Mikhail Borovoi, Appendix by Zev Rosengarten

Criterion for surjectivity of localization in Galois cohomology of a reductive group over a number field

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Abstract. Let $G$ be a connected reductive group over a number field $F$, and let $S$ be a set (finite or infinite) of places of $F$. We give a necessary and sufficient condition for the surjectivity of the localization map from $H^1(F, G)$ to the “direct sum” of the sets $H^1(F_v, G)$ where $v$ runs over $S$. In the appendices, we give a new construction of the abelian Galois cohomology of a reductive group over a field of arbitrary characteristic.

Résumé. Soit $G$ un groupe réductif connexe sur un corps de nombres $F$, et soit $S$ un ensemble (fini ou infini) de places de $F$. On donne une condition nécessaire et suffisante pour la surjectivité de l’application de localisation de $H^1(F, G)$ vers la « somme directe » des ensembles $H^1(F_v, G)$, où $v$ parcourt $S$. Dans les appendices on donne une nouvelle construction de la cohomologie galoisienne abélienne d’un groupe réductif sur un corps de caractéristique quelconque.

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Section 1.3. This is an abelian group depending functorially on \( G \) and \( F \). There is a canonical abelianization map
\[
\text{ab}: H^1(F, G) \to H^1_{\text{ab}}(F, G).
\]
We give a new, better construction of \( H^1_{\text{ab}}(F, G) \) in Section 3. Let \( S \subseteq \mathcal{V}(F) \) be a subset (finite or infinite). We consider the localization map
\[
H^1_{\text{ab}}(F, G) \to \prod_{\nu \in S} H^1_{\text{ab}}(F_\nu, G). \tag{1}
\]
In fact this map takes values in the subgroup \( \bigoplus_{\nu \in S} H^1_{\text{ab}}(F_\nu, G) \subseteq \prod_{\nu \in S} H^1_{\text{ab}}(F_\nu, G) \); see [5, Corollary 4.6]. Thus we obtain a localization map
\[
\text{loc}^{ab}_S: H^1_{\text{ab}}(F, G) \to \bigoplus_{\nu \in S} H^1_{\text{ab}}(F_\nu, G). \tag{2}
\]
Similarly, consider the localization map
\[
H^1(F, G) \to \prod_{\nu \in S} H^1(F_\nu, G). \tag{3}
\]
In fact it takes values in the subset \( \bigoplus_{\nu \in S} H^1(F_\nu, G) \) consisting of the families \( \langle \xi_\nu \rangle_{\nu \in S} \) with \( \xi_\nu \in H^1(F_\nu, G) \) and such that \( \xi_\nu = 1 \) for all \( \nu \) except maybe finitely many of them. This well-known fact follows, for instance, from the corresponding assertion for (1) together with [5, Theorem 5.11 and Corollary 5.4.1]. Thus we obtain a localization map
\[
\text{loc}^S: H^1(F, G) \to \bigoplus_{\nu \in S} H^1(F_\nu, G). \tag{3}
\]
We wish to find conditions under which the localization maps (2) and (3) are surjective.

1.2. We denote by \( M = \pi_1(G) \) the algebraic fundamental group of \( G \) (also known as the Borovoi fundamental group of \( G \)) introduced in [5, Section 1], and also introduced by Merkurjev [16, Section 10.1] and Colliot-Thélène [8, Proposition-Definition 6.1]. See Subsection 2.3 for our definition of \( \pi_1(G) \). This is a finitely generated abelian group, on which the absolute Galois group \( \text{Gal}(\overline{F}/F) \) naturally acts. Let \( E/F \) be a finite Galois extension in \( \overline{F} \) such that \( \text{Gal}(\overline{F}/E) \) acts on \( M \) trivially and that \( E \) has no real places. Then the Galois group \( \Gamma := \text{Gal}(E/F) \) naturally acts on \( M \) and on the set of places \( \mathcal{V}(E) \) of the field \( E \).

1.3. We denote by \( \mathcal{U}^1_{\text{ab}}(F, G) \) the cokernel of the homomorphism (2), that is,
\[
\mathcal{U}^1_{\text{ab}}(F, G) = \ker \left[ \text{loc}^{ab} \colon H^1_{\text{ab}}(F, G) \to \bigoplus_{\nu \in S} H^1_{\text{ab}}(F_\nu, G) \right].
\]
After explaining our notation in Section 2, we compute in Section 3 the finite abelian group \( \mathcal{U}^1_{\text{ab}}(F, G) \) in terms of the action of \( \Gamma \) on \( M \) and on \( \mathcal{V}(E) = \mathcal{V}_f(E) \cup \mathcal{V}_c(E) \); see Corollary 3.8. See Subsection 2.4 for the notations \( \mathcal{V}_f \) and \( \mathcal{V}_c \).

Concerning the map \( \text{loc}^S \) of (3), in Section 3 we compute the image of this map; see Main Theorem 3.7. Using this result, we give a criterion (necessary and sufficient condition) for the map \( \text{loc}^S \) to be surjective; see Corollary 3.9. This is also a criterion for the vanishing of \( \mathcal{U}^1_{\text{ab}}(F, G) \). Again, our criterion is given in terms of the action of \( \Gamma \) on \( M \) and on \( \mathcal{V}(E) = \mathcal{V}_f(E) \cup \mathcal{V}_c(E) \). Using this criterion, we give a simple proof of the result of Borel and Harder [2, Theorem 1.7] (see also Prasad and Rapinchuk [19, Proposition 1]) on the surjectivity of the map \( \text{loc}^S \) when \( G \) is semisimple and there exists a finite place \( v_0 \) of \( F \) outside \( S \); see Proposition 3.14 below.

Let \( \Gamma \) be a finite group. In Section 4, we construct an exact sequence arising from a short exact sequence of \( \Gamma \)-modules. In Section 5, using this exact sequence and Main Theorem 3.7, we generalize a result of Prasad and Rapinchuk giving a sufficient condition for the surjectivity of the localization map \( \text{loc}^S \) when \( G \) is reductive, in terms of the radical (largest central torus) of \( G \); see Theorem 5.1. As a particular case, we obtain the following corollary.
**Corollary 1.4 (of Theorem 5.1).** Let $G$ be a reductive group over a number field $F$, and let $C$ denote the radical of $G$ (the identity component of the center of $G$). Let $S \subset \mathcal{V}(F)$ be a set of places of $F$. Assume that the $F$-torus $C$ splits over a finite Galois extension of $F$ of prime degree $p$ and that there exists a finite place $v_0$ in the complement $S^\complement := \mathcal{V}(F) \setminus S$ such that $C$ does not split over $F_{v_0}$. Then the localization map $\text{loc}_S$ of (3) is surjective.

For $p = 2$ this assertion was earlier proved by Prasad and Rapinchuk [19, Proposition 2(b)].

**1.5.** Let $G$ be a reductive group over a field $F$ of characteristic 0. In [5], the author defined the abelian group $H^1_{ab}(F, G)$ as a set in a canonical way as the Galois hypercohomology of a certain crossed module. However, the definition of the structure of abelian group on $H^1_{ab}(F, G)$ in [5] was complicated. In Appendix A, we define $H^1_{ab}(F, G)$ (in arbitrary characteristic) following the letter of Breen to the author [7] and the article by Noohi [17] (written at the author’s request), as the Galois hypercohomology $H^1(F, G_{ab})$ of a certain stable crossed module, that is, a crossed module endowed with a symmetric braiding. The structure of abelian group comes from the symmetric braiding. Note that our specific crossed module and specific symmetric braiding were constructed by Deligne [11].

