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Criterion for surjectivity of localization in Galois cohomology of a reductive group over a number field

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Abstract. Let G be a connected reductive group over a number field F, and let S be a set (finite or infinite) of places of F. We give a necessary and sufficient condition for the surjectivity of the localization map from $H^1(F,G)$ to the "direct sum" of the sets $H^1(F_v,G)$ where v runs over S. In the appendices, we give a new construction of the abelian Galois cohomology of a reductive group over a field of arbitrary characteristic.

Résumé. Soit G un groupe réductif connexe sur un corps de nombres F, et soit S un ensemble (fini ou infini) de places de F. On donne une condition nécessaire et suffisante pour la surjectivité de l'application de localisation de $H^1(F,G)$ vers la « somme directe » des ensembles $H^1(F_v,G)$, où v parcourt S. Dans les appendices on donne une nouvelle construction de la cohomologie galoisienne abélienne d'un groupe réductif sur un corps de caractéristique quelconque.

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1. Introduction

1.1. Let G be a (connected) reductive group over a number field F (we follow the convention of SGA3, where reductive groups are assumed to be connected). Let $\overline{F}$ be a fixed algebraic closure of F. We denote by $\mathscr{V}(F)$ the set of places of F. For $v \in \mathscr{V}(F)$, we denote by F_v the completion of F at v. We refer to Serre's book [20] for the definition of the first Galois cohomology set $H^1(F,G)$.

In general, $H^1(F,G)$ is just a pointed set and has no natural groups structure. Let $H^1_{\mathrm{ab}}(F,G)$ denote the *abelian Galois cohomology group* of G introduced in [5, Section 2]; see also Labesse [14,

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Section 1.3]. This is an abelian group depending functorially on G and F. There is a canonical *abelianization map*

$$\mathrm{ab}\colon H^1(F,G)\to H^1_{\mathrm{ab}}(F,G).$$

We give a new, better construction of $H^1_{\mathrm{ab}}(F,G)$ in Appendix A.

Let $S\subseteq\mathscr{V}(F)$ be a subset (finite or infinite). We consider the localization map

$$H^1_{\mathrm{ab}}(F,G)\to\prod_{v\in S}H^1_{\mathrm{ab}}(F_v,G). \tag{1}$$

In fact this map takes values in the subgroup $\bigoplus_{v\in S}H^1_{\mathrm{ab}}(F_v,G)\subseteq\prod_{v\in S}H^1_{\mathrm{ab}}(F_v,G)$; see [5, Corollary 4.6]. Thus we obtain a localization map

$$\mathrm{loc}^{\mathrm{ab}}_S\colon H^1_{\mathrm{ab}}(F,G)\to\bigoplus_{v\in S}H^1_{\mathrm{ab}}(F_v,G). \tag{2}$$

Similarly, consider the localization map

$$H^1(F,G)\to\prod_{v\in S}H^1(F_v,G).$$

In fact it takes values in the subset $\bigoplus_{v\in S}H^1(F_v,G)$ consisting of the families $(\xi_v)_{v\in S}$ with $\xi_v\in H^1(F_v,G)$ and such that $\xi_v=1$ for all v except maybe finitely many of them. This well-known fact follows, for instance, from the corresponding assertion for (1) together with [5, Theorem 5.11 and Corollary 5.4.1]. Thus we obtain a localization map

$$\mathrm{loc}_S\colon H^1(F,G)\to\bigoplus_{v\in S}H^1(F_v,G). \tag{3}$$

We wish to find conditions under which the localization maps (2) and (3) are surjective.

1.2. We denote by $M=\pi_1(G)$ the *algebraic fundamental group of* G (also known as the Borovoi fundamental group of G) introduced in [5, Section 1], and also introduced by Merkurjev [16, Section 10.1] and Colliot-Thélène [8, Proposition-Definition 6.1]. See Subsection 2.3 for our definition of $\pi_1(G)$. This is a finitely generated abelian group, on which the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ naturally acts. Let E/F be a finite Galois extension in $\overline{F}$ such that $\mathrm{Gal}(\overline{F}/E)$ acts on M trivially and that E has no real places. Then the Galois group $\Gamma:=\mathrm{Gal}(E/F)$ naturally acts on M and on the set of places $\mathscr{V}(E)$ of the field E.

1.3. We denote by $Ч^1_S(F,G)$ the cokernel of the homomorphism (2), that is,

$$Ч^1_S(F,G)=\mathrm{coker}\left[\mathrm{loc}^{\mathrm{ab}}_S\colon H^1_{\mathrm{ab}}(F,G)\to\bigoplus_{v\in S}H^1_{\mathrm{ab}}(F_v,G)\right].$$

After explaining our notation in Section 2, we compute in Section 3 the finite abelian group $Ч^1_S(F,G)$ in terms of the action of Γ on M and on $\mathscr{V}(E)=\mathscr{V}_f(E)\cup\mathscr{V}_{\mathbb{C}}(E)$; see Corollary 3.8. See Subsection 2.4 for the notations $\mathscr{V}_f$ and $\mathscr{V}_{\mathbb{C}}$.

Concerning the map loc_S of (3), in Section 3 we compute the image of this map; see Main Theorem 3.7. Using this result, we give a *criterion* (necessary and sufficient condition) for the map loc_S to be surjective; see Corollary 3.9. This is also a criterion for the vanishing of $Ч^1_S(F,G)$. Again, our criterion is given in terms of the action of Γ on M and on $\mathscr{V}(E)=\mathscr{V}_f(E)\cup\mathscr{V}_{\mathbb{C}}(E)$. Using this criterion, we give a simple proof of the result of Borel and Harder [2, Theorem 1.7] (see also Prasad and Rapinchuk [19, Proposition 1]) on the surjectivity of the map loc_S when G is semisimple and there exists a finite place v_0 of F outside S; see Proposition 3.14 below.

Let Γ be a finite group. In Section 4, we construct an exact sequence arising from a short exact sequence of Γ-modules. In Section 5, using this exact sequence and Main Theorem 3.7, we generalize a result of Prasad and Rapinchuk giving a sufficient condition for the surjectivity of the localization map loc_S when G is reductive, in terms of the radical (largest central torus) of G; see Theorem 5.1. As a particular case, we obtain the following corollary.

Corollary 1.4 (of Theorem 5.1). *Let G be a reductive group over a number field F, and let C denote the radical of G (the identity component of the center of G). Let $S \subset \mathscr{V}(F)$ be a set of places of F. Assume that the F-torus C splits over a finite Galois extension of F of prime degree p and that there exists a finite place v_0 in the complement $S^{\complement} := \mathscr{V}(F) \smallsetminus S$ of S such that C does not split over F_{v_0}. Then the localization map* loc_S *of* (3) *is surjective.*

For $p = 2$ this assertion was earlier proved by Prasad and Rapinchuk [19, Proposition 2(b)].

1.5. Let G be a reductive group over a field F of characteristic 0. In [5], the author defined the abelian group $H^1_{\mathrm{ab}}(F,G)$ *as a set* in a canonical way as the Galois hypercohomology of a certain crossed module. However, the definition of the structure of abelian group on $H^1_{\mathrm{ab}}(F,G)$ in [5] was complicated. In Appendix A, we define $H^1_{\mathrm{ab}}(F,G)$ (in arbitrary characteristic) following the letter of Breen to the author [7] and the article by Noohi [17] (written at the author's request), as the Galois hypercohomology $\mathbb{H}^1(F,G_{\mathrm{ab}})$ of a certain *stable* crossed module, that is, a crossed module endowed with a symmetric braiding. The structure of abelian group comes from the symmetric braiding. Note that our specific crossed module and specific symmetric braiding were constructed by Deligne [11].

In Appendix B, Zev Rosengarten shows that certain equivalences of crossed modules of algebraic groups over a field F of arbitrary characteristic induce equivalences on F_s-points where F_s is a separable closure of F. This permits us to use in Appendix A the *Galois* hypercohomology of these crossed modules rather than fppf hypercohomology.

