of $\mathscr{T}^{c}$ we need to show that ${}^{\perp}(\mathscr{P}^{\perp})$ is a $\otimes^{c}$-copreaisle of $\mathscr{T}$. By [12, Theorem 3.7] an object A of $\mathscr{T}$ belongs to ${}^{\perp}(\mathscr{P}^{\perp})$ if and only if A is a summand of a homotopy colimit of a sequence.

$$0 = Y_0 \xrightarrow{f_0} Y_1 \xrightarrow{f_1} Y_2 \xrightarrow{f_2} \cdots$$

where each f_i occurs in a triangle $Y_i \to Y_{i+1} \to S_i \to \Sigma Y_i$ with $S_i \in \mathrm{Add}\mathscr{P}$. First, we observe that for any compact object T the functor $\mathscr{H}om(T,-)$ preserves small coproducts therefore $\mathscr{H}om(T,-)$ takes homotopy sequences to homotopy sequences. Since $\mathscr{P}$ is $\otimes$-copreaisle of $\mathscr{T}^{c}$ for any $T \in \mathscr{T}^{c} \cap \mathscr{T}^{\leq 0}$ we have $\mathscr{H}om(T, S_i) \in \mathrm{Add}\mathscr{P}$. Thus applying [12, Theorem 3.7] again we get $\mathscr{H}om(T, A) \in {}^{\perp}(\mathscr{P}^{\perp})$. □

4. The classification theorem for weight structures

Let X be a Noetherian separated scheme. $\mathbf{D}(\mathrm{Qcoh}\,X)$ denotes the derived category of complexes of quasi coherent $\mathscr{O}_X$-modules. The derived category $(\mathbf{D}(\mathrm{Qcoh}\,X), \otimes^{L}_{\mathscr{O}_X}, \mathscr{O}_X)$ is a tensor triangulated category with the derived tensor product $\otimes^{L}_{\mathscr{O}_X}$ and the structure sheaf $\mathscr{O}_X$ as the unit. The full subcategory of complexes whose cohomologies vanish in positive degree $\mathbf{D}^{\leq 0}(\mathrm{Qcoh}\,X)$ is a preaisle of $\mathbf{D}(\mathrm{Qcoh}\,X)$ satisfying the conditions of Definition 7. We define the $\otimes$-preaisles and $\otimes$-copreaisles of $\mathbf{D}(\mathrm{Qcoh}\,X)$ with respect to $\mathbf{D}^{\leq 0}(\mathrm{Qcoh}\,X)$. Similarly, the $\otimes$-preaisles and $\otimes$-copreaisles of $\mathrm{Perf}(X)$ are defined with respect to $\mathrm{Perf}^{\leq 0}(X)$. Note that $\mathbf{D}(\mathrm{Qcoh}\,X)$ has the property $(*)$ (see Definition 12), so we can define $\otimes^{c}$-(co)preaisles of $\mathbf{D}(\mathrm{Qcoh}\,X)$.

[1] We thank the referee for suggesting this example.

Definition 18. *A subset Z is a* specialization closed *subset of X if for each $x \in Z$ the closure of the singleton set $\{x\}$ is contained in Z, that is, $\bar{\{x\}} \subset Z$. Note that a specialization closed subset is a union of closed subsets of X.*

A subset Y is a Thomason *subset of X if $Y = \bigcup_\alpha Y_\alpha$ is a union of closed subsets Y_α such that $X \setminus Y_\alpha$ is quasi compact. Note that if X is Noetherian then the two notions coincide.*

Definition 19. *A* Thomason filtration *of X is a map $\phi : \mathbb{Z} \to 2^X$ such that $\phi(i)$ is a Thomason subset of X and $\phi(i) \supset \phi(i+1)$ for all $i \in \mathbb{Z}$.*

In our earlier work we have mentioned without proof (see [7, Remark 4.13]) about the following result, here we explicitly state it for future reference. This is a generalization of Thomason's classification [13, Theorem 3.15] of $\otimes$-ideals to $\otimes$-preaisles of $\mathrm{Perf}(X)$, for separated Noetherian scheme X.