In Appendix B, Zev Rosengarten shows that certain equivalences of crossed modules of algebraic groups over a field $F$ of arbitrary characteristic induce equivalences on $F_s$-points where $F_s$ is a separable closure of $F$. This permits us to use in Appendix A the Galois hypercohomology of these crossed modules rather than fppf hypercohomology.

**2. Notation**

**2.1.** Let $A$ be an abelian group. We denote by $A_{\text{Tors}}$ the torsion subgroup of $A$. We set $A_{\text{t.f.}} = A/A_{\text{Tors}}$, which is a torsion-free group.

**2.2.** Let $\Gamma$ be a finite group, and let $B$ be a $\Gamma$-module. We denote by $B_{\Gamma}$ the group of coinvariants of $\Gamma$ in $B$, that is,

$$B_{\Gamma} = B/\sum_{\gamma \in \Gamma} \left\{ \gamma^{-1}b_\gamma - b_\gamma \right\} \mid b_\gamma \in B.$$  

We write $B_{\Gamma, \text{Tors}} := (B_{\Gamma})_{\text{Tors}}$ (which is the torsion subgroup of $B_{\Gamma}$), $B_{\Gamma, \text{t.f.}} = B_{\Gamma}/B_{\Gamma, \text{Tors}}$ (which is a torsion-free group).

**2.3.** Let $G$ be a reductive group over a field $F$. Let $[G, G]$ denote the commutator subgroup of $G$, which is semisimple. Let $G_{\text{sc}}$ denote the universal cover of $[G, G]$, which is simply connected; see [3, Proposition (2.24)(ii)] or [10, Corollary A.4.11]. Following Deligne [11, Section 0.2], we consider the composite homomorphism

$$\rho : G_{\text{sc}} \to [G, G] \hookrightarrow G,$$

which in general is neither injective nor surjective.

For a maximal torus $T \subseteq G$, we write $T_{\text{sc}} = \rho^{-1}(T) \subseteq G_{\text{sc}}$ and consider the natural homomorphism

$$\rho : T_{\text{sc}} \to T.$$

We consider the algebraic fundamental group $M = \pi_1(G)$ of $G$ defined by

$$\pi_1(G) = X_*(T)/\rho_*X_*(T_{\text{sc}})$$

where $X_*$ denotes the cocharacter group. The Galois group $\text{Gal}(F_s/F)$ naturally acts on $M$, and the $\text{Gal}(F_s/F)$-module $M$ is well defined (does not depend on the choice of $T$ up to a transitive system of isomorphisms); see [5, Lemma 1.2].
2.4. From now on (except for the appendices), $F$ is a number field. We denote by $\mathcal{Y}(F)$, $\mathcal{Y}_f(F)$, $\mathcal{Y}_\infty(F)$, $\mathcal{Y}_\mathbb{R}(F)$, and $\mathcal{Y}_\mathbb{C}(F)$ the sets of all places of $F$, of finite places, of infinite places, of real places, and of complex places, respectively.

Let $E/F$ be a finite Galois extension of number fields with Galois group $\Gamma = \text{Gal}(E/F)$; then $\Gamma$ acts on $\mathcal{Y}(E)$. If $w \in \mathcal{Y}(E)$, we write $\Gamma_w$ for the stabilizer of $w$ in $\Gamma$; then $\Gamma_w \cong \text{Gal}(E_w/F_w)$ where $v \in \mathcal{Y}(F)$ is the restriction of $w$ to $F$.

3. Main theorem

In this section we state and prove Main Theorem 3.7 computing the images of the localization maps (2) and (3). We deduce Corollary 3.8 computing the group $\mathcal{Q}_1^g(F, G)$, and Corollary 3.9 giving a necessary and sufficient condition for the surjectivity of the localization map (3).

3.1. Let $G$ be a reductive group over a number field $F$, and let $v \in \mathcal{Y}(F)$ be a finite place of $F$. In [5] we computed $H^1_{ab}(F_v, G)$. Write $M = \pi(G)$. Let $E/F$ be a finite Galois extension in $\overline{F}$ such that $\text{Gal}(\overline{F}/E)$ acts on $M$ trivially and that $E$ has no real places. Write $\Gamma = \text{Gal}(E/F)$.

**Theorem 3.2 ([5, Proposition 4.1(i) and Corollary 5.4.1]).** With the notation and assumptions of Subsection 3.1, for any finite place $v$ of $F$ there is a canonical isomorphism of abelian groups

$$\alpha_{ab}^v : H^1_{ab}(F_v, G) \xrightarrow{\sim} M_{\Gamma_w,Tors}$$

where $w$ is a place of $E$ over $v$, and a canonical bijection

$$\alpha_{ab}^v : H^1(F_v, G) \xrightarrow{\sim} H^1_{ab}(F_v, G).$$

3.3. Let $v$ be a finite place of $F$. We have a surjective (even bijective) map

$$\alpha_v : H^1(F_v, G) \xrightarrow{\alpha_v} M_{\Gamma_w,Tors}.$$

We consider two composite maps with the same image

$$\lambda_{ab}^v : H^1_{ab}(F_v, G) \xrightarrow{\alpha_{ab}^v} M_{\Gamma_w,Tors} \xrightarrow{\omega_v} M_{\Gamma,Tors},$$

$$\lambda_v : H^1(F_v, G) \xrightarrow{\alpha_v} M_{\Gamma_w,Tors} \xrightarrow{\omega_v} M_{\Gamma,Tors},$$

where $\omega_v : M_{\Gamma_w,Tors} \rightarrow M_{\Gamma,Tors}$ is the homomorphism induced by the inclusion $\Gamma_w \rightarrow \Gamma$. Since the maps $\alpha_{ab}^v$ and $\alpha_v$ are surjective (even bijective), and $\omega_v$ is a homomorphism, we see that the set $\text{im} \lambda_{ab}^v = \text{im} \lambda_v$ is a subgroup of $M_{\Gamma,Tors}$, namely, $\text{im} \lambda_{ab}^v = \text{im} \lambda_v = \text{im} \omega_v$.

Let $v \in \mathcal{Y}_\mathbb{C}(F)$ be a complex place. We have zero maps

$$\lambda_{ab}^v : H^1_{ab}(F_v, G) = \{1\} \rightarrow \{0\} \subseteq M_{\Gamma,Tors}, \quad \lambda_v : H^1(F_v, G) = \{1\} \rightarrow \{0\} \subseteq M_{\Gamma,Tors}.$$

Clearly, the set $\text{im} \lambda_{ab}^v = \text{im} \lambda_v$ is a subgroup of $M_{\Gamma,Tors}$, namely, the subgroup $\{0\}$.

3.4. Let $v \in \mathcal{Y}_\mathbb{R}(F)$ be a real place; then $\Gamma_w$ is a group of order 2, $\Gamma_w = \{1, \gamma\}$ where $\gamma = \gamma_w$ induces the nontrivial automorphism of $E_w$ over $F_v$. We consider the Tate cohomology group

$$\hat{H}^{-1}(\Gamma_w, M) = \{m \in M \mid \gamma m = -m \} / \{m' - m \mid m' \in M\}.$$

We see immediately that the abelian group $\hat{H}^{-1}(\Gamma_w, M)$ naturally embeds into $M_{\Gamma_w}$. If $m \in M$ is a $(-1)$-cocycle, that is, $\gamma m = -m$, then $2m = m + m = -m$, whence $2 \cdot \hat{H}^{-1}(\Gamma_w, M) = 0$. We conclude that $\hat{H}^{-1}(\Gamma_w, M)$ naturally embeds into $M_{\Gamma_w,Tors}$.

