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Remarks on homogenization and $3D$-$2D$ dimension reduction of unbounded energies on thin films

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Abstract. We study periodic homogenization and $3D$-$2D$ dimension reduction by $\Gamma(\pi)$-con-vergence of heterogeneous thin films whose the stored-energy densities have no polynomial growth. In particular, our results are consistent with one of the basic facts of nonlinear elasticity, namely the necessity of an infinite amount of energy to compress a finite volume of matter into zero volume. However, our results are not consistent with the noninterpenetration of the matter.

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1. Introduction

Consider a heterogeneous thin film occupying in a reference configuration the bounded open set $\Sigma_\varepsilon \subset \mathbb{R}^3$ given by

$$\Sigma_\varepsilon := \Sigma \times \left]-\frac{\varepsilon^\gamma}{2}, \frac{\varepsilon^\gamma}{2}\right[,$$

where $\gamma \in]0,\infty[$ is fixed, $\varepsilon > 0$ and $\Sigma \subset \mathbb{R}^2$ is Lipschitz, open and bounded. The small parameters ε^γ and ε represent respectively the film thickness and the length scale of heterogeneity and material microstructure, and the meaning of the coefficient γ is as follows:

- $\gamma < 1$ means that the film thickness is much larger than heterogeneity;
- $\gamma = 1$ means that the film thickness is comparable to heterogeneity;
- $\gamma > 1$ means that the film thickness is much smaller than heterogeneity.

A point of Σ_ε is denoted by (x, x_3) with $x \in \Sigma$ and $x_3 \in \left]-\frac{\varepsilon^\gamma}{2}, \frac{\varepsilon^\gamma}{2}\right[$. In order to model x-periodic heterogenities of the material, we assume that its stored-energy density is a Borel measurable function

$$W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty]$$

with the following properties:

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- (C_1) W is p-coercive with $p > 1$, i.e. there exists $C > 0$ such that $W(x,F) \geq C|F|^p$ for all $(x,F) \in \mathbb{R}^2 \times \mathbb{M}^{3\times 3}$;
- (C_2) W is 1-periodic with respect to x, i.e. $W(x+z,F) = W(x,F)$ for all $(x,F) \in \mathbb{R}^2 \times \mathbb{M}^{3\times 3}$ and all $z \in \mathbb{Z}^2$.

In order to take into account the fact that an infinite amount of energy is required to compress a finite volume into zero volume[1], i.e.

$$W(x,F) \to \infty \quad \text{as} \quad \det F \to 0, \tag{1}$$

where $\det F$ denotes the determinant of the 3×3 matrix F, we assume that

- (C_3) W is p-ample, i.e. there exist $c > 0$ such that $\mathcal{Z}W(x,F) \leq c(1+|F|^p)$ for all $(x,F) \in \mathbb{R}^2 \times \mathbb{M}^{3\times 3}$, where $\mathcal{Z}W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty]$ is defined by

$$\mathcal{Z}W(x,F) := \inf\left\{\int_Y W(x,F+\nabla\varphi(x,x_3))\mathrm{d}x\mathrm{d}x_3 : \varphi \in W_0^{1,\infty}(Y;\mathbb{R}^3)\right\} \tag{2}$$

with $Y := \left]-\frac{1}{2},\frac{1}{2}\right[^3$.

Note that (C_3) does not imply that W is of p-polynomial growth, and is compatible with (1) (see Section 5). The object of this paper is to show that as $\varepsilon \to 0$ the three-dimensional free energy functional $E_\varepsilon : W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3) \to [0,\infty]$ (with $p > 1$) defined by

$$E_\varepsilon(u) := \frac{1}{\varepsilon^\gamma}\int_{\Sigma_\varepsilon} W\left(\frac{x}{\varepsilon},\nabla u(x,x_3)\right)\mathrm{d}x\mathrm{d}x_3 \tag{3}$$

$\Gamma(\pi)$-converges to the two-dimensional free energy functional $\overline{E} : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,\infty]$ given by

$$\overline{E}(v) := \int_\Sigma \overline{W}\big(\nabla v(x)\big)\mathrm{d}x \tag{4}$$

with $\overline{W} : \mathbb{M}^{3\times 2} \to [0,\infty]$. Usually, $\overline{E}$ is called the homogenized nonlinear membrane energy associated with the two-dimensional elastic material with respect to the reference configuration Σ. Furthermore, we wish to give a representation formula for $\overline{W}$.

