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More on lines in Euclidean Ramsey theory

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Abstract. Let ℓ_m be a sequence of m points on a line with consecutive points at distance one. Answering a question raised by Fox and the first author and independently by Arman and Tsaturian, we show that there is a natural number m and a red/blue-colouring of $\mathbb{E}^n$ for every n that contains no red copy of ℓ_3 and no blue copy of ℓ_m.

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1. Introduction

Let $\mathbb{E}^n$ denote n-dimensional Euclidean space, that is, $\mathbb{R}^n$ equipped with the Euclidean metric. Given two sets $X_1, X_2 \subset \mathbb{E}^n$, we write $\mathbb{E}^n \to (X_1, X_2)$ if every red/blue-coloring of $\mathbb{E}^n$ contains either a red copy of X_1 or a blue copy of X_2, where a copy for us will always mean an isometric copy. Conversely, $\mathbb{E}^n \not\to (X_1, X_2)$ means that there is some red/blue-coloring of $\mathbb{E}^n$ which contains neither a red copy of X_1 nor a blue copy of X_2.

The study of which sets $X_1, X_2 \subset \mathbb{E}^n$ satisfy $\mathbb{E}^n \to (X_1, X_2)$ is a particular case of the Euclidean Ramsey problem, which has a long history going back to a series of seminal papers [6–8] of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus in the 1970s. Despite the vintage of the problem, surprisingly little progress has been made since these foundational papers (though see [9, 12] for some important positive results). For instance, it is an open problem, going back to the papers of Erdős et al. [7], as to whether, for every n, there is m such that $\mathbb{E}^n \not\to (X, X)$ for every $X \subset \mathbb{E}^n$ with $|X| = m$.

Write ℓ_m for the set consisting of m points on a line with consecutive points at distance one. Perhaps because it is a little more accessible than the general problem, the question of determining which n and X satisfy the relation $\mathbb{E}^n \to (\ell_2, X)$ has received considerable attention. For instance, it is known [11, 14] that $\mathbb{E}^2 \to (\ell_2, X)$ for every four-point set $X \subset \mathbb{E}^2$ and that $\mathbb{E}^2 \to (\ell_2, \ell_5)$. On the other hand [5], there is a set X of 8 points in the plane, namely, a regular heptagon with its center, such that $\mathbb{E}^2 \not\to (\ell_2, X)$.

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In higher dimensions, by combining results of Szlam [13] and Frankl and Wilson [10], it was observed by Fox and the first author [4] that $\mathbb{E}^n \to (\ell_2, \ell_m)$ provided $m \le 2^{cn}$ for some positive constant c (see also [1, 2] for some better bounds in low dimensions). Our concern here will be with a question raised independently by Fox and the first author [4] and also by Arman and Tsaturian [1], namely, as to whether an analogous result holds with ℓ_2 replaced by ℓ_3. That is, for every natural number m, is there a natural number n such that $\mathbb{E}^n \to (\ell_3, \ell_m)$? We answer this question in the negative.

Theorem 1. *There exists a natural number m such that $\mathbb{E}^n \nrightarrow (\ell_3, \ell_m)$ for all n.*

Before our work, the best result that was known in this direction was a 50-year-old result of Erdős et al. [6], who showed that $\mathbb{E}^n \nrightarrow (\ell_6, \ell_6)$ for all n. Their proof uses a spherical colouring, where all points at the same distance from the origin receive the same colour. We will also use a spherical colouring, though, unlike the colouring in [6], which is entirely explicit, our colouring will be partly random.

2. Preliminaries

In this short section, we note two key lemmas that will be needed in our proof. The first says that certain real-valued quadratic polynomials are reasonably well-distributed modulo a prime q.

Lemma 2. *Let $p(x) = x^2 + \alpha x + \beta$, where α and β are real numbers, and let q be a prime number. Then, for $m = q^3$, the set $\{p(i)\}_{i=1}^m$ overlaps with at least $q/6$ of the intervals $[j, j+1)$ with $0 \le j \le q-1$ when considered mod q.*

Proof. By a standard argument using the pigeonhole principle, there exists some $k \le q^2$ such that $|k\alpha| \le 1/q$ mod q. We split into two cases, depending on whether k is a multiple of q or not.

