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Complex analysis and geometry / *Analyse et géométrie complexes*

A note on the weighted log canonical threshold

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Abstract. In this paper, we introduce and study a set relative to singularities of plurisubharmonic functions. We prove that this set is countable under the condition $h > 0$ on $\mathbb{B} \setminus \{0\}$.

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1. Introduction

Let Ω be a domain in $\mathbb{C}^n, z_0 \in \Omega$ and φ be a plurisubharmonic function on Ω (briefly, psh). Following Demailly and Kollár [3], we introduce the log canonical threshold of φ at z_0:

$$c_\varphi(z_0) = \sup\{c > 0 : e^{-2c\varphi} \text{ is } L^1(\mathrm{d}V_{2n}) \text{ on a neighborhood of } z_0\}, \tag{1}$$

where $\mathrm{d}V_{2n}$ denotes the Lebesgue measure of $\mathbb{C}^n$.

It is an invariant of the singularity of φ at z_0. We refer the readers to [1, 2, 4–7, 9, 10] for further information and applications to this number.

For every non-negative Radon measures μ on a neighbourhood of $z_0 \in \mathbb{C}^n$. Following Pham in [8], we introduce the weighted log canonical threshold of φ with weight μ at z_0 to be:

$$c_{\mu,\varphi}(z_0) = \sup\left\{c > 0 : \exists\, r > 0\,, \int_{\mathbb{B}(0,r)} e^{-2c\varphi(z+z_0)}\mathrm{d}\mu(z) < +\infty\right\}. \tag{2}$$

In the case if $\mu = h\mathrm{d}V_{2n}$ where $h \in L^1(\mathrm{d}V_{2n}), h > 0$ on $\mathbb{B} \setminus \{0\}, h \in L^\infty(\mathbb{B})$ then we introduce the weighted log canonical threshold of φ with weight μ at z_0 to be:

$$c_{h\mathrm{d}V_{2n},\varphi}(z_0) = \sup\left\{c > 0 : \exists\, r > 0\,, \int_{\mathbb{B}(z_0,r)} e^{-2c\varphi(z)} h(z - z_0)\mathrm{d}V_{2n}(z) < +\infty\right\}. \tag{3}$$

From the definition of $c_{h\mathrm{d}V_{2n},\varphi}(z_0)$ and $c_\varphi(z_0)$, we have $c_{h\mathrm{d}V_{2n},\varphi}(z_0) \geq c_\varphi(z_0)$. In the paper, we study properties of the set $E_{h,\varphi} = \{z \in \Omega : c_{h\mathrm{d}V_{2n},\varphi}(z) > c_\varphi(z)\}$. The main result of the paper prove that the set $E_{h,\varphi}$ is a countable set.

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2. Main Theorem

Theorem 1. *If $h \in L^1(\mathrm{d}V_{2n}), h > 0$ on $\mathbb{B} \setminus \{0\}, h \in L^\infty(\mathbb{B})$ then*

$$E_{h,\varphi} = \{z \in \Omega : c_{h\mathrm{d}V_{2n},\varphi}(z) > c_\varphi(z)\}$$

is a countable set.

Proof. We have

$$E_{h,\varphi} = \cup_{c\in\mathbb{Q}}\{z \in \Omega : c_\varphi(z) < c < c_{h\mathrm{d}V_{2n},\varphi}(z)\}.$$

It means that we need to prove the following set

$$E_{c,h,\varphi} = \{z \in \Omega : c_\varphi(z) < c < c_{h\mathrm{d}V_{2n},\varphi}(z)\}$$

is a countable set. Indeed, let $z_0 \in E_{c,h,\varphi}$. We have

$$c_\varphi(z_0) < c < c_{h\mathrm{d}V_{2n},\varphi}(z_0).$$

Since $c < c_{h\mathrm{d}V_{2n},\varphi}(z_0)$ we can find $r > 0$ such that

$$\int_{\mathbb{B}(z_0,r)} e^{-2c\varphi(z)} h(z - z_0)\mathrm{d}V = \int_{\mathbb{B}(0,r)} e^{-2c\varphi(z+z_0)} h(z)\mathrm{d}V_{2n} < +\infty.$$

Since $h > 0$ on $\mathbb{B}(0, r) \setminus \{0\}$ $(h(0) = 0)$, we have $\forall\ w \in \mathbb{B}(z_0, r) \setminus \{z_0\}$ there exists $\delta > 0$ such that $\mathbb{B}(w,\delta) \subset \mathbb{B}(z_0, r) \setminus \{z_0\}$ and

$$\int_{\mathbb{B}(0,\delta)} e^{-2c\varphi(z+w)}\mathrm{d}V_{2n}(z) < +\infty.$$

We obtain that $c_\varphi(z) > c,\ \forall\ w \in \mathbb{B}(z_0, r) \setminus \{z_0\}$. Thus, if $z_0 \in E_{c,h,\varphi}$ then

$$E_{c,h,\varphi} \cap \mathbb{B}(z_0, r) \setminus \{z_0\} = \varnothing.$$

So we have $E_{c,h,\varphi}$ is a countable set. From

$$E_{h,\varphi} = \bigcup_{c\in\mathbb{Q}} E_{c,h,\varphi}$$

we have $E_{h,\varphi}$ is a countable set. □

The following proposition shows a corollary of main theorem.

Corollary 2. *Let Ω be a domain of $\mathbb{C}^n$ and $f : \Omega \longrightarrow \mathbb{C}$ be a holomorphic function. Assuming that $h \in L^1(\mathrm{d}V_{2n}), h > 0$ on $\mathbb{B} \setminus \{0\}, h \in L^\infty(\mathbb{B})$. Then*

$$E_{h,\log|f|} \subset \{z \in \Omega : f = 0\}_{\mathrm{sing}},$$

where $\{z \in \Omega : f = 0\}_{\mathrm{sing}}$ is the singularities of the hypersurface $\{z \in \Omega : f = 0\}$.

Proof. By Theorem 1, we have $E_{h,\log|f|}$ is a countable subset of $\{z \in \Omega : f = 0\}$. Take $z_0 \in \{z \in \Omega : f = 0\}_{\mathrm{reg}}$. We only need to prove that $z_0 \not\in E_{h,\log|f|}$. Indeed, we have $f(z) = h^m$ in a neighborhood of the point z_0, where $\{z \in \Omega : f = 0\}$ defined locally at the point z_0 by h. On the other hand, from the proof of Theorem 1 we have

$$c_\varphi(z_0) \le c_{h\mathrm{d}V_{2n},\varphi}(z_0) \le \lim_{r\to 0} \min\{c_\varphi(z) :\ z \in \mathbb{B}(z_0, r)\setminus\{z_0\}\}.$$

Therefore

$$c_{\log|f|}(z_0) = c_{h\mathrm{d}V_{2n},\log|f|}(z_0) = \frac{1}{m}.$$

This implies that $z_0 \not\in E_{h,\log|f|}$. □

Example. We choose $f(z) = z_1^{m_1} + \cdots + z_n^{m_n}$ and $h(z) = \|z\|^{2t}$ $(t > 0)$. Then

(1) $E_{h,\log|f|} = \{0\}$ if $\sum_{j=1}^n \frac{1}{m_j} < 1$.

(2) $E_{h,\log|f|} = \emptyset$ if $\sum_{j=1}^n \frac{1}{m_j} \ge 1$.

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