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Volume 361 (2023), p. 969-971

Published online: 7 September 2023

https://doi.org/10.5802/crmath.456

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A note on the weighted log canonical threshold

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Abstract. In this paper, we introduce and study a set relative to singularities of plurisubharmonic functions. We prove that this set is countable under the condition $h > 0$ on $B \setminus \{0\}$.


Manuscript received 28 October 2022, revised 11 December 2022, accepted 18 December 2022.

1. Introduction

Let $\Omega$ be a domain in $\mathbb{C}^n$, $z_0 \in \Omega$ and $\varphi$ be a plurisubharmonic function on $\Omega$ (briefly, psh). Following Demailly and Kollár\cite{3}, we introduce the log canonical threshold of $\varphi$ at $z_0$:

$$c_\varphi(z_0) = \sup \{c > 0 : e^{-2c\varphi} \text{ is } L^1(dV_{2n}) \text{ on a neighborhood of } z_0\},$$

(1)

where $dV_{2n}$ denotes the Lebesgue measure of $\mathbb{C}^n$.

It is an invariant of the singularity of $\varphi$ at $z_0$. We refer the readers to\cite{1, 2, 4–7, 9, 10} for further information and applications to this number.

For every non-negative Radon measures $\mu$ on a neighbourhood of $z_0 \in \mathbb{C}^n$. Following Pham in\cite{8}, we introduce the weighted log canonical threshold of $\varphi$ with weight $\mu$ at $z_0$ to be:

$$c_{\mu, \varphi}(z_0) = \sup \{c > 0 : \exists r > 0, \int_{B(0, r)} e^{-2c\varphi(z + z_0)} d\mu(z) < +\infty\}. \tag{2}$$

In the case if $\mu = hdV_{2n}$ where $h \in L^1(dV_{2n})$, $h > 0$ on $B \setminus \{0\}$, $h \in L^\infty(\mathbb{B})$ then we introduce the weighted log canonical threshold of $\varphi$ with weight $\mu$ at $z_0$ to be:

$$c_{hdV_{2n}, \varphi}(z_0) = \sup \{c > 0 : \exists r > 0, \int_{B(z_0, r)} e^{-2c\varphi(z)} h(z - z_0) dV_{2n}(z) < +\infty\}. \tag{3}$$

From the definition of $c_{hdV_{2n}, \varphi}(z_0)$ and $c_\varphi(z_0)$, we have $c_{hdV_{2n}, \varphi}(z_0) \geq c_\varphi(z_0)$. In the paper, we study properties of the set $E_{h, \varphi} = \{z \in \Omega : c_{hdV_{2n}, \varphi}(z) > c_\varphi(z)\}$. The main result of the paper prove that the set $E_{h, \varphi}$ is a countable set.
2. Main Theorem

Theorem 1. If \( h \in L^1(\text{d}V_{2n}), h > 0 \) on \( \mathbb{B} \setminus \{0\}, h \in L^\infty(\mathbb{B}) \) then
\[
E_{h, \varphi} = \{ z \in \Omega : c_{hdV_{2n}, \varphi}(z) > c_{\varphi}(z) \}
\]
is a countable set.

Proof. We have
\[
E_{h, \varphi} = \bigcup_{c \in \mathbb{Q}} \{ z \in \Omega : c_{\varphi}(z) < c < c_{hdV_{2n}, \varphi}(z) \}.
\]
It means that we need to prove the following set
\[
E_{c, h, \varphi} = \{ z \in \Omega : c_{\varphi}(z) < c < c_{hdV_{2n}, \varphi}(z) \}
\]
is a countable set. Indeed, let \( z_0 \in E_{c, h, \varphi} \). We have
\[
c_{\varphi}(z_0) < c < c_{hdV_{2n}, \varphi}(z_0).
\]
Since \( c < c_{hdV_{2n}, \varphi}(z_0) \) we can find \( r > 0 \) such that
\[
\int_{\mathbb{B}(z_0, r)} e^{-2c\varphi(z)} h(z-z_0) \text{d}V = \int_{\mathbb{B}(0, r)} e^{-2c\varphi(z+z_0)} h(z) \text{d}V_{2n} < +\infty.
\]
Since \( h > 0 \) on \( \mathbb{B}(0, r) \setminus \{0\} \) \( h(0) = 0 \), we have \( \forall \ w \in \mathbb{B}(z_0, r) \setminus \{z_0\} \) there exists \( \delta > 0 \) such that \( \mathbb{B}(w, \delta) \subset \mathbb{B}(z_0, r) \setminus \{z_0\} \) and
\[
\int_{\mathbb{B}(0, \delta)} e^{-2c\varphi(z+w)} \text{d}V_{2n}(z) < +\infty.
\]
We obtain that \( c_{\varphi}(z) > c, \forall \ w \in \mathbb{B}(z_0, r) \setminus \{z_0\} \). Thus, if \( z_0 \in E_{c, h, \varphi} \) then
\[
E_{c, h, \varphi} \cap \mathbb{B}(z_0, r) \setminus \{z_0\} = \emptyset.
\]
So we have \( E_{c, h, \varphi} \) is a countable set. From
\[
E_{h, \varphi} = \bigcup_{c \in \mathbb{Q}} E_{c, h, \varphi}
\]
we have \( E_{h, \varphi} \) is a countable set. \( \square \)

The following proposition shows a corollary of main theorem.

Corollary 2. Let \( \Omega \) be a domain of \( \mathbb{C}^n \) and \( f : \Omega \rightarrow \mathbb{C} \) be a holomorphic function. Assuming that \( h \in L^1(\text{d}V_{2n}), h > 0 \) on \( \mathbb{B} \setminus \{0\}, h \in L^\infty(\mathbb{B}) \). Then
\[
E_{h, \log|f|} \subset \{ z \in \Omega : f = 0 \}_{\text{sing}},
\]
where \( \{ z \in \Omega : f = 0 \}_{\text{sing}} \) is the singularities of the hypersurface \( \{ z \in \Omega : f = 0 \} \).

Proof. By Theorem 1, we have \( E_{h, \log|f|} \) is a countable subset of \( \{ z \in \Omega : f = 0 \} \). Take \( z_0 \in \{ z \in \Omega : f = 0 \}_{\text{reg}} \). We only need to prove that \( z_0 \not\in E_{h, \log|f|} \). Indeed, we have \( f(z) = h^m \) in a neighborhood of the point \( z_0 \), where \( \{ z \in \Omega : f = 0 \} \) defined locally at the point \( z_0 \) by \( h \). On the other hand, from the proof of Theorem 1 we have
\[
c_{\varphi}(z_0) \leq c_{hdV_{2n}, \varphi}(z_0) \leq \lim_{r \to 0} \min\{c_{\varphi}(z) : z \in \mathbb{B}(z_0, r) \setminus \{z_0\}\}.
\]
Therefore
\[
c_{\log|f|}(z_0) = c_{hdV_{2n}, \log|f|}(z_0) = \frac{1}{m}.
\]
This implies that \( z_0 \not\in E_{h, \log|f|} \). \( \square \)

Example. We choose \( f(z) = z_1^{m_1} + \cdots + z_n^{m_n} \) and \( h(z) = ||z||^{2t} (t > 0) \). Then
\begin{enumerate}
\item \( E_{h, \log|f|} = \{0\} \) if \( \sum_{j=1}^n \frac{1}{m_j} < 1 \).
\item \( E_{h, \log|f|} = \emptyset \) if \( \sum_{j=1}^n \frac{1}{m_j} \geq 1 \).
\end{enumerate}
References