Proposition 20. *Let X be a separated Noetherian scheme. The assignment sending a Thomason filtration ϕ to $\mathscr{S}_\phi = \{E \in \mathrm{Perf}(X) \mid \mathrm{Supp}(H^i(E)) \subset \phi(i)\}$ provides a one-to-one correspondence between the following sets:*

(i) *the set of Thomason filtrations of X;*
(ii) *the set of thick $\otimes$-preaisles of* $\mathrm{Perf}(X)$.

Proof. In [7, Theorem 4.11] we have shown that sending ϕ to

$$\mathscr{U}_\phi = \{E \in \mathbf{D}(\mathrm{Qcoh}\, X) \mid \mathrm{Supp}(H^i(E)) \subset \phi(i)\}$$

provides a bijection between the set of Thomason filtration of X and the set of compactly generated tensor t-structure of $\mathbf{D}(\mathrm{Qcoh}\, X)$. From part (i) of Theorem 15, we conclude that the above assignment provides a bijection between Thomason filtrations of X and thick $\otimes$-preaisles of $\mathrm{Perf}(X)$. □

Theorem 21. *Let X be a separated Noetherian scheme. There is a one-to-one correspondence between the following sets:*

(i) *the set of Thomason filtrations of X;*
(ii) *the set of compactly generated $\otimes^c$-weight structures of* $\mathbf{D}(\mathrm{Qcoh}\, X)$.

The assignment is given by

$$\phi \mapsto (\mathscr{A}_\phi, \mathscr{B}_\phi)$$

where

$$\begin{aligned}
\mathscr{B}_\phi &= \big\{B \in \mathbf{D}(\mathrm{Qcoh}\, X) \,\big|\, \mathrm{Hom}(\mathscr{O}_X, S \otimes^L_{\mathscr{O}_X} B) = 0 \textit{ for all } S \in \mathscr{S}_\phi\big\},\\
\mathscr{S}_\phi &= \big\{S \in \mathrm{Perf}(X) \,\big|\, \mathrm{Supp}(H^i S) \subset \phi(i)\big\}, \textit{ and}\\
\mathscr{A}_\phi &= \big\{A \in \mathbf{D}(\mathrm{Qcoh}\, X) \,\big|\, \mathrm{Hom}(A, B) = 0 \textit{ for all } B \in \mathscr{B}_\phi\big\}.
\end{aligned}$$

Proof. Let ϕ be a Thomason filtration of X. By Proposition 20 we know $\phi \mapsto \mathscr{S}_\phi$ is a bijection. Now sending $\mathscr{S}_\phi$ to $\mathscr{S}_\phi^*$ is again a bijection by Lemma 10. Since $\mathscr{S}_\phi^*$ is a $\otimes$-copreaisle of $\mathrm{Perf}(X)$, the assignment $\mathscr{S}_\phi^* \mapsto ({}^\perp((\mathscr{S}_\phi^*)^\perp), (\mathscr{S}_\phi^*)^\perp)$ is a bijection by Theorem 15. We only need to show that $\mathscr{B}_\phi = (\mathscr{S}_\phi^*)^\perp$ which is the consequence of the tensor-hom adjunction. □

5. In the case of projective line

In this section, we specialize to the case of projective line $\mathbb{P}^1_k$ over a field k. By the results of earlier sections, classifying compactly generated tensor t-structures of $\mathbf{D}(\mathrm{Qcoh}\, \mathbb{P}^1_k)$ is equivalent to classifying thick $\otimes$-preaisles of $\mathrm{Perf}(\mathbb{P}^1_k)$. For any smooth Noetherian scheme X the inclusion functor from $\mathrm{Perf}(X)$ to the derived category of bounded complexes of coherent sheaves $\mathbf{D}^b(\mathrm{Coh}\, X)$ is an equivalence. Therefore, we restrict our attention to $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. Note that we define $\otimes$-preaisles of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ with respect to the standard preaisle

$$\mathbf{D}^{b,\leq 0}(\mathrm{Coh}\,\mathbb{P}^1_k) := \left\{E \in \mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k) \,\middle|\, H^i(E) = 0\ \forall\ i > 0\right\}.$$