In the homogeneous case, i.e. $W(x,F) = W(F)$, the problem of deriving membrane model as variational limit of non-linear three dimensional elasticity began at the begining of the nineties with the works of Le Dret and Raoult (see [13, 14]) who solved the problem in the case where W is bounded, i.e. $W(F) \leq c(1+|F|^p)$. In the unbounded case, the problem was solved in [3] for the constraint "$\det\nabla u \neq 0$" and in [4] for the constraint "$\det\nabla u > 0$" (see also [5, 6]). Note that the answer of the problem of deriving membrane model as variational limit of non-linear three dimensional elasticity under the constraint "$\det\nabla u > 0$" is the result of several works on the subject: mainly, the attempt of Percivale in 1991 (see [15]), the rigorous answer by Le Dret and Raoult in the p-polynomial growth case and especially the substantial contributions of Ben Belgacem (see [8–10]).

In the heterogeneous and bounded case, the problem was solved by Braides, Fonseca and Francfort (see [11]) for $\gamma = 1$ and by Shu (see [16]) for $\gamma \neq 1$. In the present paper we deal with the heterogeneous and unbounded case. Our results (see Theorem 1 and Corollary 13) are compatible with the constraint "$\det\nabla u \neq 0$" but not with the constraint "$\det\nabla u > 0$". To our knowledge, in the heterogeneous case, incorporating the constraint "$\det\nabla u > 0$" is an open problem.

The plan of the paper is as follows. In the next section we state our main result (see Theorem 1) establishing the $\Gamma(\pi)$-convergence (whose definition is recalled in Section 3.1) of E_ε in (3) to $\overline{E}$ in (4) together with a representation formula for $\overline{W}$ which depends on γ. The proof of Theorem 1 is given in Section 4 by using two results: unbounded relaxation (see Corollary 8) and bounded homogenization and $3D$-$2D$ dimension reduction (see Theorem 11). These results, proved in [1]

[1] However, we do not prevent orientation reversal.

and [11,16] respectively, are recalled in Section 3.2. In Section 5 we give applications of Theorem 1 (see Corollary 13).

Notation. For $k=2$ or 3, $\mathbb{M}^{3\times k}$ denotes the space of real $3\times k$ matrices. For $L:\mathbb{R}^2\times\mathbb{M}^{3\times k}\to[0,\infty]$, the function $\mathscr{Q}L:\mathbb{R}^2\times\mathbb{M}^{3\times k}\to[0,\infty]$ is defined as follows: for each $x\in\mathbb{R}^2$, $\mathscr{Q}L(x,\cdot)$ is the quasiconvex envelope[2] of $L(x,\cdot):\mathbb{M}^{3\times k}\to[0,\infty]$. The symbol $⨍$ stands for the mean-value integral with respect to the Lebesgue measure $\mathscr{L}^k$ on $\mathbb{R}^k$, i.e. $⨍_Q=\frac{1}{\mathscr{L}^k(Q)}\int_Q$.

2. Main result

Let $\mathscr{L}_2$ denote the class of $\lambda\in L^\infty(\mathbb{R}^2;[0,\infty[)$ such that λ is continuous almost everywhere with respect to $\mathscr{L}^2$ and let $\mathscr{I}^p$ and $\mathscr{J}^p$ be classes of Borel measurable functions $W:\mathbb{R}^2\times\mathbb{M}^{3\times 3}\to[0,\infty]$ defined by:

$$\mathscr{I}^p:=\Big\{W:\mathbb{R}^2\times\mathbb{M}^{3\times3}\to[0,\infty]:W\text{ satisfies }(\mathrm{C}_1),(\mathrm{C}_3)\text{ and }(\mathrm{C}_4)\Big\};\tag{5}$$

$$\mathscr{J}^p:=\Big\{W:\mathbb{R}^2\times\mathbb{M}^{3\times3}\to[0,\infty]:W\text{ satisfies }(\mathrm{C}_1),(\mathrm{C}_3)\text{ and }(\mathrm{C}_5)\Big\},\tag{6}$$

where (C_4) and (C_5) are given by:

(C_4) there exists $\lambda\in\mathscr{L}_2$ for every $x,x'\in\mathbb{R}^2$ and every $F\in\mathbb{M}^{3\times3}$,

$$W(x,F)\le|\lambda(x)-\lambda(x')|(1+W(x',F))+W(x',F);$$

(C_5) there exist a finite family $\{V_j\}_{j\in J}$ of open disjoint subsets of $\mathbb{R}^2$, with $\mathscr{L}^2(\partial V_j)=0$ for all $j\in J$ and $\mathscr{L}^2(\mathbb{R}^2\setminus\cup_{j\in J}V_j)=0$, and a finite family $\{H_j:\mathbb{M}^{3\times3}\to[0,\infty]\}_{j\in J}$ of Borel measurable functions such that

$$W(x,F)=\sum_{j\in J}\mathbb{1}_{V_j}(x)H_j(F).$$

Let us set:

$$\mathscr{I}^p_{\mathrm{per}}:=\Big\{W\in\mathscr{I}^p:W\text{ satisfies }(\mathrm{C}_2)\Big\};$$
$$\mathscr{J}^p_{\mathrm{per}}:=\Big\{W\in\mathscr{J}^p:W\text{ satisfies }(\mathrm{C}_2)\Big\}=\Big\{W\in\mathscr{J}^p:\mathbb{1}_{V_j}\text{ is 1-periodic for all }j\in J\Big\}.$$

In what follows, given $L:\mathbb{R}^2\times\mathbb{M}^{3\times3}\to[0,\infty]$, we consider $\widehat{L}:\mathbb{R}^2\to\mathbb{M}^{3\times2}\to[0,\infty]$ defined by

$$\widehat{L}(x,\xi):=\inf_{\zeta\in\mathbb{R}^3}L(x,[\xi\mid\zeta]),$$

and $\mathscr{H}L:\mathbb{M}^{3\times3}\to[0,\infty]$, $\widehat{\mathscr{H}L}:\mathbb{M}^{3\times2}\to[0,\infty]$ and $\mathscr{H}\widehat{L}:\mathbb{M}^{3\times2}\to[0,\infty]$ defined by:

$$\mathscr{H}L(F):=\inf_{k\in\mathbb{N}^*}\inf_{\varphi\in W^{1,p}_0(kY;\mathbb{R}^3)}⨍_{kY}L(x,F+\nabla\varphi(x,x_3))\mathrm{d}x\mathrm{d}x_3;$$

$$\widehat{\mathscr{H}L}(\xi):=\inf_{k\in\mathbb{N}^*}\inf_{\varphi\in\widehat{W}^{1,p}_0(k\widehat{Y}\times]-\frac12,\frac12[;\mathbb{R}^3)}⨍_{k\widehat{Y}\times]-\frac12,\frac12[}L(x,[\xi+\nabla_x\varphi(x,x_3)\mid\partial_3\varphi(x,x_3)])\mathrm{d}x\mathrm{d}x_3;$$

$$\mathscr{H}\widehat{L}(\xi):=\inf_{k\in\mathbb{N}^*}\inf_{\varphi\in W^{1,p}_0(k\widehat{Y};\mathbb{R}^3)}⨍_{k\widehat{Y}}\widehat{L}(x,\xi+\nabla\varphi(x))\mathrm{d}x,$$

[2]By the quasiconvex envelope of $L(x,\cdot)$ we mean the greatest quasiconvex function from $\mathbb{M}^{3\times k}$ to $[0,\infty]$ which less than or equal to $L(x,\cdot)$. Clearly, $L(x,\cdot)$ is quasiconvex if and only if $\mathscr{Q}L(x,\cdot)=L(x,\cdot)$.

where $[\xi \mid \zeta]$ denotes the element of $\mathbb{M}^{3\times 3}$ corresponding to $(\xi,\zeta)\in\mathbb{M}^{3\times 2}\times\mathbb{R}^3$ and, for each $k\in\mathbb{N}^*$, $W_0^{1,p}(kY;\mathbb{R}^3)$, $\widehat{W}_0^{1,p}(k\widehat{Y}\times]-\frac{1}{2},\frac{1}{2}[;\mathbb{R}^3)$ and $W_0^{1,p}(k\widehat{Y};\mathbb{R}^3)$ are given by:

$$W_0^{1,p}(kY;\mathbb{R}^3):=\left\{\varphi\in W^{1,p}(kY;\mathbb{R}^3):\varphi=0 \text{ on } \partial kY\right\};$$