There is a canonical surjective map of Kottwitz [13, Theorem 1.2] (see also [5, Theorem 5.4])

$$\text{ab}_v : H^1(F_v, G) \twoheadrightarrow H^1_{ab}(F_v, G),$$

a canonical isomorphism of [6, Proposition 8.21]

$$H^1_{ab}(F_v, G) \xrightarrow{\sim} \hat{H}^{-1}(\Gamma_w, M),$$
and a canonical embedding
\[ \widehat{H}^{-1}(\Gamma_w, M) \hookrightarrow M_{\Gamma_w, \text{Tors}}. \]

Thus we obtain composite maps
\[ \alpha^\text{ab}_\nu : H^1_{\text{ab}}(F_\nu, G) \xrightarrow{\alpha^\text{ab}_\nu} \widehat{H}^{-1}(\Gamma_w, M) \xrightarrow{\mu} M_{\Gamma_w, \text{Tors}}, \]
\[ \alpha_\nu : H^1(F_\nu, G) \xrightarrow{\alpha_\nu} H^1_{\text{ab}}(F_\nu, G) \xrightarrow{\alpha^\text{ab}_\nu} \widehat{H}^{-1}(\Gamma_w, M) \xrightarrow{\mu} M_{\Gamma_w, \text{Tors}}, \]

with the same image \( \text{im} \alpha^\text{ab}_\nu = \text{im} \alpha_\nu \), which is a subgroup of \( M_{\Gamma_w, \text{Tors}} \). Consider the composite maps with the same image
\[ \lambda^\text{ab}_\nu : H^1_{\text{ab}}(F_\nu, G) \xrightarrow{\alpha^\text{ab}_\nu} M_{\Gamma_w, \text{Tors}} \xrightarrow{\mu} M_{\Gamma, \text{Tors}}, \]
\[ \lambda_\nu : H^1(F_\nu, G) \xrightarrow{\alpha_\nu} \lambda^\text{ab}_\nu \xrightarrow{\mu} M_{\Gamma, \text{Tors}}. \]

Since the set \( \text{im} \alpha^\text{ab}_\nu = \text{im} \alpha_\nu \) is a subgroup of \( M_{\Gamma_w, \text{Tors}} \), and \( \omega_\nu \) is a homomorphism, we conclude that the set \( \text{im} \lambda^\text{ab}_\nu = \text{im} \lambda_\nu \) is a subgroup of \( M_{\Gamma, \text{Tors}} \), namely, \( \text{im} \lambda^\text{ab}_\nu = \text{im} \lambda_\nu = \omega_\nu \left( \widehat{H}^{-1}(\Gamma_w, M) \right) \).

**Lemma 3.5.** Let \( S \subseteq \mathcal{V}(F) \) be any subset, finite or infinite. Consider the summation maps
\[ \Sigma_S^\text{ab} : \bigoplus_{v \in S} H^1_{\text{ab}}(F_v, G) \xrightarrow{\Sigma_S} M_{\Gamma, \text{Tors}}, \quad \xi_S^\text{ab} = (\xi_v^\text{ab})_{v \in S} = \sum_{v \in S} \lambda^\text{ab}_v(\xi_v^\text{ab}), \]
\[ \Sigma_S : \bigoplus_{v \in S} H^1(F_v, G) \xrightarrow{\Sigma_S} M_{\Gamma, \text{Tors}}, \quad \xi_S = (\xi_v^\text{ab})_{v \in S} = \sum_{v \in S} \lambda_\nu(\xi_v^\text{ab}). \]

Then the sets \( \text{im} \Sigma_S^\text{ab} \) and \( \text{im} \Sigma_S \) are subgroups of \( M_{\Gamma, \text{Tors}} \), and they are equal.

**Proof.** Indeed, we have
\[ \text{im} \Sigma_S^\text{ab} = \text{im} \Sigma_S = \langle \text{im} \lambda_\nu \rangle_{v \in S} \quad \text{where} \quad \text{im} \lambda_\nu = \begin{cases} \text{im} \omega_\nu & \text{if } v \in \mathcal{V}_\text{f}(F), \\ \omega_\nu \left( \widehat{H}^{-1}(\Gamma_w, M) \right) & \text{if } v \in \mathcal{V}_\text{c}(F), \\ 0 & \text{if } v \in \mathcal{V}_\text{c}(F). \end{cases} \]

Here we write \( \langle \text{im} \lambda_\nu \rangle_{v \in S} \) for the subgroup of \( M_{\Gamma, \text{Tors}} \) generated by the subgroups \( \text{im} \lambda_\nu \) for \( v \in S \).

**Theorem 3.6.** The following sequences are exact:
\[ H^1_{\text{ab}}(F, G) \xrightarrow{\text{loc}_{\mathcal{V}}} \bigoplus_{v \in \mathcal{V}} H^1_{\text{ab}}(F_v, G) \xrightarrow{\Sigma_{\mathcal{V}}} M_{\Gamma, \text{Tors}}, \quad (4) \]
\[ H^1(F, G) \xrightarrow{\text{loc}_{\mathcal{V}}} \bigoplus_{v \in \mathcal{V}} H^1(F_v, G) \xrightarrow{\Sigma_{\mathcal{V}}} M_{\Gamma, \text{Tors}}, \quad (5) \]

where for brevity we write \( \mathcal{V} \) for \( \mathcal{V}(F) \).

**Proof.** In view of [5, Proposition 4.8], exact sequence (4) is actually a part of the exact sequence [5, (4.3.1)]. For (5), see [13, Proposition 2.6] or [5, Theorem 5.15].

**Main Theorem 3.7.** Let \( G \) be a reductive group over a number field \( F \). Let \( S \subseteq \mathcal{V} := \mathcal{V}(F) \) be a subset. Write \( S^\text{c} = \mathcal{V} \setminus S \), the complement of \( S \) in \( \mathcal{V} \). Then:
\[ \text{im} \text{loc}_S^\text{ab} = \left\{ \xi_S^\text{ab} \in \bigoplus_{v \in S} H^1_{\text{ab}}(F_v, G) \left| \Sigma_S^\text{ab}(\xi_S^\text{ab}) \in \text{im} \Sigma_S^\text{ab} \cap \text{im} \Sigma_S^\text{c} \right. \right\}, \quad (6) \]
\[ \text{im} \text{loc}_S = \left\{ \xi_S \in \bigoplus_{v \in S} H^1(F_v, G) \left| \Sigma_S(\xi_S) \in \text{im} \Sigma_S \cap \text{im} \Sigma_S^\text{c} \right. \right\}. \quad (7) \]
**Proof.** By Lemma 3.5, the sets $\text{im} \Sigma^{ab}_S$ and $\text{im} \Sigma^{\emptyset}_S$ are (equal) subgroups of $M_{1, \text{Tors}}$, and therefore it suffices to prove (6) with $(-\text{im} \Sigma^{ab}_S)$ instead of $\text{im} \Sigma^{ab}_S$, and to prove (7) with $(-\text{im} \Sigma^{\emptyset}_S)$ instead of $\text{im} \Sigma^{\emptyset}_S$. Now the corresponding assertions follow easily from the exactness of (4) and (5), respectively.