Lemma 22. *A thick preaisle $\mathscr{A}$ of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ is a $\otimes$-preaisle if and only if*

$$\mathscr{O}(-1) \otimes \mathscr{A} \subset \mathscr{A}.$$

Proof. Suppose $\mathscr{A}$ is a $\otimes$-preaisle then $\mathscr{O}(-1) \otimes \mathscr{A} \subset \mathscr{A}$ is true by definition. Conversely, suppose $\mathscr{A}$ is a preaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. Take $\mathscr{B} := \{B \in \mathbf{D}^{b,\leq 0}(\mathrm{Coh}\,\mathbb{P}^1_k) \mid B \otimes \mathscr{A} \subset \mathscr{A}\}$. From our assumption, we have $\mathscr{O}(-1) \in \mathscr{B}$. It is now easy to see that for every $n \geq 0$ we have $\mathscr{O}(-n) \in \mathscr{B}$.

As $\mathrm{Coh}\,\mathbb{P}^1_k$ has homological dimension one, every complex of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ is quasi isomorphic to the direct sum of its cohomology sheaves, see [8, Proposition 6.1]. Also, every coherent sheaf over $\mathbb{P}^1_k$ is the direct sum of line bundles and torsion sheaves. Since $\mathscr{B}$ is a preaisle, to show $\mathscr{B} = \mathbf{D}^{b,\leq 0}(\mathrm{Coh}\,\mathbb{P}^1_k)$ it is enough to show that $\mathscr{B}$ contains all the line bundles and torsion sheaves.

For any $m \geq 0$ consider the following triangle coming from the corresponding short exact sequence in $\mathrm{Coh}\,\mathbb{P}^1_k$; see for instance [8, Equation 6.3],

$$\mathscr{O}(-2)^{\oplus(m+1)} \longrightarrow \mathscr{O}(-1)^{\oplus(m+2)} \longrightarrow \mathscr{O}(m) \longrightarrow \mathscr{O}(-2)[1].$$

Since $\mathscr{B}$ is closed under extensions and positive shifts we have $\mathscr{O}(m) \in \mathscr{B}$.

Next, for any indecomposable torsion sheaf of degree d say T_x, which is supported on a closed point $x \in \mathbb{P}^1_k$, consider the following triangle coming from the corresponding short exact sequence in $\mathrm{Coh}\,\mathbb{P}^1_k$; see [8, Equation 6.5],

$$\mathscr{O}(-2)^{\oplus d} \longrightarrow \mathscr{O}(-1)^{\oplus d} \longrightarrow T_x \longrightarrow \mathscr{O}(-2)[1].$$

Again using the fact that $\mathscr{B}$ is closed under extensions and positive shifts we have $T_x \in \mathscr{B}$. $\square$

Recall that for a set of objects $\mathscr{S}$ of $\mathscr{T}$ we denote the smallest cocomplete preaisle containing $\mathscr{S}$ by $\langle\mathscr{S}\rangle^{\leq 0}$. If $\mathscr{T}$ does not have coproducts, for instance $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$, we denote $\langle\mathscr{S}\rangle^{\leq 0}$ to be the smallest preaisle containing $\mathscr{S}$. Similarly we denote $\langle\mathscr{S}\rangle^{\geq 0}$ to be the smallest copreaisle containing $\mathscr{S}$. Also recall that for any subcategory $\mathscr{U}$ we denote $\mathscr{U}^*$ the full subcategory given by

$$\mathscr{U}^* = \left\{X \in \mathscr{T} \,\middle|\, X \cong \mathscr{H}\mathrm{om}(Y, \mathbf{1}) \text{ for some } Y \in \mathscr{U}\right\}.$$

Example 23. For a fixed $n \in \mathbb{Z}$ we denote

$$\mathscr{B}_n := \langle\mathscr{O}(n)\rangle^{\leq 0}; \text{ and}$$
$$\mathscr{C}_n := \langle\mathscr{O}(n), \mathscr{O}(n+1)\rangle^{\leq 0}.$$

Using Lemma 22, we can check that $\mathscr{B}_n$ and $\mathscr{C}_n$ are not $\otimes$-preaisles of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. Similarly, $\mathscr{B}^*_n$ and $\mathscr{C}^*_n$ provide examples of copreaisles of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ which are not $\otimes$-copreaisles. This can be observed using Lemma 10.