Suppose first that $k \not\equiv 0$ mod q and consider the set of values $\{p(ki)\}_{i=1}^q$. Note first that $\{i^2\}_{i=1}^q$ is a set of $(q+1)/2$ distinct integers mod q, so, since k is not a multiple of q, the same is also true of the set $\{k^2 i^2\}_{i=1}^q$. Hence, letting $p_1(x) = x^2 + \beta$, we see that the set $\{p_1(ki)\}_{i=1}^q$ overlaps with at least $q/2$ of the intervals $[j, j+1)$ with $0 \le j \le q-1$ when considered mod q. But $|ki\alpha| \le 1$ mod q for all $1 \le i \le q$, so that $|p(ki) - p_1(ki)| \le 1$ for all $1 \le i \le q$. Therefore, since exactly three different intervals are within distance one of any particular interval, the set $\{p(ki)\}_{i=1}^q$ overlaps with at least $q/6$ of the intervals $[j, j+1)$ mod q.

Suppose now that $k = sq$ for some $s \le q$. Then $sq\alpha = rq + \epsilon$ for some $|\epsilon| \le 1/q$, which implies that $\alpha = \frac{r}{s} + \epsilon'$, where $|\epsilon'| \le 1/q^2$. Without loss of generality, we may assume that r and s have no common factors. Consider now the polynomial $p_2(x) = x^2 + \frac{r}{s}x$ and the set $\{p_2(si)\}_{i=1}^q$. Since $p_2(si) = s^2 i^2 + ri$, it is easy to check that $p_2(si) \equiv p_2(sj)$ mod q if and only if $s^2(i+j) + r \equiv 0$ mod q. Since r and s are coprime, this implies that the set $\{p_2(si)\}_{i=1}^q$ takes at least $q/2$ values mod q. Hence, letting $p_3(x) = x^2 + \frac{r}{s}x + \beta$, we see that the set $\{p_3(si)\}_{i=1}^q$ overlaps with at least $q/2$ of the intervals $[j, j+1)$ with $0 \le j \le q-1$ when considered mod q. But, since $|\alpha - r/s| \le 1/q^2$, we have that $|p(si) - p_3(si)| = |\alpha - \frac{r}{s}|si \le 1$, so that, as above, the set $\{p(si)\}_{i=1}^q$ overlaps with at least $q/6$ of the intervals $[j, j+1)$ mod q. □

Given M real polynomials $p_1, \dots, p_M$ in N variables, a vector $\sigma \in \{-1, 0, 1\}^M$ is called a sign pattern of $p_1, \dots, p_M$ if there exists some $x \in \mathbb{R}^N$ such that the sign of $p_i(x)$ is σ_i for all $1 \le i \le M$. The second result we need is the Oleinik–Petrovsky–Thom–Milnor theorem (see, for example, [3]), which, for N fixed, gives a polynomial bound for the number of sign patterns.

Lemma 3. *For $M \ge N \ge 2$, the number of sign patterns of M real polynomials in N variables, each of degree at most D, is at most $\left(\frac{50DM}{N}\right)^N$.*

3. Proof of Theorem 1

Suppose that $a_1, a_2, a_3 \in \mathbb{R}^n$ form a copy of ℓ_3 with $|a_1 - a_2| = |a_2 - a_3| = 1$. If the points are at distances x_1, x_2 and x_3, respectively, from the origin o and the angle $a_1 a_2 o$ is θ, then we have

$$x_1^2 = x_2^2 + 1 - 2x_2 \cos\theta$$

and

$$x_3^2 = x_2^2 + 1 + 2x_2 \cos\theta.$$

Adding the two gives

$$x_1^2 + x_3^2 = 2x_2^2 + 2.$$

Similarly, if $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ form a copy of ℓ_m with $|a_i - a_{i+1}| = 1$ for all $i = 1, 2, \ldots, m-1$, then, again writing x_i for the distance of a_i from the origin, we have

$$x_{i-1}^2 + x_{i+1}^2 = 2x_i^2 + 2$$

for all $i = 2, \ldots, m-1$. Given these observations, our aim will be to colour $\mathbb{R}_{\geq 0}$ so that there is no red solution to $y_1 + y_3 = 2y_2 + 2$ and no blue solution to the system $y_{i-1} + y_{i+1} = 2y_i + 2$ with $i = 2, \ldots, m-1$. Assuming that we have such a colouring χ, we can simply colour a point $a \in \mathbb{R}^n$ by $\chi(|a|^2)$ and it is easy to check that there is no red copy of ℓ_3 and no blue copy of ℓ_m.