$$\widehat{W}_0^{1,p}\left(k\widehat{Y}\times\left]-\frac{1}{2},\frac{1}{2}\right[;\mathbb{R}^3\right):=\left\{\varphi\in W^{1,p}\left(k\widehat{Y}\times\left]-\frac{1}{2},\frac{1}{2}\right[;\mathbb{R}^3\right):\varphi=0 \text{ on } \partial k\widehat{Y}\times\left]-\frac{1}{2},\frac{1}{2}\right[\right\};$$

$$W_0^{1,p}(k\widehat{Y};\mathbb{R}^3):=\left\{\varphi\in W^{1,p}(k\widehat{Y};\mathbb{R}^3):\varphi=0 \text{ on } \partial k\widehat{Y}\right\},$$

where $Y:=\left]-\frac{1}{2},\frac{1}{2}\right[^3$ and $\widehat{Y}:=\left]-\frac{1}{2},\frac{1}{2}\right[^2$. The main result of the paper is the following.

Theorem 1. *If $W\in\mathscr{I}^p_{\rm per}\cup\mathscr{J}^p_{\rm per}$ then as $\varepsilon\to 0$, E_ε in* (3) *$\Gamma(\pi)$-converges to $\overline{E}$ in* (4)*, i.e. for every $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$,*

$$\left(\Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}E_\varepsilon\right)(v)=\overline{E}(v)$$

with $\overline{W}:\mathbb{M}^{3\times 2}\to[0,\infty]$ given as follows:

(i) *if $\gamma<1$ then $\overline{W}=\mathcal{Q}\widehat{\mathcal{H}\mathcal{Z}W}$;*
(ii) *if $\gamma=1$ then $\overline{W}=\widehat{\mathcal{H}\mathcal{Z}}W$;*
(iii) *if $\gamma>1$ then $\overline{W}=\mathcal{H}\widehat{\mathcal{Z}W}$.*

Remark 2. From (C_1) and (C_3) we see that for every $(x,F)\in\mathbb{R}^2\times\mathbb{M}^{3\times 3}$,

$$C|F|^p\le\mathcal{Z}W(x,F)\le c(1+|F|^p),$$

and so, for every $(x,\xi)\in\mathbb{R}^2\times\mathbb{M}^{3\times 2}$,

$$C|\xi|^p\le\widehat{\mathcal{Z}W}(x,\xi)\le c(1+|\xi|^p).$$

Consequently, for every $\gamma\in]0,\infty[$,

$$C|\xi|^p\le\overline{W}(\xi)\le c(1+|\xi|^p).$$

On the other hand, under (C_3), $\mathcal{Z}W=\mathcal{Q}W$ by Lemma 10, and consequently in Theorem 1 we have

$$\overline{W}=\begin{cases}\mathcal{Q}\widehat{\mathcal{H}\mathcal{Q}W} & \text{if } \gamma<1\\ \widehat{\mathcal{H}\mathcal{Q}}W & \text{if } \gamma=1\\ \mathcal{H}\widehat{\mathcal{Q}W} & \text{if } \gamma>1.\end{cases}$$

The distinguishing feature of Theorem 1 is that it can be applied with stored-energy densities W having a singular behavior of type (1) (see Section 5).

3. Auxiliary results

3.1. $\Gamma(\pi)$-*convergence*

To accomplish our asymptotic analysis, we use the notion of convergence introduced by Anzellotti, Baldo and Percivale in [7] in order to deal with dimension reduction problems in mechanics. Let $\pi=\{\pi_\varepsilon\}_\varepsilon$ be the family of maps $\pi_\varepsilon:W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)\to W^{1,p}(\Sigma;\mathbb{R}^3)$ defined by

$$\pi_\varepsilon(u):=\frac{1}{\varepsilon^\gamma}\int_{-\frac{\varepsilon^\gamma}{2}}^{\frac{\varepsilon^\gamma}{2}}u(\cdot,x_3)\mathrm{d}x_3.$$

Definition 3. *We say that E_ε $\Gamma(\pi)$-converges to $\overline{E}$ as $\varepsilon\to 0$, and we write $\overline{E}=\Gamma(\pi)\text{-}\lim_{\varepsilon\to 0}E_\varepsilon$, if the following two assertions hold:*

(i) *for all $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$ and all $\{u_\varepsilon\}_\varepsilon\subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$,*

$$\text{if } \pi_\varepsilon(u_\varepsilon)\to v \text{ in } L^p(\Sigma;\mathbb{R}^3) \text{ then } \overline{E}(v)\le\varliminf_{\varepsilon\to 0}E_\varepsilon(u_\varepsilon);$$