2. Notation

2.1. Let A be an abelian group. We denote by A_{Tors} the torsion subgroup of A. We set $A_{\mathrm{t.f.}} = A/A_{\mathrm{Tors}}$, which is a torsion-free group.

2.2. Let Γ be a finite group, and let B be a Γ-module. We denote by B_Γ the group of coinvariants of Γ in B, that is,

$$B_\Gamma = B \Big/ \left\{ \sum_{\gamma\in\Gamma} \left({}^{\gamma^{-1}}b_\gamma - b_\gamma\right) \,\middle|\, b_\gamma \in B \right\}.$$

We write $B_{\Gamma,\mathrm{Tors}} := (B_\Gamma)_{\mathrm{Tors}}$ (which is the torsion subgroup of B_Γ), $B_{\Gamma,\mathrm{t.f.}} = B_\Gamma / B_{\Gamma,\mathrm{Tors}}$ (which is a torsion-free group).

2.3. Let G be a reductive group over a field F. Let $[G,G]$ denote the commutator subgroup of G, which is semisimple. Let G^{sc} denote the universal cover of $[G,G]$, which is simply connected; see [3, Proposition (2.24) (ii)] or [10, Corollary A.4.11]. Following Deligne [11, Section 0.2], we consider the composite homomorphism

$$\rho\colon G^{\mathrm{sc}} \twoheadrightarrow [G,G] \hookrightarrow G,$$

which in general is neither injective nor surjective.

For a maximal torus $T \subseteq G$, we write $T^{\mathrm{sc}} = \rho^{-1}(T) \subseteq G^{\mathrm{sc}}$ and consider the natural homomorphism

$$\rho\colon T^{\mathrm{sc}} \to T.$$

We consider the algebraic fundamental group $M = \pi_1(G)$ of G defined by

$$\pi_1(G) = \mathsf{X}_*(T)/\rho_*\mathsf{X}_*(T^{\mathrm{sc}})$$

where X_* denotes the cocharacter group. The Galois group $\mathrm{Gal}(F_s/F)$ naturally acts on M, and the $\mathrm{Gal}(F_s/F)$-module M is well defined (does not depend on the choice of T up to a transitive system of isomorphisms); see [5, Lemma 1.2].

2.4. From now on (except for the appendices), F is a number field. We denote by $\mathscr{V}(F)$, $\mathscr{V}_f(F)$, $\mathscr{V}_\infty(F)$, $\mathscr{V}_\mathbb{R}(F)$, and $\mathscr{V}_\mathbb{C}(F)$ the sets of all places of F, of finite places, of infinite places, of real places, and of complex places, respectively.

Let E/F be a finite Galois extension of number fields with Galois group $\Gamma = \mathrm{Gal}(E/F)$; then Γ acts on $\mathscr{V}(E)$. If $w \in \mathscr{V}(E)$, we write Γ_w for the stabilizer of w in Γ; then $\Gamma_w \cong \mathrm{Gal}(E_w/F_v)$ where $v \in \mathscr{V}(F)$ is the restriction of w to F.

3. Main theorem

In this section we state and prove Main Theorem 3.7 computing the images of the localization maps (2) and (3). We deduce Corollary 3.8 computing the group $Ч^1_S(F, G)$, and Corollary 3.9 giving a necessary and sufficient condition for the surjectivity of the localization map (3).

3.1. Let G be a reductive group over a number field F, and let $v \in \mathscr{V}_f(F)$ be a finite place of F. In [5] we computed $H^1_{\mathrm{ab}}(F_v, G)$. Write $M = \pi(G)$. Let E/F be a finite Galois extension in $\overline{F}$ such that $\mathrm{Gal}(\overline{F}/E)$ acts on M trivially and that E has no real places. Write $\Gamma = \mathrm{Gal}(E/F)$.

Theorem 3.2 ([5, Proposition 4.1(i) and Corollary 5.4.1]). *With the notation and assumptions of Subsection 3.1, for any* finite *place v of F there is a canonical isomorphism of abelian groups*

$$\alpha_v^{\mathrm{ab}}\colon H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\sim} M_{\Gamma_w, \mathrm{Tors}}$$

where w is a place of E over v, and a canonical bijection

$$\mathrm{ab}_v\colon H^1(F_v, G) \to H^1_{\mathrm{ab}}(F_v, G).$$

3.3. Let v be a finite place of F. We have a surjective (even bijective) map

$$\alpha_v\colon H^1(F_v, G) \xrightarrow{\mathrm{ab}_v} H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\alpha_v^{\mathrm{ab}}} M_{\Gamma_w, \mathrm{Tors}}.$$

We consider two composite maps with the same image

$$\lambda_v^{\mathrm{ab}}\colon H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\alpha_v^{\mathrm{ab}}} M_{\Gamma_w, \mathrm{Tors}} \xrightarrow{\omega_v} M_{\Gamma, \mathrm{Tors}},$$
$$\lambda_v\colon H^1(F_v, G) \xrightarrow{\alpha_v} M_{\Gamma_w, \mathrm{Tors}} \xrightarrow{\omega_v} M_{\Gamma, \mathrm{Tors}},$$

where $\omega_v\colon M_{\Gamma_w, \mathrm{Tors}} \to M_{\Gamma, \mathrm{Tors}}$ is the homomorphism induced by the inclusion $\Gamma_w \hookrightarrow \Gamma$. Since the maps α_v^{ab} and α_v are surjective (even bijective), and ω_v is a homomorphism, we see that the set $\mathrm{im}\,\lambda_v^{\mathrm{ab}} = \mathrm{im}\,\lambda_v$ is a subgroup of $M_{\Gamma, \mathrm{Tors}}$, namely, $\mathrm{im}\,\lambda_v^{\mathrm{ab}} = \mathrm{im}\,\lambda_v = \mathrm{im}\,\omega_v$.

Let $v \in \mathscr{V}_\mathbb{C}(F)$ be a complex place. We have zero maps

$$\lambda_v^{\mathrm{ab}}\colon H^1_{\mathrm{ab}}(F_v, G) = \{1\} \to \{0\} \subseteq M_{\Gamma, \mathrm{Tors}}, \qquad \lambda_v\colon H^1(F_v, G) = \{1\} \to \{0\} \subseteq M_{\Gamma, \mathrm{Tors}}.$$

Clearly, the set $\mathrm{im}\,\lambda_v^{\mathrm{ab}} = \mathrm{im}\,\lambda_v$ is a subgroup of $M_{\Gamma, \mathrm{Tors}}$, namely, the subgroup $\{0\}$.

3.4. Let $v \in \mathscr{V}_\mathbb{R}(F)$ be a real place; then Γ_w is a group of order 2, $\Gamma_w = \{1, \gamma\}$ where $\gamma = \gamma_w$ induces the nontrivial automorphism of E_w over F_v. We consider the Tate cohomology group

$$\widehat{H}^{-1}(\Gamma_w, M) = \{m \in M \mid {}^\gamma m = -m\} / \{m' - {}^\gamma m' \mid m' \in M\}.$$

We see immediately that the abelian group $\widehat{H}^{-1}(\Gamma_w, M)$ naturally embeds into M_{Γ_w}. If $m \in M$ is a (-1)-cocycle, that is, ${}^\gamma m = -m$, then $2m = m + m = m - {}^\gamma m$, whence $2 \cdot \widehat{H}^{-1}(\Gamma_w, M) = 0$. We conclude that $\widehat{H}^{-1}(\Gamma_w, M)$ naturally embeds into $M_{\Gamma_w, \mathrm{Tors}}$.