We have therefore moved our problem to one of finding a natural number m and a colouring χ of $\mathbb{R}_{\geq 0}$ with no red solution to $y_1 + y_3 = 2y_2 + 2$ and no blue solution to the system $y_{i-1} + y_{i+1} = 2y_i + 2$ with $i = 2, \ldots, m-1$. Let q be a prime number. We will take $m = q^3$ and define χ by choosing an appropriate colouring χ' of $\mathbb{Z}_q$ and then setting $\chi(y) = \chi'(\lfloor y \rfloor \bmod q)$ for all $y \in \mathbb{R}_{\geq 0}$. Our aim now is to show that there is a suitable choice for χ'. For this, we consider a random red/blue-colouring χ' of $\mathbb{Z}_q$ and show that, for q sufficiently large, the probability that χ contains either of the banned configurations is small.

Concretely, suppose that $\mathbb{Z}_q$ is coloured randomly in red and blue with each element of $\mathbb{Z}_q$ coloured red with probability $p = q^{-3/4}$ and blue with probability $1 - p$. With this choice, the expected number of solutions in red to any of the equations $y_1 + y_3 = 2y_2 + c$ with $c \in \{1, 2, 3\}$ is at most

$$3p^3 q^2 + 9p^2 q < 12q^{-1/4} < \frac{1}{2},$$

where we used that there are at most $3q$ solutions to any of our 3 equations with two of the variables $\{y_1, y_2, y_3\}$ being equal and that q is sufficiently large. Note that if there are indeed no red solutions to these three equations over $\mathbb{Z}_q$, then there is no red solution to $y_1 + y_3 = 2y_2 + 2$ in the colouring χ of $\mathbb{R}$. Indeed, if $y_i = n_i + \epsilon_i$ with $0 \leq \epsilon_i < 1$, then n_i is coloured red in χ' and

$$n_1 + n_3 = 2n_2 + 2 + 2\epsilon_2 - \epsilon_1 - \epsilon_3.$$

But $|2\epsilon_2 - \epsilon_1 - \epsilon_3| < 2$, so we must have

$$n_1 + n_3 = 2n_2 + c$$

for $c \in \{1, 2, 3\}$. However, we know that there are no red solutions to any of these equations in the colouring χ', so there is no red solution to $y_1 + y_3 = 2y_2 + 2$ in the colouring χ.

For the blue configurations, we first observe that if the y_i satisfy the equations $y_{i-1} + y_{i+1} = 2y_i + 2$ with $i = 2, \ldots, m-1$ with $y_1 = a$ and $y_2 = a + d$, then $y_i = a + (i-1)d + (i^2 - 3i + 2)$. In particular, by Lemma 2, at least $q/6$ elements of the sequence $y_1, \ldots, y_m$ lie in different intervals $[j, j+1)$ with $0 \leq j \leq q-1$ when considered mod q.

Our aim now is to apply Lemma 3 to count the number of different ways in which a set of solutions $(y_1, y_2, \ldots, y_m)$ to our system of equations can overlap the collection of intervals $[j, j+1)$ mod q. Without loss of generality, we may assume that $0 \leq a, d < q$. Since, under this assumption, any set of solutions over $\mathbb{R}$ to our system of equations is contained in the interval $[0, 2m^2)$, it will suffice to count the number of feasible overlaps with the intervals $[j, j+1)$ with $0 \leq j \leq 2m^2 - 1$.

Since we need to check at most two linear inequalities in the two variables a and d to check whether each of the m points are placed in each of the $2m^2$ intervals, we can apply Lemma 3 with $N = 2$, $D = 1$ and $M = 2 \cdot m \cdot 2m^2 = 4m^3$ to conclude that the points $y_1, \ldots, y_m$ overlap the intervals $[j, j+1)$ with $0 \le j \le 2m^2 - 1$ in at most $(100m^3)^2 = 10^4 m^6$ different ways. But now, since at least $q/6$ of the y_i must always be in distinct intervals, a union bound implies that the probability we have a blue solution to our system of equations is at most

$$10^4 m^6 (1 - q^{-3/4})^{q/6} < \frac{1}{2}$$

for m sufficiently large. Combined with our earlier estimate for the probability of a red solution to $y_1 + y_3 = 2y_2 + 2$, we see that for m sufficiently large ($m = 10^{50}$ will suffice) there exists a colouring with no red ℓ_3 and no blue ℓ_m, as required.