For the reader’s convenience, we provide an easy proof of (7) with $(-\text{im} \Sigma^{\emptyset}_S)$ instead of $\text{im} \Sigma^{\emptyset}_S$. Let

$$\eta_S = \sum_{v \in S} \left( \xi_v \right)_{v \in S} \in \text{im} \text{loc}_S \subseteq \bigoplus_{v \in S} H^1(F_v, G),$$

that is, $\eta_S = \text{loc}_S(\xi)$ for some $\xi \in H^1(F, G)$. Write $\eta_S^{\emptyset} = \left( \eta_v \right)_{v \in S^{\emptyset}} = \text{loc}^{\emptyset}_S(\xi)$. Since the sequence (5) is exact, we have $\left( \Sigma^r \circ \text{loc}^r \right)(\xi) = 0$, whence

$$\Sigma_S(\Sigma_S(\xi)) + \Sigma^{\emptyset}_S(\eta^{\emptyset}_S) = 0 \quad \text{and} \quad \Sigma_S(\xi) = -\Sigma^{\emptyset}_S(\eta^{\emptyset}_S).$$

We conclude that $\Sigma_S(\xi) \in \text{im} \Sigma_S \cap (-\text{im} \Sigma^{\emptyset}_S)$, as required.

Conversely, let an element $\xi_S = \left( \xi_v \right)_{v \in S} \in \bigoplus_{v \in S} H^1(F_v, G)$ be such that $\Sigma_S(\xi_S) \in \text{im} \Sigma_S \cap (-\text{im} \Sigma^{\emptyset}_S)$. Write $a = \Sigma_S(\xi_S)$. Then $-a \in \text{im} \Sigma^{\emptyset}_S$, that is,

$$-a = \Sigma^{\emptyset}_S(\eta^{\emptyset}_S) \quad \text{for some} \quad \eta^{\emptyset}_S = \left( \eta_v \right)_{v \in S^{\emptyset}} \in \bigoplus_{v \in S^{\emptyset}} H^1(F_v, G).$$

Define

$$\xi' = \left( \xi_v \right)_{v \in S'}, \quad \xi_v = \begin{cases} \xi_v & \text{if } v \in S, \\ \eta_v & \text{if } v \in S^{\emptyset}. \end{cases}$$

Then

$$\Sigma_{S'}(\xi') = a + (-a) = 0.$$

Since the sequence (5) is exact, we have $\xi' = \text{loc}_{S'}(\xi)$ for some $\xi \in H^1(F, G)$. Then $\text{loc}_S(\xi) = \xi_S$, whence $\xi_S \in \text{im} \text{loc}_S$, as required. \hfill \Box

**Corollary 3.8.** The homomorphism

$$\chi^{ab}_S : \bigoplus_{v \in S'} H^1(F_v, G) \xrightarrow{\Sigma^{ab}} \text{im} \Sigma^{ab}_S \longrightarrow \text{im} \Sigma^{ab}_S / (\text{im} \Sigma^{ab}_S \cap \text{im} \Sigma^{ab}_{S^{\emptyset}})$$

induces a canonical isomorphism

$$\Psi^1_S(F, G) \xrightarrow{\cong} \text{im} \Sigma^{ab}_S / (\text{im} \Sigma^{ab}_S \cap \text{im} \Sigma^{ab}_{S^{\emptyset}}).$$

**Proof.** The homomorphism $\chi^{ab}_S$ is clearly surjective, and by Theorem 3.7 its kernel is the image $\text{im} \text{loc}^{ab}_S$ of the localization homomorphism $\text{loc}^{ab}_S$ of (2). The corollary follows. \hfill \Box

**Corollary 3.9.** The localization map $\text{loc}_S$ of (3) is surjective if and only if

$$\text{im} \Sigma_S \subseteq \text{im} \Sigma^{\emptyset}_S.$$ \hfill (8)

**Proof.** Consider the map

$$\chi_S : \bigoplus_{v \in S} H^1(F_v, G) \xrightarrow{\Sigma} \text{im} \Sigma_S \longrightarrow \text{im} \Sigma_S / (\text{im} \Sigma_S \cap \text{im} \Sigma^{\emptyset}_S).$$

By Lemma 3.5 the sets $\text{im} \Sigma_S = \text{im} \Sigma^{ab}_S$ and $\text{im} \Sigma_S \cap \text{im} \Sigma^{\emptyset}_S = \text{im} \Sigma^{ab}_S \cap \text{im} \Sigma^{ab}_{S^{\emptyset}}$ are abelian groups. The morphism of pointed sets $\chi_S$ is clearly surjective, and by Theorem 3.7 its kernel is $\text{im} \text{loc}_S$. We see that the following assertions are equivalent:

(a) the map $\text{loc}_S$ is surjective, that is, $\text{im} \text{loc}_S = \bigoplus_{v \in S} H^1(F_v, G)$;

(b) $\ker \chi_S = \bigoplus_{v \in S} H^1(F_v, G)$;

(c) $\#(\text{im} \chi_S) = 1$;

(d) $\text{im} \Sigma_S \cap \text{im} \Sigma^{\emptyset}_S = \text{im} \Sigma_S$;

(e) $\text{im} \Sigma S = \text{im} \Sigma^{ab}_S$.
Remark 3.10. Since by Lemma 3.5 we have \( \text{im} \Sigma_S^\text{ab} = \text{im} \Sigma_S \) and \( \text{im} \Sigma_S^\text{ab} = \text{im} \Sigma_S^\text{c} \), we see from (d) in the proof above and from Corollary 3.8 that the localization map \( \text{loc}_S \) of (3) is surjective if and only if \( \Upsilon^1_S(F, G) = \{1\} \).

Corollary 3.11. Let \( v_0 \in \mathcal{V}(F) \), \( S = \mathcal{V}(F) \setminus \{v_0\} \). Then the localization map \( \text{loc}_S \) of (3) is surjective if and only if

\[
\text{im} \lambda_v \subseteq \text{im} \lambda_{v_0} \quad \text{for all } v \in \mathcal{V}(F).
\]

Proof. Indeed, in our case condition (9) is equivalent to (8), and we conclude by Corollary 3.9.

Corollary 3.12. For a subset \( S \subset \mathcal{V}(F) \), let \( v_0 \in S^\text{c} \), and assume that

\[
\text{im} \lambda_v \subseteq \text{im} \lambda_{v_0} \quad \text{for all } v \in S.
\]

Then the localization map \( \text{loc}_S \) of (3) is surjective.

Proof. Indeed, (10) implies (8), and we conclude by Corollary 3.9.

Corollary 3.13. Let \( v_0 \in S^\text{c} \), and assume that the map \( \lambda_{v_0} : H^1(F_{v_0}, G) \to M_{\Gamma, \text{Tors}} \) is surjective. Then the localization map \( \text{loc}_S \) of (3) is surjective.

Proof. Indeed, then

\[
\text{im} \Sigma_S \subseteq M_{\Gamma, \text{Tors}} = \text{im} \lambda_{v_0} \subseteq \text{im} \Sigma_S^\text{c},
\]

and we conclude by Corollary 3.9.