Recall that a Thomason filtration of X is a map $\phi : \mathbb{Z} \to 2^X$ such that $\phi(i)$ is a Thomason subset of X and $\phi(i) \supset \phi(i+1)$ for all $i \in \mathbb{Z}$. We say ϕ is of *type-1* if $\bigcup_i \phi(i) \neq X$; and we say ϕ is of *type-2* if $\bigcup_i \phi(i) = X$ but not all $\phi(i) = X$.

Let $x \in \mathbb{P}^1_k$ be a closed point. We denote the simple torsion sheaf supported on x by $k(x)$. Now, we give an explicit description of the $\otimes$-preaisles of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ in terms of simple torsion sheaves and line bundles.

Proposition 24. *Any proper thick* $\otimes$*-preaisle of* $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ *is one of the following forms:*

(i) $\langle k(x)[-i] \mid x \in \phi(i)\rangle^{\leq 0}$;
where ϕ *is a type-1 Thomason filtration of* $\mathbb{P}^1_k$.

(ii) $\langle \mathscr{O}(n)[-i_0], k(x)[-i] \mid \forall\ n \in \mathbb{Z}$ *and* $x \in \phi(i)\rangle^{\leq 0}$
where ϕ *is a type-2 Thomason filtration of* $\mathbb{P}^1_k$ *and* i_0 *a fixed integer.*

Proof. Suppose $\mathscr{A}$ is a thick $\otimes$-preaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. By Proposition 20 there is a unique Thomason filtration ϕ such that

$$\mathscr{A} = \left\{E \in \mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k) \,\middle|\, \mathrm{Supp}\, H^i(E) \subset \phi(i)\right\}.$$

Since $\mathrm{Coh}\,\mathbb{P}^1_k$ has homological dimension one, every complex of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ is quasi isomorphic to the direct sum of its cohomology sheaves. Therefore, we can write $\mathscr{A}$ in terms of coherent sheaves alone,

$$\mathscr{A} = \left\langle F[-i] \,\middle|\, F \in \mathrm{Coh}\,\mathbb{P}^1_k \text{ and } \mathrm{Supp}\, F \subset \phi(i)\right\rangle^{\leq 0}.$$

Case 1. ϕ **is type-1.** Note that $\phi(i) \subsetneq \mathbb{P}^1_k$ for all i. Every coherent sheaf over $\mathbb{P}^1_k$ is the direct sum of line bundles and torsion sheaves. Since the support of any line bundle is whole $\mathbb{P}^1_k$. In this case, $\mathscr{A}$ contains only torsion sheaves. As torsion sheaves can be generated by simple torsion sheaves we have,

$$\mathscr{A} = \langle k(x)[-i] \mid x \in \phi(i)\rangle^{\leq 0}.$$

Case 2. ϕ **is type-2.** Since $\bigcup_i \phi(i) = X$ there is an integer i_0 such that $\phi(i_0)$ contains the generic point of $\mathbb{P}^1_k$. We can take i_0 to be the largest such integer. Observe that $\phi(i) = \mathbb{P}^1_k$ for all $i \leq i_0$ and $\phi(i_0+1) \subsetneq \mathbb{P}^1_k$. Here we can check that

$$\mathscr{A} = \langle \mathscr{O}(n)[-i_0], k(x)[-i] \mid \forall\ n \in \mathbb{Z} \text{ and } x \in \phi(i)\rangle^{\leq 0}. \qquad \square$$

Next, we will show which of these $\otimes$-preaisles of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ are t-structures on $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. First, we prove a few lemmas.

Lemma 25. *Let* E *be a complex in* $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ *such that* $H^{-1}(E)$ *is a torsion sheaf. Let* $\mathscr{L}$ *be a line bundle over* $\mathbb{P}^1_k$ *and* $\delta : E \to \mathscr{L}[1]$ *be a map in* $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. *Then the following statements hold:*

(i) *If* $T \in \mathrm{Coh}\,\mathbb{P}^1_k$ *is a torsion sheaf then* $\mathrm{Hom}(T, \mathrm{cone}(\delta)) \neq 0$.

(ii) *If* $\mathscr{A}$ *is a subcategory of* $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ *containing all line bundles, then* $\mathrm{cone}(\delta) \notin \mathscr{A}^{\perp}$.