(ii) *for all* $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$, *there exists* $\{u_\varepsilon\}_\varepsilon \subset W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$ *such that:*

$$\pi_\varepsilon(u_\varepsilon) \to v \text{ in } L^p(\Sigma;\mathbb{R}^3) \text{ and } \overline{E}(v) \geq \overline{\lim_{\varepsilon\to 0}} E_\varepsilon(u_\varepsilon).$$

In fact, Definition 3 is a variant of De Giorgi's Γ-convergence. This is made clear by Proposition 5. Consider $\widehat{E}_\varepsilon : W^{1,p}(\Sigma;\mathbb{R}^3) \to [0,\infty]$ defined by

$$\widehat{E}_\varepsilon(v) := \inf\{E_\varepsilon(u) : \pi_\varepsilon(u) = v\}.$$

Definition 4. *We say that* $\widehat{E}_\varepsilon$ Γ*-converges to* $\overline{E}$ *as* $\varepsilon \to 0$, *and we write* $\overline{E} = \Gamma\text{-}\lim_{\varepsilon\to 0} \widehat{E}_\varepsilon$ *if for every* $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\left(\Gamma\text{-}\underline{\lim_{\varepsilon\to 0}} \widehat{E}_\varepsilon\right)(v) = \left(\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} \widehat{E}_\varepsilon\right)(v) = \overline{E}(v),$$

where:

$$\left(\Gamma\text{-}\underline{\lim_{\varepsilon\to 0}} \widehat{E}_\varepsilon\right)(v) := \inf\left\{\underline{\lim_{\varepsilon\to 0}} \widehat{E}_\varepsilon(v_\varepsilon) : v_\varepsilon \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\right\};$$
$$\left(\Gamma\text{-}\overline{\lim_{\varepsilon\to 0}} \widehat{E}_\varepsilon\right)(v) := \inf\left\{\overline{\lim_{\varepsilon\to 0}} \widehat{E}_\varepsilon(v_\varepsilon) : v_\varepsilon \to v \text{ in } L^p(\Sigma;\mathbb{R}^3)\right\}.$$

For a deeper discussion of the Γ-convergence theory we refer to the book [12]. Clearly, Definition 4 is equivalent to assertions (i) and (ii) in Definition 3 with "$\pi(u_\varepsilon) \to v$" replaced by "$v_\varepsilon \to v$". It is then obvious that

Proposition 5. $\overline{E} = \Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} E_\varepsilon$ *if and only if* $\overline{E} = \Gamma\text{-}\lim_{\varepsilon\to 0} \widehat{E}_\varepsilon$.

As, for every $\varepsilon > 0$, we have

$$\inf\{E_\varepsilon(u) : \pi_\varepsilon(u) = v\} = \inf\left\{\overline{E}_\varepsilon(u) : \pi_\varepsilon(u) = v\right\},$$

where $\overline{E}_\varepsilon : W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3) \to [0,\infty]$ is the relaxed functional of $E_\varepsilon : W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3) \to [0,\infty]$, i.e. for every $u \in W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$,

$$\overline{E}_\varepsilon(u) = \inf\left\{\underline{\lim_{n\to\infty}} E_\varepsilon(u_n) : u_n \to u \text{ in } L^p(\Sigma_\varepsilon;\mathbb{R}^3)\right\}, \tag{7}$$

from proposition 5 we deduce that the following result which is used in the proof of Theorem 1.

Proposition 6. *The* $\Gamma(\pi)$*-limit is stable by substituting* E_ε *by its relaxed functional* $\overline{E}_\varepsilon$.

3.2. *Relaxation in the heterogeneous and unbounded case*

Let $W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty]$ be a Borel measurable function, let $p > 1$, let $\Omega \subset \mathbb{R}^3$ be a bounded open set such that $\mathscr{L}^3(\partial\Omega) = 0$, let $I : W^{1,p}(\Omega;\mathbb{R}^3) \to [0,\infty]$ be defined by

$$I(u) := \int_\Omega W(x, \nabla u(x,x_3))\mathrm{d}x\mathrm{d}x_3$$

and let $\overline{I} : W^{1,p}(\Omega;\mathbb{R}^3) \to [0,\infty]$ the relaxed functional of I, i.e.

$$\overline{I}(u) := \inf\left\{\underline{\lim_{n\to\infty}} I(u_n) : u_n \to u \text{ in } L^p(\Omega;\mathbb{R}^3)\right\}.$$

In [1, Theorems 3.8 and 3.15] we proved the following integral representation theorem.