Proposition 3.14 (Borel and Harder [2, Theorem 1.7]). Let \( G \) be a semisimple group over a number field \( F \), and let \( S \subset \mathcal{V}(F) \) be a subset such that the complement \( S^\text{c} \) of \( S \) contains a finite place \( v_0 \in \mathcal{V}_f(F) \). Then the localization map \( \text{loc}_S \) of (3) is surjective.

Proof. Since \( G \) is semisimple, the \( \Gamma \)-module \( M \) is finite, and so are the groups \( M_{\Gamma} \) and \( M_{\Gamma_w} \) where \( w \) is a place of \( E \) over \( v_0 \). It follows that

\[
M_{\Gamma_w, \text{Tors}} = M_{\Gamma_w} \quad \text{and} \quad M_{\Gamma, \text{Tors}} = M_{\Gamma}.
\]

The natural homomorphism \( M_{\Gamma_w} \to M_{\Gamma} \) is clearly surjective. Therefore, the homomorphism

\[
\omega_{v_0} : M_{\Gamma_w, \text{Tors}} = M_{\Gamma_w} \to M_{\Gamma} = M_{\Gamma, \text{Tors}}
\]

is surjective. Since \( v_0 \) is finite, we have \( \text{im} \lambda_{v_0} = \text{im} \omega_{v_0} \), whence the map \( \lambda_{v_0} \) is surjective. We conclude by Corollary 3.13.

4. Exact sequence

In this section we construct an exact sequence that we shall use in Section 5.

Theorem 4.1. A finite group \( \Gamma \) and a short exact sequence of \( \Gamma \)-modules

\[
0 \to B_1 \overset{i}{\longrightarrow} B_2 \overset{j}{\longrightarrow} B_3 \to 0
\]

give rises to an exact sequence

\[
(B_1)_{\Gamma, \text{Tors}} \overset{i_*}{\longrightarrow} (B_2)_{\Gamma, \text{Tors}} \overset{j_*}{\longrightarrow} (B_3)_{\Gamma, \text{Tors}} \overset{\delta}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (B_1)_{\Gamma} \overset{i_*}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (B_2)_{\Gamma} \overset{j_*}{\longrightarrow} \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (B_3)_{\Gamma} \to 0
\]

depending functorially on \( \Gamma \) and on the sequence (11).
4.2. We specify the homomorphism $\delta$. Let $x_3 \in B_3$ be such that the image $(x_3)_\Gamma$ of $x_3$ in $(B_3)_\Gamma$ is contained in $(B_3)_{\Gamma, \text{Tors}}$. This means that there exist $n \in \mathbb{Z}_{>0}$ and $y_{3, \gamma} \in B_3$ such that

$$nx_3 = \sum_{\gamma \in \Gamma} \left( y_{3, \gamma} - y_{3, \gamma} \right).$$

We lift $x_3$ to some $x_2 \in B_2$, we lift each $y_{3, \gamma}$ to some $y_{2, \gamma} \in B_2$, and we consider the element

$$z_2 = nx_2 - \sum_{\gamma \in \Gamma} \left( y_{2, \gamma} - y_{2, \gamma} \right).$$

Then $j(z_2) = 0 \in B_3$, whence $z_2 = \delta(z_1)$ for some $z_1 \in B_1$. We consider the image $(z_1)_{\Gamma, \text{t.f.}}$ of $z_1 \in B_1$ in $(B_1)_{\Gamma, \text{t.f.}}$, and we put

$$\delta((x_3)_\Gamma) = \frac{1}{n} \otimes (z_1)_{\Gamma, \text{t.f.}} \in \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (B_1)_{\Gamma, \text{t.f.}} = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (B_1)_\Gamma$$

where we write $\frac{1}{n}$ for the image in $\mathbb{Q}/\mathbb{Z}$ of $\frac{1}{n} \in \mathbb{Q}$.

Below we give the proof of Theorem 4.1 suggested by Vladimir Hinich (private communication). For another proof, due to Alexander Petrov, see [18].

**Proof of Theorem 4.1 due to Vladimir Hinich.** The functor from the category of $\Gamma$-modules to the category of abelian groups

$$B \mapsto \mathbb{Q}/\mathbb{Z} \otimes_{\Lambda} B_{\Gamma}$$

is the same as

$$B \mapsto \mathbb{Q}/\mathbb{Z} \otimes_{\Lambda} B$$

where $\Lambda = \mathbb{Z}[\Gamma]$ is the group ring of $\Gamma$. From the short exact sequence of $\Gamma$-modules (11), we obtain a long exact sequence

$$\cdots \to \text{Tor}^1_\Lambda(\mathbb{Q}/\mathbb{Z}, B_1) \xrightarrow{i_*} \text{Tor}^1_\Lambda(\mathbb{Q}/\mathbb{Z}, B_2) \xrightarrow{j_*} \text{Tor}^1_\Lambda(\mathbb{Q}/\mathbb{Z}, B_3) \xrightarrow{\delta} \mathbb{Q}/\mathbb{Z} \otimes_{\Lambda} B_1 \xrightarrow{i_*} \mathbb{Q}/\mathbb{Z} \otimes_{\Lambda} B_2 \xrightarrow{j_*} \mathbb{Q}/\mathbb{Z} \otimes_{\Lambda} B_3 \to 0$$

depending functorially on $\Gamma$ and on (11); see Weibel [21]. Now Theorem 4.1 follows from the next proposition.

**Proposition 4.3.** For a finite group $\Gamma$ and a $\Gamma$-module $B$, there is a canonical and functorial isomorphism

$$\text{Tor}^1_\Lambda(\mathbb{Q}/\mathbb{Z}, B) \cong B_{\Gamma, \text{Tors}}$$

where $\Lambda = \mathbb{Z}[\Gamma]$.

**Proof.** Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

regarded as a short exact sequence of $\Gamma$-modules with trivial action of $\Gamma$. Tensoring with $B$, we obtain a long exact sequence

$$\cdots \to \text{Tor}^1_\Lambda(\mathbb{Q}, B) \to \text{Tor}^1_\Lambda(\mathbb{Q}/\mathbb{Z}, B) \to \mathbb{Z} \otimes_{\Lambda} B \to \mathbb{Q} \otimes_{\Lambda} B \to \mathbb{Q}/\mathbb{Z} \otimes_{\Lambda} B \to 0. \tag{13}$$

We have canonical isomorphisms

$$\mathbb{Z} \otimes_{\Lambda} B = B_{\Gamma} \quad \text{and} \quad \ker \left( \mathbb{Z} \otimes_{\Lambda} B \to \mathbb{Q} \otimes_{\Lambda} B \right) = B_{\Gamma, \text{Tors}}.$$

By Lemma 4.4 below, we have $\text{Tor}^1_\Lambda(\mathbb{Q}, B) = 0$, and the proposition follows from (13).

**Lemma 4.4.** For a finite group $\Gamma$ and any $\Gamma$-module $B$, we have

$$\text{Tor}^1_\Lambda(\mathbb{Q}, B) = 0$$

where $\Lambda = \mathbb{Z}[\Gamma]$.
Proof. Let
\[ P_\ast : \cdots \to P_2 \to P_1 \to P_0 \to Z \to 0 \]
be a \( \Lambda \)-free resolution of the trivial \( \Gamma \)-module \( Z \), for example, the standard complex; see Atiyah and Wall [1, Section 2]. Tensoring with \( Q \otimes Z \), we obtain a flat resolution of \( Q \)
\[ \cdots \to Q \otimes Z P_2 \to Q \otimes Z P_1 \to Q \otimes Z P_0 \to Q \to 0. \]

Tensoring with \( B \) over \( \Lambda = \mathbb{Z}[\Gamma] \), we obtain the complex \( (Q \otimes Z P_\ast) \otimes_\Lambda B \):
\[ \cdots \to (Q \otimes Z P_2) \otimes_\Lambda B \to (Q \otimes Z P_1) \otimes_\Lambda B \to (Q \otimes Z P_0) \otimes_\Lambda B \to Q \otimes_\Lambda B \to 0. \quad (14) \]
By definition, \( \text{Tor}_1^\Lambda(Q, B) \) is the first homology group of this complex.