Proof. Let $\delta : E \to \mathscr{L}[1]$ be a map. The abelian category $\mathrm{Coh}\,\mathbb{P}^1_k$ is hereditary, hence E is quasi-isomorphic to $\oplus_i H^i(E)[-i]$, see [8, Proposition 6.1]. Therefore, $\mathrm{Hom}(E, \mathscr{L}[1]) = \oplus_i \mathrm{Hom}(H^i(E)[-i], \mathscr{L}[1]) = \oplus_i \mathrm{Ext}^{i+1}(H^i(E), \mathscr{L})$. Again since $\mathrm{Ext}^n(-,-)$ groups vanish for all $n > 1$ as well as for $n < 0$, it is enough to consider the cone of the map $H^{-1}(E)[1] \oplus H^0(E) \to \mathscr{L}[1]$; other factors being mapped to zero. By assumption the factor $H^{-1}(E)$ is a torsion sheaf, hence the map $H^{-1}(E)[1] \to \mathscr{L}[1]$, from a torsion sheaf to torsion-free is zero.

Let us denote the map $H^0(E) \to \mathscr{L}[1]$ by δ and $H^0(E)$ by A. As we know $\mathrm{Hom}(A, \mathscr{L}[1]) \cong \mathrm{Ext}^1(A, \mathscr{L})$, a map $\delta : A \to \mathscr{L}[1]$ corresponds to an element of the group $\mathrm{Ext}^1(A, \mathscr{L})$. By abuse of notation, we denote the corresponding element in $\mathrm{Ext}^1(A, \mathscr{L})$ by δ.

Now we take the short exact sequence corresponding to $\delta \in \mathrm{Ext}^1(A, \mathscr{L})$, say

$$0 \longrightarrow \mathscr{L} \longrightarrow B \longrightarrow A \longrightarrow 0.$$

Since $\mathscr{L}$ injects into B, and every coherent sheaf over $\mathbb{P}^1_k$ is direct sum of its torsion and torsion-free part, B has a torsion-free summand, in particular has a line bundle as a summand, say $\mathscr{M}$. And the short exact sequence gives rise to a distinguished triangle

$$\mathscr{L} \longrightarrow B \longrightarrow A \xrightarrow{\delta} \mathscr{L}[1].$$

Therefore, cone(δ) being isomorphic to $B[1]$ has a summand isomorphic to $\mathscr{M}[1]$. Now claim (i) follows, since $\mathrm{Ext}^1(T,\mathscr{M}) \neq 0$ is factor of $\mathrm{Hom}(T,\mathrm{cone}(\delta))$. And claim (ii) follows, since by assumption $\mathscr{A}$ contains all the line bundles, in particular, contains twists of $\mathscr{M}$ and $\mathrm{Ext}^1(\mathscr{M}(2),\mathscr{M}) \neq 0$ implies $\mathrm{Hom}(\mathscr{M}(2),\mathrm{cone}(\delta)) \neq 0$. □

Recall that a preaisle $\mathscr{A}$ is an aisle if $(\mathscr{A},\mathscr{A}^{\perp}[1])$ is a t-structure.

Lemma 26. *Let $\mathscr{A}$ be a $\otimes$-preaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ and ϕ be its corresponding Thomason filtration. If $\mathscr{A}$ is an aisle and $\phi(i) \neq \emptyset$ for some i, then $\phi(i-1) = \mathbb{P}^1_k$.*

Proof. Without loss of generality, we may assume $i = 0$. If $\phi(0) = \mathbb{P}^1_k$ then $\phi(-1) = \mathbb{P}^1_k$ and there is nothing to prove. Let $\mathscr{L}$ be a line bundle on $\mathrm{Coh}\,\mathbb{P}^1_k$. If $\phi(-1) \neq \mathbb{P}^1_k$ then $\mathscr{L}[1] \notin \mathscr{A}$. Since $\phi(0) \neq \emptyset$ there is a closed point $x \in \phi(0)$ and $k(x) \in \mathscr{A}$. As $\mathrm{Hom}(k(x),\mathscr{L}[1]) = \mathrm{Ext}^1(k(x),\mathscr{L}) \neq 0$ we also have $\mathscr{L}[1] \notin \mathscr{A}^{\perp}$. We must have a t-decomposition of $\mathscr{L}[1]$ as $\mathscr{A}$ is given to be an aisle.