Theorem 7. *If* $W \in \mathscr{I}^p \cup \mathscr{J}^p$, *where* $\mathscr{I}^p$ *and* $\mathscr{J}^p$ *are defined in* (5) *and* (6) *respectively, then*

$$\overline{I}(u) = \int_\Omega \mathcal{Z}W(x, \nabla u(x))\mathrm{d}x$$

for all $u \in W^{1,p}(\Omega;\mathbb{R}^m)$.

Given $\varepsilon > 0$, set $W_\varepsilon(x,F) = \frac{1}{\varepsilon^\gamma} W\left(\frac{x}{\varepsilon}, F\right)$. It is easy to see that:

- if (C_1) holds then $W_\varepsilon(x,F)\geq \frac{C}{\varepsilon^\gamma}|F|^p$ for all $(x,F)\in\mathbb{R}^2\times\mathbb{M}^{3\times3}$, i.e. W_ε is p-coercive;
- if (C_3) holds then $\mathcal{Z}W_\varepsilon(x,F)\leq\frac{c}{\varepsilon^\gamma}(1+|F|^p)$ for all $(x,F)\in\mathbb{R}^2\times\mathbb{M}^{3\times3}$, i.e. $\mathcal{Z}W_\varepsilon$ is p-ample;
- if (C_4) holds then $W_\varepsilon(x,F)\leq|\lambda_\varepsilon(x)-\lambda_\varepsilon(x')|(1+W_\varepsilon(x',F))+W_\varepsilon(x',F)$ for all $x,x'\in\mathbb{R}^2$ and all $F\in\mathbb{M}^{3\times3}$ where $\lambda_\varepsilon(\cdot):=\frac{1}{\varepsilon^\gamma}\lambda(\frac{\cdot}{\varepsilon})\in\mathscr{L}_2$;
- if (C_5) holds then $W_\varepsilon(x,F)=\sum_{j\in J}\mathbb{1}_{V_j^\varepsilon}(x)H_j^\varepsilon(F)$ where, for each $j\in J$, $H_j^\varepsilon:=\frac{1}{\varepsilon^\gamma}H_j$ and $V_j^\varepsilon:=\varepsilon V_j$; moreover, as $\mathscr{L}^2(\partial V_j)=0$ for all $j\in J$ and $\mathscr{L}^2(\mathbb{R}^2\setminus\bigcup_{j\in J}V_j)=0$, we have $\mathscr{L}^2(\partial\varepsilon V_j)=0$ for all $j\in J$ and $\mathscr{L}^2(\mathbb{R}^2\setminus\bigcup_{j\in J}\varepsilon V_j)=0$.

Hence if $W\in\mathscr{I}^p\cup\mathscr{J}^p$ then $W_\varepsilon\in\mathscr{I}^p\cup\mathscr{J}^p$. So, by applying Theorem 7 with $\Omega=\Sigma_\varepsilon$ and $I=E_\varepsilon$, with E_ε defined in (3), we obtain the following result which is used in the proof of Theorem 1.

Corollary 8. *If $W\in\mathscr{I}^p\cup\mathscr{J}^p$ then, for every $\varepsilon>0$,*

$$\overline{E}_\varepsilon(u)=\frac{1}{\varepsilon^\gamma}\int_{\Sigma_\varepsilon}\mathcal{Z}W\Big(\frac{x}{\varepsilon},\nabla u(x,x_3)\Big)\mathrm{d}x\mathrm{d}x_3$$

for all $u\in W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$ where $\overline{E}_\varepsilon$, defined in (7)*, is the relaxed functional of E_ε.*

Remark 9. Because of the following lemma (see [2, Theorem 2.3-bis]) which makes clear the link between the quasiconvex envelope $\mathcal{Q}W$ of W and $\mathcal{Z}W$ defined in (2), in Theorem 7 and Corollary 8, $\mathcal{Z}W$ can be replaced by $\mathcal{Q}W$.