4. Concluding remarks

We say that a set $X \subset \mathbb{E}^d$ is *Ramsey* if for every natural number r there exists n such that every r-colouring of $\mathbb{E}^n$ contains a monochromatic copy of X. In [4], it was shown that a set X is Ramsey if and only if for every natural number m and every fixed $K \subset \mathbb{E}^m$ there exists n such that $\mathbb{E}^n \to (X, K)$. We suspect that there may be an even simpler characterisation.

Conjecture 4. *A set X is Ramsey if and only if for every natural number m there exists n such that $\mathbb{E}^n \to (X, \ell_m)$.*

Of course, by the result mentioned above, we already know that if X is Ramsey, then $\mathbb{E}^n \to (X, \ell_m)$ for n sufficiently large. It therefore remains to show that if X is not Ramsey, then there exists m such that $\mathbb{E}^n \not\to (X, \ell_m)$ for all n. To prove this in full generality might be difficult. However, an important result of Erdős et al. [6] says that if X is Ramsey, then it must be spherical, in the sense that it must be contained in the surface of a sphere of some dimension. Thus, a first step towards Conjecture 4 might be to prove the following.

Conjecture 5. *For every non-spherical set X, there exists a natural number m such that $\mathbb{E}^n \not\to (X, \ell_m)$ for all n.*

The simplest example of a non-spherical set is the line ℓ_3, so our main result may be seen as a verification of Conjecture 5 in this particular case. The next case of interest seems to be when X consists of three points a_1, a_2, a_3 on a line, but now with $|a_1 - a_2| = 1$ and $|a_2 - a_3| = \alpha$ for some irrational α.

References

[1] A. Arman, S. Tsaturian, "Equally spaced collinear points in Euclidean Ramsey theory", https://arxiv.org/abs/1705.04640, 2017.

[2] ———, "A result in asymmetric Euclidean Ramsey theory", *Discrete Math.* **341** (2018), no. 5, p. 1502-1508.

[3] S. Basu, R. Pollack, M.-F. Roy, *Algorithms in Real Algebraic Geometry*, 2nd ed., Algorithms and Computation in Mathematics, vol. 10, Springer, 2006.

[4] D. Conlon, J. Fox, "Lines in Euclidean Ramsey theory", *Discrete Comput. Geom.* **61** (2019), no. 1, p. 218-225.

[5] G. Csizmadia, G. Tóth, "Note on a Ramsey-type problem in geometry", *J. Comb. Theory, Ser. A* **65** (1994), no. 2, p. 302-306.

[6] P. Erdős, R. L. Graham, P. Montgomery, B. L. Rothschild, J. Spencer, E. G. Straus, "Euclidean Ramsey theorems I", *J. Comb. Theory, Ser. A* **14** (1973), p. 341-363.

[7] ———, "Euclidean Ramsey theorems II", in *Infinite and finite sets (Colloq., Keszthely, 1973), Vol. I*, Colloquia Mathematica Societatis János Bolyai, vol. 529, North-Holland, 1973, p. 529-557.

[8] ———, "Euclidean Ramsey theorems III", in *Infinite and finite sets (Colloq., Keszthely, 1973), Vol. I*, Colloquia Mathematica Societatis János Bolyai, vol. 529, North-Holland, 1973, p. 559-583.

[9] P. Frankl, V. Rödl, "A partition property of simplices in Euclidean space", *J. Am. Math. Soc.* **3** (1990), no. 1, p. 1-7.
[10] P. Frankl, R. M. Wilson, "Intersection theorems with geometric consequences", *Combinatorica* **1** (1981), p. 357-368.
[11] R. Juhasz, "Ramsey type theorems in the plane", *J. Comb. Theory, Ser. A* **27** (1979), p. 152-160.
[12] I. Kříž, "Permutation groups in Euclidean Ramsey Theory", *Proc. Am. Math. Soc.* **112** (1991), no. 3, p. 899-907.
[13] A. D. Szlam, "Monochromatic translates of configurations in the plane", *J. Comb. Theory, Ser. A* **93** (2001), no. 1, p. 173-176.
[14] S. Tsaturian, "A Euclidean Ramsey result in the plane", *Electron. J. Comb.* **24** (2017), no. 4, article no. P4.35 (9 pages).