There is a canonical surjective map of Kottwitz [13, Theorem 1.2] (see also [5, Theorem 5.4])

$$\mathrm{ab}_v\colon H^1(F_v, G) \twoheadrightarrow H^1_{\mathrm{ab}}(F_v, G),$$

a canonical isomorphism of [6, Proposition 8.21]

$$H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\sim} \widehat{H}^{-1}(\Gamma_w, M),$$

and a canonical embedding

$$\widehat{H}^{-1}(\Gamma_w, M) \hookrightarrow M_{\Gamma_w,\mathrm{Tors}}.$$

Thus we obtain composite maps

$$\alpha_v^{\mathrm{ab}}\colon H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\sim} \widehat{H}^{-1}(\Gamma_w, M) \hookrightarrow M_{\Gamma_w,\mathrm{Tors}},$$
$$\alpha_v\colon H^1(F_v, G) \twoheadrightarrow H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\sim} \widehat{H}^{-1}(\Gamma_w, M) \hookrightarrow M_{\Gamma_w,\mathrm{Tors}},$$

with the same image $\operatorname{im}\alpha_v^{\mathrm{ab}} = \operatorname{im}\alpha_v$, which is a subgroup of $M_{\Gamma_w,\mathrm{Tors}}$. Consider the composite maps with the same image

$$\lambda_v^{\mathrm{ab}}\colon H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\alpha_v^{\mathrm{ab}}} M_{\Gamma_w,\mathrm{Tors}} \xrightarrow{\omega_v} M_{\Gamma,\mathrm{Tors}},$$
$$\lambda_v\colon H^1(F_v, G) \xrightarrow{\alpha_v} M_{\Gamma_w,\mathrm{Tors}} \xrightarrow{\omega_v} M_{\Gamma,\mathrm{Tors}}.$$

Since the set $\operatorname{im}\alpha_v^{\mathrm{ab}} = \operatorname{im}\alpha_v$ is a subgroup of $M_{\Gamma_w,\mathrm{Tors}}$, and ω_v is a homomorphism, we conclude that the set $\operatorname{im}\lambda_v^{\mathrm{ab}} = \operatorname{im}\lambda_v$ is a subgroup of $M_{\Gamma,\mathrm{Tors}}$, namely, $\operatorname{im}\lambda_v^{\mathrm{ab}} = \operatorname{im}\lambda_v = \omega_v\big(\widehat{H}^{-1}(\Gamma_w, M)\big)$.

Lemma 3.5. *Let $S \subseteq \mathcal{V}(F)$ be any subset, finite or infinite. Consider the summation maps*

$$\Sigma_S^{\mathrm{ab}}\colon \bigoplus_{v\in S} H^1_{\mathrm{ab}}(F_v, G) \longrightarrow M_{\Gamma,\mathrm{Tors}}, \quad \xi_S^{\mathrm{ab}} = \big(\xi_v^{\mathrm{ab}}\big)_{v\in S} \longmapsto \sum_{v\in S} \lambda_v^{\mathrm{ab}}(\xi_v^{\mathrm{ab}}),$$
$$\Sigma_S\colon \bigoplus_{v\in S} H^1(F_v, G) \longrightarrow M_{\Gamma,\mathrm{Tors}}, \quad \xi_S = \big(\xi_v\big)_{v\in S} \longmapsto \sum_{v\in S} \lambda_v(\xi_v).$$

Then the sets $\operatorname{im}\Sigma_S^{\mathrm{ab}}$ and $\operatorname{im}\Sigma_S$ are subgroups of $M_{\Gamma,\mathrm{Tors}}$, and they are equal.

Proof. Indeed, we have

$$\operatorname{im}\Sigma_S^{\mathrm{ab}} = \operatorname{im}\Sigma_S = \langle \operatorname{im}\lambda_v\rangle_{v\in S} \quad \text{where } \operatorname{im}\lambda_v = \begin{cases} \operatorname{im}\omega_v & \text{if } v\in\mathcal{V}_f(F),\\ \omega_v\big(\widehat{H}^{-1}(\Gamma_w, M)\big) & \text{if } v\in\mathcal{V}_{\mathbb{R}}(F),\\ 0 & \text{if } v\in\mathcal{V}_{\mathbb{C}}(F).\end{cases}$$

Here we write $\langle \operatorname{im}\lambda_v\rangle_{v\in S}$ for the subgroup of $M_{\Gamma,\mathrm{Tors}}$ generated by the subgroups $\operatorname{im}\lambda_v$ for $v\in S$. □

Theorem 3.6. *The following sequences are exact:*

$$H^1_{\mathrm{ab}}(F, G) \xrightarrow{\mathrm{loc}^{\mathrm{ab}}_{\mathcal{V}}} \bigoplus_{v\in\mathcal{V}} H^1_{\mathrm{ab}}(F_v, G) \xrightarrow{\Sigma^{\mathrm{ab}}_{\mathcal{V}}} M_{\Gamma,\mathrm{Tors}}, \tag{4}$$
$$H^1(F, G) \xrightarrow{\mathrm{loc}_{\mathcal{V}}} \bigoplus_{v\in\mathcal{V}} H^1(F_v, G) \xrightarrow{\Sigma_{\mathcal{V}}} M_{\Gamma,\mathrm{Tors}}, \tag{5}$$

where for brevity we write $\mathcal{V}$ for $\mathcal{V}(F)$.

Here (4) is an exact sequence of abelian groups, and (5) is an exact sequence of pointed sets.

Proof. In view of [5, Proposition 4.8], exact sequence (4) is actually a part of the exact sequence [5, (4.3.1)]. For (5), see [13, Proposition 2.6] or [5, Theorem 5.15]. □

Main Theorem 3.7. *Let G be a reductive group over a number field F. Let $S \subseteq \mathcal{V} := \mathcal{V}(F)$ be a subset. Write $S^{\complement} = \mathcal{V} \smallsetminus S$, the complement of S in $\mathcal{V}$. Then:*

$$\operatorname{im}\mathrm{loc}_S^{\mathrm{ab}} = \left\{ \xi_S^{\mathrm{ab}} \in \bigoplus_{v\in S} H^1_{\mathrm{ab}}(F_v, G) \;\middle|\; \Sigma_S^{\mathrm{ab}}(\xi_S^{\mathrm{ab}}) \in \operatorname{im}\Sigma_S^{\mathrm{ab}} \cap \operatorname{im}\Sigma_{S^{\complement}}^{\mathrm{ab}} \right\}, \tag{6}$$
$$\operatorname{im}\mathrm{loc}_S = \left\{ \xi_S \in \bigoplus_{v\in S} H^1(F_v, G) \;\middle|\; \Sigma_S(\xi_S) \in \operatorname{im}\Sigma_S \cap \operatorname{im}\Sigma_{S^{\complement}} \right\}. \tag{7}$$

Proof. By Lemma 3.5, the sets $\mathrm{im}\,\Sigma^{\mathrm{ab}}_{S^{\complement}}$ and $\mathrm{im}\,\Sigma_{S^{\complement}}$ are (equal) subgroups of $M_{\Gamma,\mathrm{Tors}}$, and therefore it suffices to prove (6) with $\left(-\mathrm{im}\,\Sigma^{\mathrm{ab}}_{S^{\complement}}\right)$ instead of $\mathrm{im}\,\Sigma^{\mathrm{ab}}_{S^{\complement}}$, and to prove (7) with $\left(-\mathrm{im}\,\Sigma_{S^{\complement}}\right)$ instead of $\mathrm{im}\,\Sigma_{S^{\complement}}$. Now the corresponding assertions follow easily from the exactness of (4) and (5), respectively.

For the reader's convenience, we provide an easy proof of (7) with $\left(-\mathrm{im}\,\Sigma_{S^{\complement}}\right)$ instead of $\mathrm{im}\,\Sigma_{S^{\complement}}$. Let

$$\xi_S = \left(\xi_v\right)_{v\in S} \in \mathrm{im}\,\mathrm{loc}_S \subseteq \bigoplus_{v\in S} H^1(F_v,G),$$

that is, $\xi_S = \mathrm{loc}_S(\xi)$ for some $\xi \in H^1(F,G)$. Write $\eta_{S^{\complement}} = \left(\eta_v\right)_{v\in S^{\complement}} = \mathrm{loc}_{S^{\complement}}(\xi)$. Since the sequence (5) is exact, we have $(\Sigma_{\mathscr{V}}\circ \mathrm{loc}_{\mathscr{V}})(\xi) = 0$, whence

$$\Sigma_S(\xi_S) + \Sigma_{S^{\complement}}(\eta_{S^{\complement}}) = 0 \quad \text{and} \quad \Sigma_S(\xi_S) = -\Sigma_{S^{\complement}}(\eta_{S^{\complement}}).$$

We conclude that $\Sigma_S(\xi_S) \in \mathrm{im}\,\Sigma_S \cap \left(-\mathrm{im}\,\Sigma_{S^{\complement}}\right)$, as required.