Lemma 10. *If $\mathcal{Z}W$ is finite then $\mathcal{Q}W=\mathcal{Z}W$.*

3.3. *Homogenization and 3D-2D dimension reduction in the bounded case*

In the bounded case, instead of (C_3), we consider the following condition:

(C_3^{b}) W is of p-polynomial growth, i.e. there exists $c>0$ such that $W(x,F)\leq c(1+|F|^p)$ for all $(x,F)\in\mathbb{R}^2\times\mathbb{M}^{3\times3}$,

and we also assume that

(C_3^{lip}) W is p-locally lipschitz with respect to F in the following sense: there exists $\theta>0$ such that $|W(x,F)-W(x,F')|\leq\theta|F-F'|(1+|F|^{p-1}+|F'|^{p-1})$ for all $x\in\mathbb{R}^2$ and all $F,F'\in\mathbb{M}^{3\times3}$.

To establish Theorem 1 we need the following result which was proved by Braides, Fonseca and Francfort (see [11]) for $\gamma=1$ and by Shu (see [16]) for $\gamma\neq1$.

Theorem 11. *If (C_1), (C_2), (C_3^{b}) and (C_3^{lip}) hold then as $\varepsilon\to0$, E_ε in* (3) *$\Gamma(\pi)$-converges to $\overline{E}$ in* (4)*, i.e. for every $v\in W^{1,p}(\Sigma;\mathbb{R}^3)$,*

$$\Big(\Gamma(\pi)\text{-}\lim_{\varepsilon\to0}E_\varepsilon\Big)(v)=\overline{E}(v)$$

with $\overline{W}:\mathbb{M}^{3\times2}\to[0,\infty]$ given as follows:

(i) *if $\gamma<1$ then $\overline{W}=\mathcal{Q}\widehat{\mathcal{H}W}$;*
(ii) *if $\gamma=1$ then $\overline{W}=\widehat{\mathcal{H}}W$;*
(iii) *if $\gamma>1$ then $\overline{W}=\mathcal{H}\widehat{W}$.*

Contrary to Theorem 1, due to the fact that W is of p-polynomial growth, Theorem 11 is not compatible with (1).

4. Proof of Theorem 1

As, by Proposition 6, the $\Gamma(\pi)$-limit is stable by substituting E_ε by its relaxed functional $\overline{E}_\varepsilon$, i.e.

$$\begin{aligned}\overline{E}_\varepsilon(u)&=\inf\Big\{\varliminf_{n\to\infty}E_\varepsilon(u_n):u_n\to u\text{ in }L^p(\Sigma_\varepsilon;\mathbb{R}^3)\Big\}\\&=\frac{1}{\varepsilon^\gamma}\inf\Big\{\varliminf_{n\to\infty}\int_{\Sigma_\varepsilon}W\Big(\frac{x}{\varepsilon},\nabla u_n(x,x_3)\Big)\mathrm{d}x\mathrm{d}x_3:u_n\to u\text{ in }L^p(\Sigma_\varepsilon;\mathbb{R}^3)\Big\},\end{aligned}$$

it suffices to prove that for every $v \in W^{1,p}(\Sigma;\mathbb{R}^3)$,

$$\left(\Gamma(\pi)\text{-}\lim_{\varepsilon\to 0} \overline{E}_\varepsilon\right)(v) = \int_\Sigma \overline{W}(\nabla v(x))\mathrm{d}x \tag{8}$$

with $\overline{W} : \mathbb{M}^{3\times 2} \to [0,\infty]$ given by (i), (ii) or (iii). Since $W \in \mathscr{I}^p \cup \mathscr{J}^p$, by Corollary 8 we have

$$\overline{E}_\varepsilon(u) = \frac{1}{\varepsilon^\gamma} \int_{\Sigma_\varepsilon} \mathcal{Z}W\left(\frac{x}{\varepsilon}, \nabla u(x, x_3)\right) \mathrm{d}x\mathrm{d}x_3$$

for all $\varepsilon > 0$ and all $u \in W^{1,p}(\Sigma_\varepsilon;\mathbb{R}^3)$ with $\mathcal{Z}W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty]$ given by (2). It is clear that $\mathcal{Z}W$ is p-coercive and 1-periodic with respect to x. Moreover, $\mathcal{Z}W$ is of p-polynomial growth and so, by Lemma 10, $\mathcal{Z}W = \mathcal{Q}W$, hence, for each $x \in \mathbb{R}^2$, $\mathcal{Z}W(x,\cdot)$ is quasiconvex. Consequently $\mathcal{Z}W$ is p-locally lipschitz and the result follows by applying Theorem 11 to $\mathcal{Z}W$. □