Let $E \in \mathscr{A}$ and $\delta : E \to \mathscr{L}[1]$ be a map. As $\phi(-1) \neq \mathbb{P}^1_k$, the cohomology sheaf $H^{-1}(E)$ is a torsion sheaf, so by Lemma 25(i), $\mathrm{Hom}(k(x),\mathrm{cone}(\delta)) \neq 0$ which implies $\mathrm{cone}(\delta) \notin \mathscr{A}^{\perp}$. This shows that there is no distinguished triangle

$$E \longrightarrow \mathscr{L}[1] \longrightarrow F \longrightarrow E[1]$$

such that $E \in \mathscr{A}$ and $F \in \mathscr{A}^{\perp}$. This contradicts the fact that $\mathscr{A}$ is an aisle. □

Definition 27. *We say ϕ is a* one-step *Thomason filtration of $\mathbb{P}^1_k$ if there is an integer i_0 and a Thomson subset Z_{i_0} such that*

$$\phi(j) = \begin{cases} \mathbb{P}^1_k & \text{if } j < i_0; \\ Z_{i_0} & \text{if } j = i_0; \\ \emptyset & \text{if } j > i_0. \end{cases}$$

Proposition 28. *Let $\mathscr{A}$ be a $\otimes$-preaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. Then, $\mathscr{A}$ is an aisle if and only if the corresponding Thomason filtration is a one-step filtration.*

Proof. If $\mathscr{A}$ is a $\otimes$-preaisle which is also an aisle then by Lemma 26 the corresponding filtration is a one-step filtration.Conversely, suppose the filtration is one step, we will show that every complex of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ can be decomposed into a triangle where the first term is in $\mathscr{A}$ and the third term is in $\mathscr{A}^{\perp}$. Without loss of generality we may assume the one step occurs at $i_0 = 0$, and $\phi(0) = Z_0$ is a Thomason subset.

If $Z_0 = \mathbb{P}^1_k$, then $\mathscr{A} = \mathbf{D}^{b,\leq 0}(\mathrm{Coh}\,\mathbb{P}^1_k)$ and we get the standard t-structure. Now, suppose $Z_0 \neq \mathbb{P}^1_k$. Since the filtration is one step we only need to show sheaves at degree zero have t-decompositions, all other shifted sheaves have obvious t-decompositions. The functor $\Gamma_{Z_0}(-)$ gives a t-decomposition of sheaves at degree zero. □

Next, we give an explicit description of $\otimes$-copreaisles of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ in terms of simple torsion sheaves and line bundles.

Proposition 29. *Any proper thick $\otimes$-copreaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ is one of the following forms:*

(i) $\langle k(x)^*[i] \mid x \in \phi(i)\rangle^{\geq 0}$;
where ϕ is a type-1 Thomason filtration of $\mathbb{P}^1_k$.

(ii) $\langle \mathscr{O}(n)[i_0], k(x)^*[i] \mid \forall\, n \in \mathbb{Z} \text{ and } x \in \phi(i)\rangle^{\geq 0}$
where ϕ is a type-2 Thomason filtration of $\mathbb{P}^1_k$ and i_0 a fixed integer.

Proof. By the proof of Lemma 10, we know that every $\otimes$-copreaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ is of the form $\mathscr{A}^*$ where $\mathscr{A}$ is a $\otimes$-preaisle. Now using the description given in Proposition 24 we conclude our result. □

The trivial $\otimes$-copreaisles $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ and 0 give rise to tensor weight structures on $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. In contrast to the case of t-structures (see Proposition 28), the next result shows that, there are no other tensor weight structures on $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$.

Proposition 30. *The trivial weight structures are the only tensor weight structures on* $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$.

Proof. Suppose $\mathscr{A}$ is a non-zero $\otimes$-copreaisle of $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$ which induces a weight structure. We first claim that $\mathscr{A}$ can not be a copreaisle containing only torsion sheaves. Indeed, since $\mathscr{A}$ is non-zero it contains some indecomposable torsion sheaf (up to shift), say $T_x[i]$. Without loss of generality we can assume $i = 0$, that is, $T_x \in \mathscr{A}$. By our assumption $\mathscr{A}$ contains only torsion sheaves hence $\mathscr{L}[1] \notin \mathscr{A}$, and since $\mathrm{Ext}^1(T_x, \mathscr{L}) \neq 0$, we also have $\mathscr{L}[1] \notin \mathscr{A}^{\perp}$. Suppose $\delta : E \to \mathscr{L}[1]$ is a map with $E \in \mathscr{A}$, then by Lemma 25 (i) $\mathrm{Hom}(T_x, \mathrm{cone}(\delta)) \neq 0$ which implies $\mathrm{cone}(\delta) \notin \mathscr{A}^{\perp}$. This shows $\mathscr{L}[1]$ does not have a weight decomposition. Hence, $\mathscr{A}$ can not be a copreaisle containing only torsion sheaves.