Conversely, let an element $\xi_S = \left(\xi_v\right)_{v\in S} \in \bigoplus_{v\in S} H^1(F_v,G)$ be such that

$$\Sigma_S(\xi_S) \in \mathrm{im}\,\Sigma_S \cap \left(-\mathrm{im}\,\Sigma_{S^{\complement}}\right).$$

Write $a = \Sigma_S(\xi_S)$. Then $-a \in \mathrm{im}\,\Sigma_{S^{\complement}}$, that is,

$$-a = \Sigma_{S^{\complement}}(\eta_{S^{\complement}}) \quad \text{for some} \quad \eta_{S^{\complement}} = \left(\eta_v\right)_{v\in S^{\complement}} \in \bigoplus_{v\in S^{\complement}} H^1(F_v,G).$$

Define

$$\zeta_{\mathscr{V}} = \left(\zeta_v\right)_{v\in\mathscr{V}} \in \bigoplus_{v\in\mathscr{V}} H^1(F_v,G), \qquad \zeta_v = \begin{cases} \xi_v & \text{if } v\in S,\\ \eta_v & \text{if } v\in S^{\complement}.\end{cases}$$

Then

$$\Sigma_{\mathscr{V}}(\zeta_{\mathscr{V}}) = a + (-a) = 0.$$

Since the sequence (5) is exact, we have $\zeta_{\mathscr{V}} = \mathrm{loc}_{\mathscr{V}}(\zeta)$ for some $\zeta \in H^1(F,G)$. Then $\mathrm{loc}_S(\zeta) = \xi_S$, whence $\xi_S \in \mathrm{im}\,\mathrm{loc}_S$, as required. □

Corollary 3.8. *The homomorphism*

$$\chi^{\mathrm{ab}}_S\colon \bigoplus_{v\in\mathscr{V}} H^1_{\mathrm{ab}}(F_v,G) \xrightarrow{\Sigma^{\mathrm{ab}}_S} \mathrm{im}\,\Sigma^{\mathrm{ab}}_S \longrightarrow \mathrm{im}\,\Sigma^{\mathrm{ab}}_S/\left(\mathrm{im}\,\Sigma^{\mathrm{ab}}_S \cap \mathrm{im}\,\Sigma^{\mathrm{ab}}_{S^{\complement}}\right)$$

induces a canonical isomorphism

$$Ч^1_S(F,G) \xrightarrow{\sim} \mathrm{im}\,\Sigma^{\mathrm{ab}}_S/\left(\mathrm{im}\,\Sigma^{\mathrm{ab}}_S \cap \mathrm{im}\,\Sigma^{\mathrm{ab}}_{S^{\complement}}\right).$$

Proof. The homomorphism χ^{ab}_S is clearly surjective, and by Theorem 3.7 its kernel is the image $\mathrm{im}\,\mathrm{loc}^{\mathrm{ab}}_S$ of the localization homomorphism $\mathrm{loc}^{\mathrm{ab}}_S$ of (2). The corollary follows. □

Corollary 3.9. *The localization map* loc_S *of* (3) *is surjective if and only if*

$$\mathrm{im}\,\Sigma_S \subseteq \mathrm{im}\,\Sigma_{S^{\complement}}. \tag{8}$$

Proof. Consider the map

$$\chi_S\colon \bigoplus_{v\in S} H^1(F_v,G) \xrightarrow{\Sigma_S} \mathrm{im}\,\Sigma_S \longrightarrow \mathrm{im}\,\Sigma_S/\left(\mathrm{im}\,\Sigma_S \cap \mathrm{im}\,\Sigma_{S^{\complement}}\right).$$

By Lemma 3.5 the sets $\mathrm{im}\,\Sigma_S = \mathrm{im}\,\Sigma^{\mathrm{ab}}_S$ and $\mathrm{im}\,\Sigma_S \cap \mathrm{im}\,\Sigma_{S^{\complement}} = \mathrm{im}\,\Sigma^{\mathrm{ab}}_S \cap \mathrm{im}\,\Sigma^{\mathrm{ab}}_{S^{\complement}}$ are abelian groups. The morphism of pointed sets χ_S is clearly surjective, and by Theorem 3.7 its kernel is $\mathrm{im}\,\mathrm{loc}_S$. We see that the following assertions are equivalent:

(a) the map loc_S is surjective, that is, $\mathrm{im}\,\mathrm{loc}_S = \bigoplus_{v\in S} H^1(F_v,G)$;
(b) $\ker \chi_S = \bigoplus_{v\in S} H^1(F_v,G)$;
(c) $\#(\mathrm{im}\,\chi_S) = 1$;
(d) $\mathrm{im}\,\Sigma_S \cap \mathrm{im}\,\Sigma_{S^{\complement}} = \mathrm{im}\,\Sigma_S$;

(e) $\operatorname{im}\Sigma_S \subseteq \operatorname{im}\Sigma_{S^{\complement}}$.

This completes the proof. □

Remark 3.10. Since by Lemma 3.5 we have $\operatorname{im}\Sigma_S^{\mathrm{ab}} = \operatorname{im}\Sigma_S$ and $\operatorname{im}\Sigma_{S^{\complement}}^{\mathrm{ab}} = \operatorname{im}\Sigma_{S^{\complement}}$, we see from (d) in the proof above and from Corollary 3.8 that the localization map loc_S of (3) is surjective if and only if $Ч_S^1(F,G) = \{1\}$.

Corollary 3.11. *Let $v_0 \in \mathcal{V}(F)$, $S = \mathcal{V}(F) \smallsetminus \{v_0\}$. Then the localization map* loc_S *of* (3) *is surjective if and only if*

$$\operatorname{im}\lambda_v \subseteq \operatorname{im}\lambda_{v_0} \quad \textit{for all } v \in \mathcal{V}(F). \tag{9}$$

Proof. Indeed, in our case condition (9) is equivalent to (8), and we conclude by Corollary 3.9. □

Corollary 3.12. *For a subset $S \subset \mathcal{V}(F)$, let $v_0 \in S^{\complement}$, and assume that*

$$\operatorname{im}\lambda_v \subseteq \operatorname{im}\lambda_{v_0} \quad \textit{for all } v \in S. \tag{10}$$

Then the localization map loc_S *of* (3) *is surjective.*

Proof. Indeed, (10) implies (8), and we conclude by Corollary 3.9. □

Corollary 3.13. *Let $v_0 \in S^{\complement}$, and assume that the map $\lambda_{v_0}\colon H^1(F_{v_0}, G) \to M_{\Gamma,\mathrm{Tors}}$ is surjective. Then the localization map* loc_S *of* (3) *is surjective.*

Proof. Indeed, then

$$\operatorname{im}\Sigma_S \subseteq M_{\Gamma,\mathrm{Tors}} = \operatorname{im}\lambda_{v_0} \subseteq \operatorname{im}\Sigma_{S^{\complement}},$$

and we conclude by Corollary 3.9. □

Proposition 3.14 (Borel and Harder [2, Theorem 1.7]). *Let G be a* semisimple *group over a number field F, and let $S \subset \mathcal{V}(F)$ be a subset such that the complement $S^{\complement}$ of S contains a* finite *place $v_0 \in \mathcal{V}_f(F)$. Then the localization map* loc_S *of* (3) *is surjective.*

Proof. Since G is semisimple, the Γ-module M is finite, and so are the groups M_Γ and M_{Γ_w} where w is a place of E over v_0. It follows that

$$M_{\Gamma_w,\mathrm{Tors}} = M_{\Gamma_w} \quad \text{and} \quad M_{\Gamma,\mathrm{Tors}} = M_\Gamma.$$

The natural homomorphism $M_{\Gamma_w} \to M_\Gamma$ is clearly surjective. Therefore, the homomorphism

$$\omega_{v_0}\colon M_{\Gamma_w,\mathrm{Tors}} = M_{\Gamma_w} \longrightarrow M_\Gamma = M_{\Gamma,\mathrm{Tors}}$$

is surjective. Since v_0 is finite, we have $\operatorname{im}\lambda_{v_0} = \operatorname{im}\omega_{v_0}$, whence the map λ_{v_0} is surjective. We conclude by Corollary 3.13. □

4. Exact sequence

In this section we construct an exact sequence that we shall use in Section 5.