5. Applications

Let $\mathscr{K}$ be the class of Borel measurable function $H : \mathbb{M}^{3\times 3} \to [0,\infty]$ defined by

$$\mathscr{K} := \left\{ H : \mathbb{M}^{3\times 3} \to [0,\infty] : H \text{ is } p\text{-coercive and satisfies } (\mathrm{C}_6)\right\},$$

where (C_6) is given by

(C_6) there exist $\alpha, \beta > 0$ such that for every $F \in \mathbb{M}^{3\times 3}$,

$$\text{if } |\det \xi| \geq \alpha \text{ then } H(F) \leq \beta(1 + |F|^p).$$

Note that (C_6) is compatible with the singular behavior

$$H(F) \to \infty \text{ as } \det F \to 0. \tag{9}$$

A typical example of a function belonging to the class $\mathscr{K}$ is given by

$$H(F) = |F|^p + h(\det F)$$

where $h : \mathbb{R} \to [0,\infty]$ is a Borel measurable function for which there exist $\delta, \delta' > 0$ such that $h(t) \leq \delta'$ for all $|t| \geq \delta$. For example, given $s > 0$ and $T \geq 0$ (possibly very large), this latter condition is satisfied with $\delta = 2T$ and $\delta' = \max\left\{\frac{1}{(2T)^s}, T\right\}$ when h is of type

$$h(t) = \begin{cases} T & \text{if } t < -T \\ \infty & \text{if } t \in [-T, 0] \\ \dfrac{1}{t^s} & \text{if } t > 0. \end{cases}$$

Let $\mathscr{S}_1$, $\mathscr{S}_2$ and $\mathscr{S}_3$ be classes of Borel measurable functions $W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty]$ defined by:

$$\mathscr{S}_1 := \left\{ W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty] : W \text{ satisfies } (\mathrm{C}_7)\right\};$$
$$\mathscr{S}_2 := \left\{ W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty] : W \text{ satisfies } (\mathrm{C}_8)\right\};$$
$$\mathscr{S}_3 := \left\{ W : \mathbb{R}^2 \times \mathbb{M}^{3\times 3} \to [0,\infty] : W \text{ satisfies } (\mathrm{C}_5) \text{ with } H_j \in \mathscr{K} \text{ for all } j \in J\right\},$$

where (C_7) and (C_8) are given by:

(C_7) there exist $H \in \mathscr{K}$ and a 1-periodic $a \in \mathscr{L}_2$ with $a(x) \geq \eta$ for all $x \in \mathbb{R}^2$ and some $\eta > 0$ such that

$$W(x, F) = a(x)H(F);$$

(C_8) there exist Borel measurable functions $H_1, H_2 : \mathbb{M}^{3\times 3} \to [0,\infty]$ with

$$\begin{cases} H_2 \in \mathscr{K} \\ H_2 \leq H_1 \leq \gamma H_2 \quad \text{for some } \gamma > 1 \end{cases}$$

such that

$$W(x,F) = \mathbb{1}_{E_1}(x) H_1(F) + \mathbb{1}_{E_2}(x) H_2(F),$$

where E_1 is a 1-periodic open subset of $\mathbb{R}^2$ such that $|\partial E_1| = 0$ and $E_2 := \mathbb{R}^2 \setminus E_1$, with $\mathbb{1}_{E_1}$ and $\mathbb{1}_{E_2}$ denoting the characteristic functions of E_1 and E_2 respectively.

Due to the fact that any $H \in \mathscr{K}$ is compatible with (9), for every $i \in \{1,2,3\}$, any $W \in \mathscr{S}_i$ is compatible with (1). The following result was proved in [1, Lemmas 2.11, 2.16 and 2.21].

Proposition 12. *The following inclusions hold:* $\mathscr{S}_1 \subset \mathscr{I}^p_{\mathrm{per}}$, $\mathscr{S}_2 \subset \mathscr{I}^p_{\mathrm{per}}$ *and* $\mathscr{S}_3 \subset \mathscr{J}^p_{\mathrm{per}}$.

As a consequence of Theorem 1 and Proposition 12 we have

Corollary 13. *If* $W \in \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3$ *then as* $\varepsilon \to 0$, E_ε *in* (3) $\Gamma(\pi)$*-converges to* $\overline{E}$ *in* (4) *with* $\overline{W}$ *given by (i), (ii) or (iii) in Theorem 1.*

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