If $\mathscr{L}[i]$ is in $\mathscr{A}$ for some $\mathscr{L}$, then for any line bundle $\mathscr{M}$, the tensor property of $\mathscr{A}$ implies $\mathscr{M}[i] = \mathscr{H}om(\widecheck{\mathscr{M}} \otimes \mathscr{L}, \mathscr{L}[i])$ is in $\mathscr{A}$. Now, there are two cases:

(1) either there is an integer i such that for any line bundle $\mathscr{L}$, $\mathscr{L}[i] \in \mathscr{A}$ and $\mathscr{L}[i+1] \notin \mathscr{A}$, or

(2) there is no such i, and $\mathscr{A}$ contains all line bundles and their shifts.

Case (1). Suppose there is an integer i, then without loss of generality we can assume $i = 0$. By our assumption, for any line bundle $\mathscr{L}$ we have $\mathscr{L}[1] \notin \mathscr{A}$. Since $\mathscr{A}$ contains all the line bundles, in particular, contains twists of $\mathscr{L}$ and $\mathrm{Ext}^1(\mathscr{L}(2), \mathscr{L}) \neq 0$ implies $\mathrm{Hom}(\mathscr{L}(2), \mathscr{L}[1]) \neq 0$, so $\mathscr{L}[1] \notin \mathscr{A}^{\perp}$. Again by our assumption on $\mathscr{A}$, for any $E \in \mathscr{A}$ the cohomology sheaf $H^{-1}(E)$ is a torsion sheaf. For any map $\delta : E \to \mathscr{L}[1]$ with $E \in \mathscr{A}$, Lemma 25 (ii) implies $\mathrm{cone}(\delta) \notin \mathscr{A}^{\perp}$. This shows that $\mathscr{L}[1]$ does not have a weight decomposition. This is a contradiction.

Case (2). Now, suppose $\mathscr{A}$ contains all line bundles and their shifts. In particular, it contains $\mathscr{O}(-1)$, $\mathscr{O}(-2)$ and all their shifts. Now, for any indecomposable torsion sheaf of degree d say T_x supported on a closed point $x \in \mathbb{P}^1_k$, consider the following distinguished triangle coming from the corresponding short exact sequence in $\mathrm{Coh}\,\mathbb{P}^1_k$,

$$\mathscr{O}(-2)^{\oplus d} \longrightarrow \mathscr{O}(-1)^{\oplus d} \longrightarrow T_x \longrightarrow \mathscr{O}(-2)[1].$$

As $\mathscr{A}$ is closed under extensions, T_x is in $\mathscr{A}$. This proves $\mathscr{A}$ contains all the torsion sheaves and their shifts. Therefore, $\mathscr{A}$ must be equal to $\mathbf{D}^b(\mathrm{Coh}\,\mathbb{P}^1_k)$. □

Remark 31. The above proposition says when $X = \mathbb{P}^1_k$, the bounded derived category $\mathbf{D}^b(\mathrm{Coh}\,X)$ has no non-trivial tensor weight structure. However, when $X = \mathrm{Spec}\,R$ for a regular Noetherian ring R, the derived category $\mathbf{D}^b(\mathrm{Coh}\,X)$ has tensor weight structures. In this case, $\mathbf{D}^b(\text{mod-}R)$ is equivalent to the homotopy category $\mathbf{K}^b(\text{proj-}R)$, and the brutal truncations provide non-trivial tensor weight structures.

Acknowledgement

The authors are grateful for the excellent work environment and the assistance of the support staff of HRI, Prayagraj. The second author acknowledges the financial aid provided by the INFOSYS scholarship. We thank the referee for the valuable suggestions and comments which greatly improved the exposition.

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