Theorem 4.1. *A finite group Γ and a short exact sequence of Γ-modules*

$$0 \to B_1 \xrightarrow{\ i\ } B_2 \xrightarrow{\ j\ } B_3 \to 0 \tag{11}$$

give rises to an exact sequence

$$\begin{aligned}(B_1)_{\Gamma,\mathrm{Tors}} &\xrightarrow{i_*} (B_2)_{\Gamma,\mathrm{Tors}} \xrightarrow{j_*} (B_3)_{\Gamma,\mathrm{Tors}} \\ &\xrightarrow{\delta} \mathbb{Q}/\mathbb{Z}\otimes_{\mathbb{Z}}(B_1)_\Gamma \xrightarrow{i_*} \mathbb{Q}/\mathbb{Z}\otimes_{\mathbb{Z}}(B_2)_\Gamma \xrightarrow{j_*} \mathbb{Q}/\mathbb{Z}\otimes_{\mathbb{Z}}(B_3)_\Gamma \to 0\end{aligned} \tag{12}$$

depending functorially on Γ and on the sequence (11).

4.2. We specify the homomorphism δ. Let $x_3 \in B_3$ be such that the image $(x_3)_\Gamma$ of x_3 in $(B_3)_\Gamma$ is contained in $(B_3)_{\Gamma,\mathrm{Tors}}$. This means that there exist $n \in \mathbb{Z}_{>0}$ and $y_{3,\gamma} \in B_3$ such that

$$nx_3 = \sum_{\gamma\in\Gamma} \big({}^\gamma y_{3,\gamma} - y_{3,\gamma}\big).$$

We lift x_3 to some $x_2 \in B_2$, we lift each $y_{3,\gamma}$ to some $y_{2,\gamma} \in B_2$, and we consider the element

$$z_2 = nx_2 - \sum_{\gamma\in\Gamma} \big({}^\gamma y_{2,\gamma} - y_{2,\gamma}\big).$$

Then $j(z_2) = 0 \in B_3$, whence $z_2 = i(z_1)$ for some $z_1 \in B_1$. We consider the image $(z_1)_{\Gamma,\,\mathrm{t.f.}}$ of $z_1 \in B_1$ in $(B_1)_{\Gamma,\,\mathrm{t.f.}}$, and we put

$$\delta\big((x_3)_\Gamma\big) = \frac{1}{n} \otimes (z_1)_{\Gamma,\,\mathrm{t.f.}} \in \mathbb{Q}/\mathbb{Z} \otimes_\mathbb{Z} (B_1)_{\Gamma,\,\mathrm{t.f.}} = \mathbb{Q}/\mathbb{Z} \otimes_\mathbb{Z} (B_1)_\Gamma$$

where we write $\frac{1}{n}$ for the image in $\mathbb{Q}/\mathbb{Z}$ of $\frac{1}{n} \in \mathbb{Q}$.

Below we give the proof of Theorem 4.1 suggested by Vladimir Hinich (private communication). For another proof, due to Alexander Petrov, see [18].

Proof of Theorem 4.1 due to Vladimir Hinich. The functor from the category of Γ-modules to the category of abelian groups

$$B \rightsquigarrow \mathbb{Q}/\mathbb{Z} \otimes_\mathbb{Z} B_\Gamma$$

is the same as

$$B \rightsquigarrow \mathbb{Q}/\mathbb{Z} \otimes_\Lambda B$$

where $\Lambda = \mathbb{Z}[\Gamma]$ is the group ring of Γ. From the short exact sequence of Γ-modules (11), we obtain a long exact sequence

$$\cdots \to \mathrm{Tor}_1^\Lambda(\mathbb{Q}/\mathbb{Z}, B_1) \xrightarrow{i_*} \mathrm{Tor}_1^\Lambda(\mathbb{Q}/\mathbb{Z}, B_2) \xrightarrow{j_*} \mathrm{Tor}_1^\Lambda(\mathbb{Q}/\mathbb{Z}, B_3)$$
$$\xrightarrow{\delta} \mathbb{Q}/\mathbb{Z} \otimes_\Lambda B_1 \xrightarrow{i_*} \mathbb{Q}/\mathbb{Z} \otimes_\Lambda B_2 \xrightarrow{j_*} \mathbb{Q}/\mathbb{Z} \otimes_\Lambda B_3 \to 0$$

depending functorially on Γ and on (11); see Weibel [21]. Now Theorem 4.1 follows from the next proposition. □

Proposition 4.3. *For a finite group* Γ *and a* Γ*-module* B*, there is a canonical and functorial isomorphism*

$$\mathrm{Tor}_1^\Lambda(\mathbb{Q}/\mathbb{Z}, B) \xrightarrow{\sim} B_{\Gamma,\mathrm{Tors}}$$

where $\Lambda = \mathbb{Z}[\Gamma]$.

Proof. Consider the short exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

regarded as a short exact sequence of Γ-modules with trivial action of Γ. Tensoring with B, we obtain a long exact sequence

$$\cdots \to \mathrm{Tor}_1^\Lambda(\mathbb{Q}, B) \to \mathrm{Tor}_1^\Lambda(\mathbb{Q}/\mathbb{Z}, B) \to \mathbb{Z} \otimes_\Lambda B \to \mathbb{Q} \otimes_\Lambda B \to \mathbb{Q}/\mathbb{Z} \otimes_\Lambda B \to 0. \tag{13}$$

We have canonical isomorphisms

$$\mathbb{Z} \otimes_\Lambda B = B_\Gamma \quad \text{and} \quad \ker\big[\mathbb{Z} \otimes_\Lambda B \to \mathbb{Q} \otimes_\Lambda B\big] = B_{\Gamma,\mathrm{Tors}}.$$

By Lemma 4.4 below, we have $\mathrm{Tor}_1^\Lambda(\mathbb{Q}, B) = 0$, and the proposition follows from (13). □

Lemma 4.4. *For a finite group* Γ *and any* Γ*-module* B*, we have*

$$\mathrm{Tor}_1^\Lambda(\mathbb{Q}, B) = 0$$

where $\Lambda = \mathbb{Z}[\Gamma]$.

Proof. Let

$$P_\bullet : \quad \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

be a Λ-free resolution of the trivial Γ-module $\mathbb{Z}$, for example, the standard complex; see Atiyah and Wall [1, Section 2]. Tensoring with $\mathbb{Q}$ over $\mathbb{Z}$, we obtain a flat resolution of $\mathbb{Q}$

$$\cdots \to \mathbb{Q} \otimes_{\mathbb{Z}} P_2 \to \mathbb{Q} \otimes_{\mathbb{Z}} P_1 \to \mathbb{Q} \otimes_{\mathbb{Z}} P_0 \to \mathbb{Q} \to 0.$$

Tensoring with B over $\Lambda = \mathbb{Z}[\Gamma]$, we obtain the complex $(\mathbb{Q} \otimes_{\mathbb{Z}} P_\bullet) \otimes_\Lambda B$:

$$\cdots \to (\mathbb{Q} \otimes_{\mathbb{Z}} P_2) \otimes_\Lambda B \to (\mathbb{Q} \otimes_{\mathbb{Z}} P_1) \otimes_\Lambda B \to (\mathbb{Q} \otimes_{\mathbb{Z}} P_0) \otimes_\Lambda B \to \mathbb{Q} \otimes_\Lambda B \to 0. \tag{14}$$

By definition, $\mathrm{Tor}_1^\Lambda(\mathbb{Q}, B)$ is the first homology group of this complex.