However, we can obtain the complex (14) from \( P_\ast \) by tensoring first with \( B \) over \( \Lambda \), and after that with \( Q \) over \( Z \):
\[ Q \otimes_\Lambda (P_\ast \otimes_\Lambda B) \cong (Q \otimes Z P_\ast) \otimes_\Lambda B. \]
Since \( Q \) is a flat \( Z \)-module, we obtain canonical isomorphisms
\[ \text{Tor}_1^\Lambda(Q, B) \cong Q \otimes_\Lambda \text{Tor}_1^\Lambda(Z, B) = Q \otimes_\Lambda H_1(\Gamma, B). \]

Now, since the group \( \Gamma \) is finite, the abelian group \( H_1(\Gamma, B) \) is killed by multiplication by \#\( \Gamma \); see, for instance, Atiyah and Wall [1, Section 6, Corollary 1 of Proposition 8]. It follows that \( Q \otimes_\Lambda H_1(\Gamma, B) = 0 \). Thus \( \text{Tor}_1^\Lambda(Q, B) = 0 \), which completes the proofs of Lemma 4.4, Proposition 4.3, and Theorem 4.1. \( \square \)

Alternatively, one can check directly that the map \( \delta \) constructed in Subsection 4.2 is well-defined (does not depend on the choices made) and that the sequence (12) is exact.

5. Surjectivity for a reductive group with nice radical

In this section we prove the following theorem that gives a sufficient condition for the surjectivity of the localization map (3) for a reductive \( F \)-group \( G \) in terms of the radical (largest central torus) of \( G \).

Theorem 5.1. Let \( G \) be a reductive group over a number field \( F \), and let \( C \) denote the radical of \( G \). Write \( \overline{G} = G/C \), which is a semisimple group, and consider the short exact sequence of fundamental groups [5, Lemma 1.5]
\[ 0 \to M_C \to M \to \overline{M} \to 0 \]
where
\[ M_C = \pi_1(C) = X_*(C), \quad M = \pi_1(G), \quad \overline{M} = \pi_1(\overline{G}). \]
We define \( \Gamma = \text{Gal}(E/F) \) for \( M \) as in Subsection 3.1. Let \( S \subset \mathcal{V}(F) \) be a subset, and assume that \( S \) contains a finite place \( v_0 \) such that
\[ \mathrm{im} \{ \Gamma_{w} \to \text{Aut} M_C \} = \mathrm{im} \{ \Gamma \to \text{Aut} M_C \} \quad (15) \]
where \( w \) is a place of \( E \) over \( v_0 \). Then the localization map \( \text{loc}_S \) of (3) is surjective.

Proof. It follows from (15) that \( (M_C)_{\Gamma_w} = (M_C)_{\Gamma} \), whence
\[ (M_C)_{\Gamma_{w}, \text{Tors}} = (M_C)_{\Gamma, \text{Tors}} \quad \text{and} \quad Q/\mathbb{Z} \otimes Z (M_C)_{\Gamma_w} = Q/\mathbb{Z} \otimes Z (M_C)_{\Gamma}. \]
Using Theorem 4.1, we construct an exact commutative diagram
\[
\begin{array}{cccccc}
(M_C)_{\Gamma_{w}, \text{Tors}} & \longrightarrow & M_{\Gamma_{w}, \text{Tors}} & \longrightarrow & \overline{M}_{\Gamma_{w}, \text{Tors}} & \longrightarrow & Q/\mathbb{Z} \otimes Z (M_C)_{\Gamma_w} \\
\downarrow \omega & & \downarrow & & \downarrow & & \\
(M_C)_{\Gamma, \text{Tors}} & \longrightarrow & M_{\Gamma, \text{Tors}} & \longrightarrow & \overline{M}_{\Gamma, \text{Tors}} & \longrightarrow & Q/\mathbb{Z} \otimes Z (M_C)_{\Gamma} 
\end{array}
\]
Since $G$ is semisimple, its algebraic fundamental group $\tilde{M}$ is finite, and therefore the homomorphism $\tilde{\omega}$ in the diagram above is surjective; see the proof of Proposition 3.14. By a four lemma, the homomorphism
\[ \omega = \omega_{v_0} : M_{\Gamma,\text{Tors}} \to M_{\Gamma,\text{Tors}} \]
is surjective as well. Since $v_0$ is finite, the map
\[ \alpha_{v_0} : H^1(F_{v_0}, G) \to H^1_{\text{ab}}(F_{v_0}, G) \to M_{\Gamma,\text{Tors}} \]
is bijective, and therefore the map
\[ \lambda_{v_0} : H^1(F_{v_0}, G) \to H^1_{\text{ab}}(F_{v_0}, G) \to M_{\Gamma,\text{Tors}} \to M_{\Gamma,\text{Tors}} \]
is surjective. We conclude by Corollary 3.13.

Corollary 5.2 (Prasad and Rapinchuk [19, Proposition 2(a)]). Let $G$ be a reductive group over a number field $F$, and let $C$ denote the radical of $G$. Assume that the $F$-torus $C$ is split and that $S^G$ contains a finite place $v_0$. Then the localization map $\text{loc}_S$ of (3) is surjective.

Proof. We define $E$, $\Gamma$, and $\Gamma_w$ for $M = \pi_1(G)$ as in Subsection 3.1. Then $\text{im} [\Gamma \to \text{Aut} M_C] = \{1\}$, and hence (15) holds. We conclude by Theorem 5.1.

Proof of Corollary 1.4. We define $E$, $\Gamma$, and $\Gamma_w$ for $M = \pi_1(G)$ as in Subsection 3.1. We have
\[ \text{im} [\Gamma_w : \text{Aut} M_C] \subseteq \text{im} [\Gamma : \text{Aut} M_C], \quad \# \text{im} [\Gamma_w : \text{Aut} M_C] \neq 1. \]
It follows that (15) holds. We conclude by Theorem 5.1.

Appendix A. Abelianization

A.1. Let $G$ be a reductive group over a field $F$ of arbitrary characteristic. We consider the homomorphism $\rho : G^{sc} \to G$ of Subsection 2.3.

