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Absolutely minimum attaining Toeplitz and absolutely norm attaining Hankel operators

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Abstract. In this article, we completely characterize absolutely norm attaining Hankel operators and absolutely minimum attaining Toeplitz operators. We also improve [19, Theorem 2.1], by characterizing the absolutely norm attaining Toeplitz operator T_φ in terms of the symbol $\varphi \in L^\infty$.

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1. Introduction

Let L^2 denote the Lebesgue space of all square integrable functions on the unit circle \mathbb{T} in the complex plane with respect to the normalized Lebesgue measure μ and L^∞ be the Banach space of all complex valued essentially bounded μ -measurable functions on \mathbb{T} . Let H^2 denote the closed subspace of L^2 consisting of all those functions whose negative Fourier co-efficients vanish.

For any $\varphi \in L^\infty$, the Toeplitz operator $T_\varphi : H^2 \rightarrow H^2$ is defined by

$$T_\varphi f = P(\varphi f) \quad \text{for all } f \in H^2,$$

and the Hankel operator $H_\varphi : H^2 \rightarrow H^2$ is defined by

$$H_\varphi f = J(I - P)\varphi f \quad \text{for all } f \in H^2,$$

where $(\varphi f)(z) = \varphi(z)f(z)$ for all $z \in \mathbb{T}$, P is the orthogonal projection of L^2 onto H^2 and $J(z^{-n}) = z^{n-1}$, $n = 0, \pm 1, \pm 2, \dots$ is the unitary operator on L^2 .

Toeplitz and Hankel operators are well studied in matrix theory, operator theory, function theory etc., as they find applications in many fields. The initial study of Toeplitz operators was done by Otto Toeplitz and thereafter Brown and Halmos in [4] discussed some of the important properties of them. Further, H. Widom, R. G. Douglas, D. Sarason, etc., also worked on Toeplitz

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operators and most of their results can be found in [7]. We refer to [13, 17] for the results on Hankel operators.

One of the important properties of any bounded linear operator is its norm attaining property. Such a property is vastly studied on Banach spaces (for example, see [1, 2, 12]). A few developments of norm attaining operators on Hilbert spaces can be found in [5, 8]. For more details on norm attaining Toeplitz and Hankel operators we refer to [3, 21]. It is important to note that the set of all compact operators and isometries defined between the Hilbert spaces are always norm attaining. Moreover, on restriction of such operators to any non-zero closed subspace of the Hilbert space is also norm attaining. This observation initiated the study of new class of operators by Carvajal and Neves [5] and called them the absolutely norm attaining operators or \mathcal{AN} -operators.

A natural counterpart of the norm attaining and the absolutely norm attaining operators are the minimum attaining and the absolutely minimum attaining operators or \mathcal{AM} -operators, which were also introduced by Carvajal and Neves in [6].

Due to the lack of non-zero compact Toeplitz operators on H^2 , the authors in [19] were interested in looking at \mathcal{AN} -Toeplitz operators, since this set is non-empty as it contains isometric Toeplitz operators. The authors also characterized absolutely minimum attaining Hankel operators in the same article.

In this article, we address the following question which is posed in [19].

Question 1. *Characterize all the \mathcal{AM} -Toeplitz operators and all the \mathcal{AN} -Hankel operators.*

In [19, Theorem 2.1], a characterization of an \mathcal{AN} -Toeplitz operator $T_\varphi, \varphi \in L^\infty$ is given as follows:

Theorem 2. *Let $\varphi \in L^\infty$. Then T_φ is an \mathcal{AN} -operator if and only if $\|\varphi\|_\infty^2 I - T_\varphi^* T_\varphi$ is a finite rank operator.*

Here we characterize such an operator in terms of the symbol φ itself, which improves the above result.

We also characterize \mathcal{AM} -Toeplitz operators and further prove that there are no non-compact \mathcal{AN} -Hankel operators. All these results are found in Section 2.

In the remaining part of this section, we give a few notations and definitions which are necessary for developing the article.

1.1. Preliminaries

Let H_1, H_2 be infinite dimensional complex Hilbert spaces and $\mathcal{B}(H_1, H_2)$ denote the Banach space of all bounded linear operators from H_1 to H_2 . If $T \in \mathcal{B}(H_1, H_2)$, then T is said to be finite rank if its range space is finite dimensional and compact if T maps any bounded set in H_1 to a pre-compact set in H_2 .

Definition 3 ([5, Definitions 1.1, 1.2]). *An operator $T \in \mathcal{B}(H_1, H_2)$ is called norm attaining if there exists a unit vector $x \in H_1$ such that $\|Tx\| = \|T\|$. If for every non-zero closed subspace M of H_1 , $T|_M : M \rightarrow H_2$ is norm attaining, then T is called absolutely norm attaining or an \mathcal{AN} -operator.*

Definition 4 ([6, Definitions 1.1, 1.4]). *An operator $T \in \mathcal{B}(H_1, H_2)$ is said to be minimum attaining if there exists a unit vector $x \in H_1$ such that $m(T) = \|Tx\|$, where*

$$m(T) := \inf\{\|Tx\| : x \in H_1, \|x\| = 1\}.$$

If for every non-zero closed subspace M of H_1 , $T|_M : M \rightarrow H_2$ is minimum attaining, then T is said to be absolutely minimum attaining or an \mathcal{AM} -operator.