However, we can obtain the complex (14) from $P_\bullet$ by tensoring first with B over Λ, and after that with $\mathbb{Q}$ over $\mathbb{Z}$:

$$\mathbb{Q} \otimes_{\mathbb{Z}} \big(P_\bullet \otimes_\Lambda B\big) \cong (\mathbb{Q} \otimes_{\mathbb{Z}} P_\bullet) \otimes_\Lambda B.$$

Since $\mathbb{Q}$ is a flat $\mathbb{Z}$-module, we obtain canonical isomorphisms

$$\mathrm{Tor}_1^\Lambda(\mathbb{Q}, B) \cong \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Tor}_1^\Lambda(\mathbb{Z}, B) = \mathbb{Q} \otimes_{\mathbb{Z}} H_1(\Gamma, B).$$

Now, since the group Γ is finite, the abelian group $H_1(\Gamma, B)$ is killed by multiplication by $\#\Gamma$; see, for instance, Atiyah and Wall [1, Section 6, Corollary 1 of Proposition 8]. It follows that $\mathbb{Q} \otimes_{\mathbb{Z}} H_1(\Gamma, B) = 0$. Thus $\mathrm{Tor}_1^\Lambda(\mathbb{Q}, B) = 0$, which completes the proofs of Lemma 4.4, Proposition 4.3, and Theorem 4.1. □

Alternatively, one can check directly that the map δ constructed in Subsection 4.2 is well-defined (does not depend on the choices made) and that the sequence (12) is exact.

5. Surjectivity for a reductive group with nice radical

In this section we prove the following theorem that gives a sufficient condition for the surjectivity of the localization map (3) for a reductive F-group G in terms of the radical (largest central torus) of G.

Theorem 5.1. *Let G be a reductive group over a number field F, and let C denote the radical of G. Write $\overline{G} = G/C$, which is a semisimple group, and consider the short exact sequence of fundamental groups [5, Lemma 1.5]*

$$0 \to M_C \to M \to \overline{M} \to 0$$

where

$$M_C = \pi_1(C) = \mathsf{X}_*(C), \quad M = \pi_1(G), \quad \overline{M} = \pi_1(\overline{G}).$$

We define $\Gamma = \mathrm{Gal}(E/F)$ for M as in Subsection 3.1. Let $S \subset \mathcal{V}(F)$ be a subset, and assume that $S^{\complement}$ contains a finite *place v_0 such that*

$$\mathrm{im}\,[\Gamma_w \to \mathrm{Aut}\, M_C] = \mathrm{im}\,[\Gamma \to \mathrm{Aut}\, M_C] \tag{15}$$

where w is a place of E over v_0. Then the localization map loc_S of (3) is surjective.

Proof. It follows from (15) that $(M_C)_{\Gamma_w} = (M_C)_\Gamma$, whence

$$(M_C)_{\Gamma_w,\mathrm{Tors}} = (M_C)_{\Gamma,\mathrm{Tors}} \quad \text{and} \quad \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (M_C)_{\Gamma_w} = \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (M_C)_\Gamma.$$

Using Theorem 4.1, we construct an exact commutative diagram

$$\begin{array}{ccccccc}
(M_C)_{\Gamma_w,\mathrm{Tors}} & \longrightarrow & M_{\Gamma_w,\mathrm{Tors}} & \longrightarrow & \overline{M}_{\Gamma_w,\mathrm{Tors}} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (M_C)_{\Gamma_w} \\
\| & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\overline{\omega}} & & \| \\
(M_C)_{\Gamma,\mathrm{Tors}} & \longrightarrow & M_{\Gamma,\mathrm{Tors}} & \longrightarrow & \overline{M}_{\Gamma,\mathrm{Tors}} & \longrightarrow & \mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} (M_C)_\Gamma
\end{array}$$

Since $\overline{G}$ is semisimple, its algebraic fundamental group $\overline{M}$ is finite, and therefore the homomorphism $\overline{\omega}$ in the diagram above is surjective; see the proof of Proposition 3.14. By a four lemma, the homomorphism

$$\omega = \omega_{v_0}\colon\ M_{\Gamma_w,\mathrm{Tors}} \to M_{\Gamma,\mathrm{Tors}}$$

is surjective as well. Since v_0 is finite, the map

$$\alpha_{v_0}\colon\ H^1(F_{v_0}, G) \to H^1_{\mathrm{ab}}(F_{v_0}, G) \to M_{\Gamma_w,\mathrm{Tors}}$$

is bijective, and therefore the map

$$\lambda_{v_0}\colon\ H^1(F_{v_0}, G) \to H^1_{\mathrm{ab}}(F_{v_0}, G) \to M_{\Gamma_w,\mathrm{Tors}} \xrightarrow{\omega} M_{\Gamma,\mathrm{Tors}}$$

is surjective. We conclude by Corollary 3.13. □

Corollary 5.2 (Prasad and Rapinchuk [19, Proposition 2(a)]). *Let G be a reductive group over a number field F, and let C denote the radical of G. Assume that the F-torus C is split and that $S^{\complement}$ contains a finite place v_0. Then the localization map* loc_S *of* (3) *is surjective.*

Proof. We define E, Γ, and Γ_w for $M = \pi_1(G)$ as in Subsection 3.1. Then $\mathrm{im}[\Gamma \to \mathrm{Aut}\, M_C] = \{1\}$, and hence (15) holds. We conclude by Theorem 5.1. □

Proof of Corollary 1.4. We define E, Γ, and Γ_w for $M = \pi_1(G)$ as in Subsection 3.1. We have

$$\mathrm{im}\big[\Gamma_w \to \mathrm{Aut}\, M_C\big] \subseteq \mathrm{im}\big[\Gamma \to \mathrm{Aut}\, M_C\big], \quad \#\mathrm{im}\big[\Gamma_w \to \mathrm{Aut}\, M_C\big] \,\big|\, p, \quad \mathrm{im}\big[\Gamma_w \to \mathrm{Aut}\, M_C\big] \neq \{1\}.$$

It follows that (15) holds. We conclude by Theorem 5.1. □

Appendix A. Abelianization

A.1. Let G be a reductive group over a field F *of arbitrary characteristic*. We consider the homomorphism $\rho\colon G^{\mathrm{sc}} \to G$ of Subsection 2.3.

The group G acts by conjugation on itself on the left, and by functoriality G acts on G^{sc}. We obtain an action

$$\theta\colon G \times G^{\mathrm{sc}} \to G^{\mathrm{sc}}, \quad (g, s) \mapsto {}^g s.$$

On $\overline{F}$-points, if $s \in G^{\mathrm{sc}}(\overline{F})$, $g_1 \in G(\overline{F})$, $g_1 = \rho(s_1)\cdot z_1$ with $s_1 \in G^{\mathrm{sc}}(\overline{F})$, $z_1 \in Z_G(\overline{F})$, then

$$\theta(g_1, s) = {}^{g_1}s = s_1 s s_1^{-1}.$$

Since the groups G and G^{sc} are smooth, this formula uniquely determines θ. The action θ has the following properties:

$$^{\rho(s)}s' = s s' s^{-1},$$
$$\rho({}^{g_1}s') = g_1\, \rho(s')\, g_1^{-1}$$

for $g_1 \in G(\overline{F}),\ s, s' \in G^{\mathrm{sc}}(\overline{F})$. In other words, $(G^{\mathrm{sc}}, G, \rho, \theta)$ is a (left) *crossed module of algebraic groups*; see for instance [5, Definition 3.2.1]. We write it as $\big(G^{\mathrm{sc}} \xrightarrow{\rho} G, \theta\big)$, and we regard it as a complex in degrees $-1, 0$. On F_s-points we obtain a $\mathrm{Gal}(F_s/F)$-equivariant crossed module $\big(G^{\mathrm{sc}}(F_s) \xrightarrow{\rho} G(F_s), \theta\big)$ where F_s is the separable closure of F in $\overline{F}$.