The group $G$ acts by conjugation on itself on the left, and by functoriality $G$ acts on $G^{sc}$. We obtain an action
\[ \theta : G \times G^{sc} \to G^{sc}, \quad (g, s) \mapsto \overline{g} s. \]
On $\overline{F}$-points, if $s \in G^{sc}(\overline{F})$, $g_1 \in G(\overline{F})$, $g_1 = \rho(s_1) \cdot z_1$ with $s_1 \in G^{sc}(\overline{F})$, $z_1 \in Z_G(\overline{F})$, then
\[ \theta(g_1, s) = \overline{g_1}s = s_1\overline{s_1}^{-1}. \]
Since the groups $G$ and $G^{sc}$ are smooth, this formula uniquely determines $\theta$. The action $\theta$ has the following properties:
\[ \rho(s s') = s s' s^{-1}, \]
\[ \rho(g_1 s') = g_1 \rho(s') g_1^{-1} \]
for $g_1 \in G(\overline{F})$, $s, s' \in G^{sc}(\overline{F})$. In other words, $(G^{sc}, G, \rho, \theta)$ is a (left) crossed module of algebraic groups; see for instance [5, Definition 3.2.1]. We write it as $(G^{sc} \xrightarrow{\rho} G, \theta)$, and we regard it as a complex in degrees $-1, 0$. On $F$-points we obtain a $\text{Gal}(F_s/F)$-equivariant crossed module $(G^{sc}(F_s) \xrightarrow{\rho} G(F_s), \theta)$ where $F_s$ is the separable closure of $F$ in $\overline{F}$.

A.2. Deligne [11, Section 2.0.2] noticed that the commutator map
\[ \{\cdot, \cdot\} : G \times G \to G, \quad g_1, g_2 \mapsto [g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1} \]
lifts to a certain map (morphism of $F$-varieties)
\[ \{\cdot, \cdot\} : G \times G \to G^{sc}, \quad g_1, g_2 \mapsto \{g_1, g_2\} \]
as follows. The commutator map
\[ G^{sc} \times G^{sc} \to G^{sc}, \quad s_1, s_2 \mapsto [s_1, s_2] := s_1 s_2 s_1^{-1} s_2^{-1} \]
clearly factors via a morphism of $F$-varieties

$$(G^{sc})^{\text{ad}} \times (G^{sc})^{\text{ad}} \to G^{sc}$$

where $(G^{sc})^{\text{ad}} = G^{sc}/Z_{G^{sc}}$ and $Z_{G^{sc}}$ denotes the center of $G^{sc}$. Identifying $(G^{sc})^{\text{ad}}$ with $G^{ad} := G/Z_G$, we obtain the desired morphism of $F$-varieties

$$\{\cdot, \cdot\} : G \times G \to G^{ad} \times G^{ad} \to G^{sc}.$$  

On $F$-points, if $g_1, g_2 \in G(F)$, $g_1 = \rho(s_1)z_1$, $g_2 = \rho(s_2)z_2$ where $s_1, s_2 \in G^{sc}(\overline{F})$, $z_1, z_2 \in Z_G(\overline{F})$, then

$$\{g_1, g_2\} = [s_1, s_2] = s_1s_2^{-1}s_1^{-1}.$$  

Since $G$ and $G^{sc}$ are smooth, this formula uniquely determines $\{\cdot, \cdot\}$. The constructed map $\{\cdot, \cdot\}$ satisfies the following equalities of Conduché [9, (3.11)]:

$$\rho([g_1, g_2]) = [g_1, g_2];$$

$$\{\rho(s_1), \rho(s_2)\} = [s_1, s_2];$$

$$\{g_1, g_2\} = (g_2, g_1)^{-1};$$

$$\{g_1g_2, g_3\} = (g_1g_2g_1^{-1}, g_1g_3g_1^{-1})\{g_1, g_3\}.$$  

In other words, the map $\{\cdot\}$ is a symmetric braiding of the crossed module $(G^{sc}, G, \rho, \theta)$. We denote by $G_{ab}$ the corresponding stable (=symmetrically braided) crossed module:

$$G_{ab} = \{G^{sc} \overset{\rho}{\to} G, \theta, \{\cdot, \cdot\}\}.$$  

Let $\varphi : G \to H$ be a homomorphism of reductive $F$-groups. It induces a homomorphism $\varphi^{sc} : G^{sc} \to H^{sc}$. It is easy to see that

$$\varphi^{sc}(g)\varphi^{sc}(s) = \varphi^{sc}(gs) \quad \text{for all} \quad g \in G(F), s \in G^{sc}(\overline{F}).$$  

Thus we obtain a morphism of crossed modules

$$(G^{sc} \to G, \theta_G) \to (H^{sc} \to H, \theta_H)$$

with obvious notations. Moreover, we have

$$\{\varphi(g_1), \varphi(g_2)\}_H = \varphi^{sc}(\{g_1, g_2\}_G) \quad \text{for all} \quad g_1, g_2 \in G(F)$$

with obvious notations; see [4] for a proof. Thus we obtain a morphism of stable crossed modules

$$(G^{sc} \to G, \theta_G, \{\cdot, \cdot\}_G) \to (H^{sc} \to H, \theta_H, \{\cdot, \cdot\}_H). \quad (16)$$  

A.3. In this appendix, we denote by $H^1$ and $\mathbb{H}^1$ the first Galois cohomology and hypercohomology. One can define the first Galois (hyper)cohomology of the $\text{Gal}(F_s/F)$-equivariant crossed module

$$\mathbb{H}^1(F, G^{sc} \overset{\rho}{\to} G, \theta) := \mathbb{H}^1(\text{Gal}(F_s/F), G^{sc}(F_s) \overset{\rho}{\to} G(F_s), \theta); \quad (17)$$

see [5, Section 3] or Noohi [17, Section 4]. A priori it is just a pointed set. However, using the symmetric braiding $\{\cdot, \cdot\}$, one can define a structure of abelian group on the pointed set (17); see Noohi [17, Corollaries 4.2 and 4.5]. We denote the obtained abelian group by

$$H^1_{ab}(F, G) = \mathbb{H}^1(F, G_{ab}) := \mathbb{H}^1(\text{Gal}(F_s/F), G^{sc}(F_s) \overset{\rho}{\to} G(F_s), \theta, \{\cdot, \cdot\}).$$

A homomorphism of reductive $F$-groups $\varphi : G \to H$ induces a morphism of stable crossed modules (16), which in turn induces a homomorphism of abelian groups

$$\varphi_{ab} : H^1_{ab}(F, G) \to H^1_{ab}(F, H).$$

Thus $G \rightsquigarrow H^1_{ab}(F, G)$ is a functor from the category of reductive $F$-group to the category of abelian groups.
A.4. The morphism of crossed modules (but not of stable crossed modules)

$$i_G: (1 \to G) \to (G_{sc}^{\text{fppf}} \to G)$$

induces a morphism of pointed sets

$$(i_G)_*: \mathbb{H}^1(F, 1 \to G) \to \mathbb{H}^1(F, G_{sc}^{\text{fppf}} \to G).$$

The abelianization map is the composite morphism of pointed sets

$$\text{ab}: H^1(F, G) = \mathbb{H}^1(F, 1 \to G) \xrightarrow{(i_G)_*} \mathbb{H}^1(F, G_{sc}^{\text{fppf}} \to G, \theta) = \mathbb{H}^1(F, G_{ab}) =: H^1_{ab}(F, G).$$

Here $\mathbb{H}^1(F, G_{sc}^{\text{fppf}} \to G, \theta)$ and $\mathbb{H}^1(F, G_{ab})$ are the same sets, but $\mathbb{H}^1(F, G_{ab})$ is endowed with the structure of abelian group coming from the symmetric braiding $\{\cdot, \cdot\}$.