For more details on \mathcal{AN} -operators we refer to [5, 14, 16, 18] and for that on \mathcal{AM} -operators we refer to [6, 9, 15].

Let $\mathcal{B}(H^2)$ be the Banach space of all bounded linear operators on H^2 and let $\mathcal{F}(H^2)$ and $\mathcal{K}(H^2)$ denote the set of all finite rank and compact operators on H^2 , respectively. The set of all positive elements (operators) in $\mathcal{F}(H^2)$ and $\mathcal{K}(H^2)$ are denoted by $\mathcal{F}(H^2)_+$ and $\mathcal{K}(H^2)_+$, respectively. The set of all \mathcal{AN} and \mathcal{AM} -operators on H^2 are denoted by $\mathcal{AN}(H^2)$ and $\mathcal{AM}(H^2)$, respectively.

Let $C(\mathbb{T})$ denote the set of all complex valued continuous functions on \mathbb{T} . The Sarason algebra $H^\infty + C(\mathbb{T})$ is a closed subalgebra of L^∞ , where $H^\infty = H^2 \cap L^\infty$. The notations and terminologies used in [19] will be adhered throughout this article.

2. Main Results

We start this section by recalling a result which connects both Toeplitz and Hankel operators.

Proposition 5 ([21, Proposition 8]). *For any $\varphi, \psi \in L^\infty$, we have $H_\psi^* H_\varphi = T_{\overline{\psi}\varphi} - T_\psi^* T_\varphi$.*

The following results characterize all finite rank and compact Hankel operators.

Theorem 6 ([17, Corollary 3.2, p. 21]). *Let $\varphi \in L^\infty$. Then $H_\varphi \in \mathcal{F}(H^2)$ if and only if $(I - P)\varphi$ is a rational function.*

Theorem 7 ([13, Corollary 4.3.3, p. 145]). *Let $\varphi \in L^\infty$. Then $H_\varphi \in \mathcal{K}(H^2)$ if and only if $\varphi \in H^\infty + C(\mathbb{T})$.*

Now we characterize an \mathcal{AN} -Toeplitz operator T_φ in terms of φ .

Theorem 8. *Let $\varphi \in L^\infty$. Then $T_\varphi \in \mathcal{AN}(H^2)$ if and only if $|\varphi| = \|\varphi\|_\infty$ a.e. (μ) and $(I - P)\varphi$ is a rational function.*

Proof. By Theorem 2 and [19, Corollary 2.3], we have $\|\varphi\|_\infty^2 I - T_\varphi^* T_\varphi = F$, where $F \in \mathcal{F}(H^2)_+$ and $|\varphi| = \|\varphi\|_\infty$ a.e. (μ) . Hence by Proposition 5, we get $H_\varphi^* H_\varphi = F \in \mathcal{F}(H^2)_+$. Therefore by Theorem 6, it follows that $(I - P)\varphi$ is a rational function.

Conversely, if $|\varphi| = \|\varphi\|_\infty$ a.e. (μ) and $(I - P)\varphi$ is a rational function, then $H_\varphi^* H_\varphi = F$, where $F \in \mathcal{F}(H^2)_+$. Again by Proposition 5, we get $\|\varphi\|_\infty^2 I - T_\varphi^* T_\varphi = F$. Hence by Theorem 2, $T_\varphi \in \mathcal{AN}(H^2)$. □

Corollary 9. *Let $\varphi \in L^\infty$. Then $T_\varphi \in \mathcal{AN}(H^2)$ if and only if $|\varphi| = \|\varphi\|_\infty$ a.e. (μ) and $H_\varphi \in \mathcal{AM}(H^2)$.*

Proof. By [19, Theorem 3.3], we have $H_\varphi \in \mathcal{AM}(H^2)$ if and only if $H_\varphi \in \mathcal{F}(H^2)$. Now, from Theorems 6 and 8, we get the required conclusion. □

Remark 10. If $T_\varphi \in \mathcal{AN}(H^2)$, then it satisfies $T_\varphi^* T_\varphi + H_\varphi^* H_\varphi = \|\varphi\|_\infty^2 I$.

Next we characterize all \mathcal{AM} -Toeplitz operators. For this, we need the following definition.