A.2. Deligne [11, Section 2.0.2] noticed that the commutator map

$$[\cdot,\cdot]\colon G \times G \to G, \quad g_1, g_2 \mapsto [g_1, g_2] := g_1 g_2 g_1^{-1} g_2^{-1}$$

lifts to a certain map (morphism of F-varieties)

$$\{\cdot,\cdot\}\colon G \times G \to G^{\mathrm{sc}}, \quad g_1, g_2 \mapsto \{g_1, g_2\}$$

as follows. The commutator map

$$G^{\mathrm{sc}} \times G^{\mathrm{sc}} \to G^{\mathrm{sc}}, \quad s_1, s_2 \mapsto [s_1, s_2] := s_1 s_2 s_1^{-1} s_2^{-1}$$

clearly factors via a morphism of F-varieties

$$(G^{\mathrm{sc}})^{\mathrm{ad}} \times (G^{\mathrm{sc}})^{\mathrm{ad}} \to G^{\mathrm{sc}}$$

where $(G^{\mathrm{sc}})^{\mathrm{ad}} = G^{\mathrm{sc}}/Z_{G^{\mathrm{sc}}}$ and $Z_{G^{\mathrm{sc}}}$ denotes the center of G^{sc}. Identifying $(G^{\mathrm{sc}})^{\mathrm{ad}}$ with $G^{\mathrm{ad}} := G/Z_G$, we obtain the desired morphism of F-varieties

$$\{\cdot,\cdot\}\colon G \times G \to G^{\mathrm{ad}} \times G^{\mathrm{ad}} \to G^{\mathrm{sc}}.$$

On $\overline{F}$-points, if $g_1, g_2 \in G(\overline{F})$, $g_1 = \rho(s_1)z_1$, $g_2 = \rho(s_2)z_2$ where $s_1, s_2 \in G^{\mathrm{sc}}(\overline{F})$, $z_1, z_2 \in Z_G(\overline{F})$, then

$$\{g_1, g_2\} = [s_1, s_2] = s_1 s_2 s_1^{-1} s_2^{-1}.$$

Since G and G^{sc} are smooth, this formula uniquely determines $\{\cdot,\cdot\}$. The constructed map $\{\cdot,\cdot\}$ satisfies the following equalities of Conduché [9, (3.11)]):

$$\begin{aligned}
&\rho\big(\{g_1, g_2\}\big) = [g_1, g_2];\\
&\big\{\rho(s_1), \rho(s_2)\big\} = [s_1, s_2];\\
&\{g_1, g_2\} = \{g_2, g_1\}^{-1};\\
&\{g_1 g_2, g_3\} = \{g_1 g_2 g_1^{-1}, g_1 g_3 g_1^{-1}\}\{g_1, g_3\}.
\end{aligned}$$

In other words, the map $\{,\}$ is a *symmetric braiding* of the crossed module $(G^{\mathrm{sc}}, G, \rho, \theta)$. We denote by G_{ab} the corresponding *stable* (=symmetrically braided) crossed module:

$$G_{\mathrm{ab}} = \big(G^{\mathrm{sc}} \xrightarrow{\rho} G, \theta, \{\cdot,\cdot\}\big).$$

Let $\varphi\colon G \to H$ be a homomorphism of reductive F-groups. It induces a homomorphism $\varphi^{\mathrm{sc}}\colon G^{\mathrm{sc}} \to H^{\mathrm{sc}}$. It is easy to see that

$$^{\varphi(g)}\varphi^{\mathrm{sc}}(s) = \varphi^{\mathrm{sc}}(^{g}s) \quad \text{for all } \ g \in G(\overline{F}),\, s \in G^{\mathrm{sc}}(\overline{F}).$$

Thus we obtain a morphism of crossed modules

$$(G^{\mathrm{sc}} \to G, \theta_G) \to (H^{\mathrm{sc}} \to H, \theta_H)$$

with obvious notations. Moreover, we have

$$\big\{\varphi(g_1), \varphi(g_2)\big\}_H = \varphi^{\mathrm{sc}}\big(\{g_1, g_2\}_G\big) \quad \text{for all } \ g_1, g_2 \in G(\overline{F})$$

with obvious notations; see [4] for a proof. Thus we obtain a morphism of stable crossed modules

$$\big(G^{\mathrm{sc}} \to G, \theta_G, \{\cdot,\cdot\}_G\big) \longrightarrow \big(H^{\mathrm{sc}} \to H, \theta_H, \{\cdot,\cdot\}_H\big). \tag{16}$$

A.3. In this appendix, we denote by H^1 and $\mathbb{H}^1$ the first *Galois* cohomology and hypercohomology. One can define the first Galois (hyper)cohomology of the $\mathrm{Gal}(F_s/F)$-equivariant crossed module

$$\mathbb{H}^1(F, G^{\mathrm{sc}} \xrightarrow{\rho} G, \theta) := \mathbb{H}^1(\mathrm{Gal}(F_s/F), G^{\mathrm{sc}}(F_s) \xrightarrow{\rho} G(F_s), \theta); \tag{17}$$

see [5, Section 3] or Noohi [17, Section 4]. A priori it is just a pointed set. However, using the symmetric braiding $\{\cdot,\cdot\}$, one can define a structure of abelian group on the pointed set (17); see Noohi [17, Corollaries 4.2 and 4.5]. We denote the obtained abelian group by

$$H^1_{\mathrm{ab}}(F, G) = \mathbb{H}^1(F, G_{\mathrm{ab}}) := \mathbb{H}^1\big(\mathrm{Gal}(F_s/F), G^{\mathrm{sc}}(F_s) \xrightarrow{\rho} G(F_s), \theta, \{\cdot,\cdot\}\big).$$

A homomorphism of reductive F-groups $\varphi\colon G \to H$ induces a morphism of stable crossed modules (16), which in turn induces a homomorphism of abelian groups

$$\varphi_{\mathrm{ab}}\colon H^1_{\mathrm{ab}}(F, G) \to H^1_{\mathrm{ab}}(F, H).$$

Thus $G \rightsquigarrow H^1_{\mathrm{ab}}(F, G)$ is a functor from the category of reductive F-group to the category of abelian groups.

A.4. The morphism of crossed modules (but not of stable crossed modules)

$$i_G\colon (1\to G)\hookrightarrow (G^{\mathrm{sc}}\to G)$$

induces a morphism of pointed sets

$$(i_G)_*\colon \mathbb{H}^1(F,1\to G)\to \mathbb{H}^1(F,G^{\mathrm{sc}}\xrightarrow{\rho} G).$$

The *abelianization map* is the composite morphism of pointed sets

$$\mathrm{ab}\colon H^1(F,G)=\mathbb{H}^1(F,1\to G)\xrightarrow{(i_G)_*}\mathbb{H}^1\big(F,G^{\mathrm{sc}}\xrightarrow{\rho}G,\theta\big)=\mathbb{H}^1(F,G_{\mathrm{ab}})=:H^1_{\mathrm{ab}}(F,G).$$

Here $\mathbb{H}^1\big(F,G^{\mathrm{sc}}\xrightarrow{\rho}G,\theta\big)$ and $\mathbb{H}^1(F,G_{\mathrm{ab}})$ are the same sets, but $\mathbb{H}^1(F,G_{\mathrm{ab}})$ is endowed with the structure of abelian group coming from the symmetric braiding $\{\cdot,\cdot\}$.