A.5. For a maximal torus $T \subseteq G$, we consider the homomorphism

$$\rho: T_{sc} \to T$$

of Subsection 2.3, which we regard as a stable crossed module with the trivial action $\theta_T$ of $T$ on $T_{sc}$ and the trivial symmetric braiding $\{\cdot, \cdot\}_T: T \times T \to T_{sc}$. We may and shall identify the first Galois hypercohomology of this stable crossed module with the usual first Galois hypercohomology of the complex $T_{sc} \xrightarrow{\rho} T$ in degrees $-1, 0$:

$$\mathbb{H}^1(F, T_{sc} \xrightarrow{\rho} T, \theta_T, \{\cdot, \cdot\}_T) = \mathbb{H}^1(F, T_{sc} \xrightarrow{\rho} T).$$

The morphism of stable crossed modules

$$j_T: \left(T_{sc} \xrightarrow{\rho} T, \theta_T, \{\cdot, \cdot\}_T\right) \to \left(G_{sc}^{\text{fppf}} \xrightarrow{\rho} G, \theta, \{\cdot, \cdot\}\right)$$

is an equivalence (quasi-isomorphism), that is, it induces isomorphisms of $F$-group schemes

$$\ker[T_{sc} \to T] \xrightarrow{\sim} \ker[G_{sc} \to G] \quad \text{and} \quad \coker[T_{sc} \to T] \xrightarrow{\sim} \coker[G_{sc} \to G].$$

Following an idea sketched by Labesse and Lemaire [15], we observe that (18) induces isomorphisms on groups of $F$-points

$$\ker[T_{sc}(F) \to T(F)] \xrightarrow{\sim} \ker[G_{sc}(F) \to G(F)]$$

$$\coker[T_{sc}(F) \to T(F)] \xrightarrow{\sim} \coker[G_{sc}(F) \to G(F)]$$

(in arbitrary characteristic); see Theorem B.1 in Appendix B below. It follows that the induced map on Galois hypercohomology

$$(j_T)_*: \mathbb{H}^1(F, T_{sc} \to T) \to \mathbb{H}^1(F, G_{sc}^{\text{fppf}} \xrightarrow{\rho} G, \theta, \{\cdot, \cdot\}) =: H^1_{ab}(F, G)$$

is an isomorphism of abelian groups; see Noohi [17, Proposition 5.6]. This shows that the abelian group structure on the pointed set $\mathbb{H}^1(F, G_{sc}^{\text{fppf}} \xrightarrow{\rho} G, \theta)$ defined using the bijection $(j_T)_*$ (as in [5, Section 3.8]) coincides with the abelian group structure defined by the symmetric braiding $\{\cdot, \cdot\}$.

Remark A.6. González-Avilés [12] defined the abelian fppf cohomology group $H^1_{\text{fppf, ab}}(X, G)$ and the abelianization map

$$\text{ab}: H^1_{\text{fppf}}(X, G) \to H^1_{\text{fppf, ab}}(X, G)$$

for a reductive group scheme $G$ over an arbitrary base scheme $X$, which includes the case of a reductive group over a field $F$ of arbitrary characteristic. However, his definition uses the center $Z_G$ of $G$, and hence it is functorial only with respect to the normal homomorphisms $G_1 \to G_2$ (homomorphisms with normal image, hence sending $Z_{G_1}$ to $Z_{G_2}$), whereas our definition above (over a field only) is functorial with respect to all homomorphisms.
Appendix B. Equivalence on $F_s$-points in arbitrary characteristic  
written by Zev Rosengarten

In this appendix we prove the following theorem:

**Theorem B.1.** Let $F$ be a field of arbitrary characteristic and let $F_s$ be a fixed separable closure of $F$. Let

$$\rho: G^{sc} \to [G, G] \hookrightarrow G$$

be as in Subsection 2.3. Let $T \subseteq G$ be a maximal torus. We write $T^{sc} = \rho^{-1}(T)$. Then the morphism of crossed modules

$$\begin{array}{c}
\{T^{sc}(F_s) \to T(F_s)\} \longrightarrow \{G^{sc}(F_s) \to G(F_s)\}
\end{array}$$

is an equivalence (quasi-isomorphism).

**Proof.** We must show that the maps

$$i_{\ker}: \ker[T^{sc}(F_s) \to T(F_s)] \longrightarrow \ker[G^{sc}(F_s) \to G(F_s)] \tag{19}$$

and

$$i_{\coker}: \coker[T^{sc}(F_s) \to T(F_s)] \longrightarrow \coker[G^{sc}(F_s) \to G(F_s)] \tag{20}$$

are isomorphisms.

For (19), the injectivity is obvious. Moreover, any element of ker $[G^{sc}(F_s) \to G(F_s)]$ lies in the preimage $T^{sc}$ of $T$, hence it is an element of $T^{sc}(F_s)$ and of ker $[T^{sc}(F_s) \to T(F_s)]$, which gives the surjectivity of $i_{\ker}$.

We prove the injectivity of (20). Let $[t] \in \coker[T^{sc}(F_s) \to T(F_s)]$, $t \in T(F_s)$, and $[t] \in \ker i_{\coker}$; then $t = \rho(s)$ for some $s \in G^{sc}(F_s)$. Since $T^{sc} = \rho^{-1}(T)$, we see that $s \in T^{sc}(F_s)$, whence $[t] = 1$, as required.

We prove the surjectivity of (20). Let $C \subseteq G$ denote the radical (largest central torus) of $G$. Then the map

$$\psi: C \times G^{sc} \to G, \quad (c, s) \mapsto c \cdot \rho(s) \quad \text{for} \ c \in C, \ s \in G^{sc}$$

is surjective with central kernel $Z \cong \rho^{-1}(C \cap [G, G])$ (which might be non-smooth). We have an exact commutative diagram of $F$-group schemes

$$\begin{array}{cccccc}
1 & \longrightarrow & Z & \longrightarrow & C \times T^{sc} & \xrightarrow{\psi_T} & T & \longrightarrow & 1 \\
& & \parallel & & \downarrow & & \parallel & & \\
1 & \longrightarrow & Z & \longrightarrow & C \times G^{sc} & \xrightarrow{\psi} & G & \longrightarrow & 1
\end{array}$$

in which the maps on $F_s$-points

$$\psi_T: C(F_s) \times T^{sc}(F_s) \to T(F_s) \quad \text{and} \quad \psi: C(F_s) \times G^{sc}(F_s) \to G(F_s)$$

might not be surjective. This diagram gives rise to an exact commutative diagram of fppf cohomology groups

$$\begin{array}{cccc}
C(F_s) \times T^{sc}(F_s) & \xrightarrow{\psi_T} & T(F_s) & \longrightarrow & H^1_{fppf}(F_s, Z) & \longrightarrow & H^1_{fppf}(F_s, C \times T^{sc}) = 1 \\
\downarrow & & \downarrow & & \parallel & & \\
C(F_s) \times G^{sc}(F_s) & \xrightarrow{\psi} & G(F_s) & \longrightarrow & H^1_{fppf}(F_s, Z) & \longrightarrow & H^1_{fppf}(F_s, C \times G^{sc}) = 1
\end{array}$$

in which the rightmost term in both rows is trivial because $F_s$ is separably closed and the $F$-groups $C \times T^{sc}$, $C \times G^{sc}$ are smooth. The latter diagram shows that

$$G(F_s) = T(F_s) \cdot \psi[C(F_s) \times G^{sc}(F_s)] = T(F_s) \cdot C(F_s) \cdot \rho[G^{sc}(F_s)] = T(F_s) \cdot \rho[G^{sc}(F_s)],$$

whence the surjectivity of (20).  □
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