A.5. For a maximal torus $T\subseteq G$, we consider the homomorphism

$$\rho\colon T^{\mathrm{sc}}\to T$$

of Subsection 2.3, which we regard as a stable crossed module with the trivial action θ_T of T on T^{sc} and the trivial symmetric braiding $\{\cdot,\cdot\}_T\colon T\times T\to T^{\mathrm{sc}}$. We may and shall identify the first Galois hypercohomology of this stable crossed module with the usual first Galois hypercohomology of the complex $T^{\mathrm{sc}}\xrightarrow{\rho}T$ in degrees -1, 0:

$$\mathbb{H}^1\big(F,T^{\mathrm{sc}}\xrightarrow{\rho}T,\theta_T,\{\cdot,\cdot\}_T\big)=\mathbb{H}^1(F,T^{\mathrm{sc}}\xrightarrow{\rho}T).$$

The morphism of stable crossed modules

$$j_T\colon \big(T^{\mathrm{sc}}\xrightarrow{\rho}T,\theta_T,\{\cdot,\cdot\}_T\big)\hookrightarrow\big(G^{\mathrm{sc}}\xrightarrow{\rho}G,\theta,\{\cdot,\cdot\}\big)\tag{18}$$

is an *equivalence* (quasi-isomorphism), that is, it induces isomorphisms of F-group schemes

$$\ker[T^{\mathrm{sc}}\to T]\xrightarrow{\sim}\ker[G^{\mathrm{sc}}\to G]\quad\text{and}\quad\operatorname{coker}[T^{\mathrm{sc}}\to T]\xrightarrow{\sim}\operatorname{coker}[G^{\mathrm{sc}}\to G].$$

Following an idea sketched by Labesse and Lemaire [15], we observe that (18) induces isomorphisms on groups of F_s-points

$$\ker\big[T^{\mathrm{sc}}(F_s)\to T(F_s)\big]\xrightarrow{\sim}\ker\big[G^{\mathrm{sc}}(F_s)\to G(F_s)\big]$$
$$\operatorname{coker}\big[T^{\mathrm{sc}}(F_s)\to T(F_s)\big]\xrightarrow{\sim}\operatorname{coker}\big[G^{\mathrm{sc}}(F_s)\to G(F_s)\big]$$

(in arbitrary characteristic); see Theorem B.1 in Appendix B below. It follows that the induced map on Galois gypercohomology

$$(j_T)_*\colon \mathbb{H}^1(F,T^{\mathrm{sc}}\to T)\longrightarrow\mathbb{H}^1\big(F,G^{\mathrm{sc}}\xrightarrow{\rho}G,\theta,\{\cdot,\cdot\}\big)=:H^1_{\mathrm{ab}}(F,G)$$

is an isomorphism of abelian groups; see Noohi [17, Proposition 5.6]. This shows that the abelian group structure on the pointed set $\mathbb{H}^1\big(F,G^{\mathrm{sc}}\xrightarrow{\rho}G,\theta\big)$ defined using the bijection $(j_T)_*$ (as in [5, Section 3.8]) coincides with the abelian group structure defined by the symmetric braiding $\{\cdot,\cdot\}$.

Remark A.6. González-Avilés [12] defined the abelian fppf cohomology group $H^1_{\mathrm{fppf,ab}}(X,G)$ and the abelianization map

$$\mathrm{ab}\colon H^1_{\mathrm{fppf}}(X,G)\to H^1_{\mathrm{fppf,ab}}(X,G)$$

for a reductive group scheme G over an arbitrary base scheme X, which includes the case of a reductive group over a field F of arbitrary characteristic. However, his definition uses the center Z_G of G, and hence it is functorial only with respect to the *normal* homomorphisms $G_1\to G_2$ (homomorphisms with normal image, hence sending Z_{G_1} to Z_{G_2})), whereas our definition above (over a field only) is functorial with respect to all homomorphisms.

Appendix B. Equivalence on F_s-points in arbitrary characteristic *written by Zev Rosengarten*

In this appendix we prove the following theorem:

Theorem B.1. *Let F be a field of arbitrary characteristic and let F_s be a fixed separable closure of F. Let*

$$\rho\colon G^{\mathrm{sc}} \twoheadrightarrow [G,G] \hookrightarrow G$$

be as in Subsection 2.3. Let $T \subseteq G$ be a maximal torus. We write $T^{\mathrm{sc}} = \rho^{-1}(T)$. Then the morphism of crossed modules

$$\big(T^{\mathrm{sc}}(F_s) \to T(F_s)\big) \longrightarrow \big(G^{\mathrm{sc}}(F_s) \to G(F_s)\big)$$

is an equivalence (quasi-isomorphism).

Proof. We must show that the maps

$$i_{\mathrm{ker}}\colon \ker\big[T^{\mathrm{sc}}(F_s) \to T(F_s)\big] \longrightarrow \ker\big[G^{\mathrm{sc}}(F_s) \to G(F_s)\big] \tag{19}$$

and

$$i_{\mathrm{cok}}\colon \operatorname{coker}\big[T^{\mathrm{sc}}(F_s) \to T(F_s)\big] \longrightarrow \operatorname{coker}\big[G^{\mathrm{sc}}(F_s) \to G(F_s)\big] \tag{20}$$

are isomorphisms.

For (19), the injectivity is obvious. Moreover, any element of $\ker\big[G^{\mathrm{sc}}(F_s) \to G(F_s)\big]$ lies in the preimage T^{sc} of T, hence it is an element of $T^{\mathrm{sc}}(F_s)$ and of $\ker\big[T^{\mathrm{sc}}(F_s) \to T(F_s)\big]$, which gives the surjectivity of i_{ker}.

We prove the injectivity of (20). Let $[t] \in \operatorname{coker}\big[T^{\mathrm{sc}}(F_s) \to T(F_s)\big]$, $t \in T(F_s)$, and $[t] \in \ker i_{\mathrm{cok}}$; then $t = \rho(s)$ for some $s \in G^{\mathrm{sc}}(F_s)$. Since $T^{\mathrm{sc}} = \rho^{-1}(T)$, we see that $s \in T^{\mathrm{sc}}(F_s)$, whence $[t] = 1$, as required.

We prove the surjectivity of (20). Let $C \subseteq G$ denote the radical (largest central torus) of G. Then the map

$$\psi\colon C \times G^{\mathrm{sc}} \to G, \qquad (c,s) \mapsto c\cdot\rho(s) \text{ for } c \in C,\ s \in G^{\mathrm{sc}}$$

is surjective with central kernel $Z \cong \rho^{-1}(C \cap [G,G])$ (which might be non-smooth). We have an exact commutative diagram of F-group schemes

$$\begin{array}{ccccccccc}
1 & \longrightarrow & Z & \longrightarrow & C \times T^{\mathrm{sc}} & \xrightarrow{\psi_T} & T & \longrightarrow & 1 \\
 & & \| & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Z & \longrightarrow & C \times G^{\mathrm{sc}} & \xrightarrow{\psi} & G & \longrightarrow & 1
\end{array}$$

in which the maps on F_s-points

$$\psi_T\colon C(F_s) \times T^{\mathrm{sc}}(F_s) \to T(F_s) \quad \text{and} \quad \psi\colon C(F_s) \times G^{\mathrm{sc}}(F_s) \to G(F_s)$$

might not be surjective. This diagram gives rise to an exact commutative diagram of fppf cohomology groups

$$\begin{array}{ccccccc}
C(F_s) \times T^{\mathrm{sc}}(F_s) & \xrightarrow{\psi_T} & T(F_s) & \longrightarrow & H^1_{\mathrm{fppf}}(F_s, Z) & \longrightarrow & H^1_{\mathrm{fppf}}(F_s, C \times T^{\mathrm{sc}}) = 1 \\
\downarrow & & \downarrow & & \| & & \downarrow \\
C(F_s) \times G^{\mathrm{sc}}(F_s) & \xrightarrow{\psi} & G(F_s) & \longrightarrow & H^1_{\mathrm{fppf}}(F_s, Z) & \longrightarrow & H^1_{\mathrm{fppf}}(F_s, C \times G^{\mathrm{sc}}) = 1
\end{array}$$

in which the rightmost term in both rows is trivial because F_s is separably closed and the F-groups $C \times T^{\mathrm{sc}}$, $C \times G^{\mathrm{sc}}$ are smooth. The latter diagram shows that

$$G(F_s) = T(F_s)\cdot\psi\big(C(F_s) \times G^{\mathrm{sc}}(F_s)\big) = T(F_s)\cdot C(F_s)\cdot\rho\big(G^{\mathrm{sc}}(F_s)\big) = T(F_s)\cdot\rho\big(G^{\mathrm{sc}}(F_s)\big),$$

whence the surjectivity of (20). □

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