Definition 11. *For $T \in \mathcal{B}(H^2)$,*

(1) *the spectrum of T is defined by*

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathcal{B}(H^2)\},$$

(2) *the essential spectrum of T is defined by $\sigma_{\text{ess}}(T) = \sigma(\pi(T))$, where $\pi : \mathcal{B}(H^2) \rightarrow \mathcal{B}(H^2) / \mathcal{K}(H^2)$ is the canonical quotient map.*

Theorem 12. *Let $\varphi \in L^\infty$. Then $T_\varphi \in \mathcal{AM}(H^2)$ if and only if $|\varphi| = \alpha$ a.e. (μ) and $\varphi \in H^\infty + C(\mathbb{T})$, where $\alpha \in \sigma_{\text{ess}}(|T_\varphi|)$ and $|T_\varphi|$ is the positive square root of $T_\varphi^* T_\varphi$.*

Proof. If $T_\varphi \in \mathcal{AM}(H^2)$, then by [9, Theorem 5.14], $T_\varphi^* T_\varphi \in \mathcal{AM}(H^2)$. From [9, Theorem 5.8], we have $T_{\bar{\varphi}} T_\varphi = \alpha^2 I - K + F$, where $\alpha^2 \in \sigma_{\text{ess}}(T_{\bar{\varphi}} T_\varphi)$, $\alpha \geq 0$, $K \in \mathcal{K}(H^2)_+$, $F \in \mathcal{F}(H^2)_+$ with $K \leq \alpha^2 I$ and $KF = 0$. Hence by [19, Proposition 2.18], we get $|\varphi| = \alpha$ a.e. (μ) and by [11, Lemma 3], $\alpha \in \sigma_{\text{ess}}(|T_\varphi|)$. By Proposition 5, we get $H_\varphi^* H_\varphi = K - F \in \mathcal{K}(H^2)$. Therefore $H_\varphi \in \mathcal{K}(H^2)$ and by Theorem 7, $\varphi \in H^\infty + C(\mathbb{T})$.

Conversely, assume that $|\varphi| = \alpha$ a.e. (μ) and $\varphi \in H^\infty + C(\mathbb{T})$. Then $H_\varphi^* H_\varphi = K$, where $K \in \mathcal{K}(H^2)_+$. By Proposition 5, we get $T_{\bar{\varphi}} T_\varphi = \alpha^2 I - K$. Then by [19, Proposition 3.5], $T_{\bar{\varphi}} T_\varphi \in \mathcal{AM}(H^2)$ and hence $T_\varphi \in \mathcal{AM}(H^2)$ by [9, Theorem 5.14]. □

In general, the set of all \mathcal{AN} -operators (or \mathcal{AM} -operators) is not closed. As $\{\varphi \in L^\infty : (I - P)\varphi \text{ is a rational function}\}$ is not a closed set, the set of all \mathcal{AN} -Toeplitz operators is not closed. But the set of all \mathcal{AM} -Toeplitz operators is closed.

Let $\mathfrak{T}_\infty(H^2)$ denote the set of all Toeplitz operators on H^2 induced by the symbols from L^∞ .

Proposition 13. *The set of all \mathcal{AM} -Toeplitz operators is closed in $\mathfrak{T}_\infty(H^2)$.*

Proof. Let T_ψ be in the closure of set of all \mathcal{AM} -Toeplitz operators. Then there exists a sequence of \mathcal{AM} -Toeplitz operators $\{T_{\varphi_n}\}$ such that T_{φ_n} converges to T_ψ in the operator norm. So by Theorem 12, we have $\varphi_n \in H^\infty + C(\mathbb{T})$ and $|\varphi_n| = \alpha_n$, where $\alpha_n \in \sigma_{\text{ess}}(|T_{\varphi_n}|)$ which further implies $\alpha_n^2 \in \sigma_{\text{ess}}(T_{\varphi_n}^* T_{\varphi_n})$ for all $n \in \mathbb{N}$ ([11, Lemma 3]). Since $H^\infty + C(\mathbb{T})$ is a closed subalgebra of L^∞ , we get $\psi \in H^\infty + C(\mathbb{T})$. Now by [9, Theorem 5.14], we have $T_{\varphi_n}^* T_{\varphi_n} \in \mathcal{AM}(H^2)$ and by [9, Theorem 5.8], we write $T_{\varphi_n}^* T_{\varphi_n} = \beta_n^2 I - K_n + F_n$, where $\beta_n^2 \in \sigma_{\text{ess}}(T_{\varphi_n}^* T_{\varphi_n})$, $\beta_n \geq 0$, $K_n \in \mathcal{K}(H^2)_+$, $F_n \in \mathcal{F}(H^2)_+$ with $K_n \leq \beta_n^2 I$ and $K_n F_n = 0$ for all $n \in \mathbb{N}$. From [15, Theorem 3.10], we have $\sigma_{\text{ess}}(T_{\varphi_n}^* T_{\varphi_n}) = \{\beta_n^2\} = \{\alpha_n^2\}$, so $\alpha_n = \beta_n$ for all $n \in \mathbb{N}$. As $\alpha_n^2 \leq \|T_{\varphi_n}^* T_{\varphi_n}\| \leq \sup_{n \in \mathbb{N}} \|T_{\varphi_n}\|^2$, we have $\{\alpha_n^2\}$ is bounded. Then there exists a subsequence $\{\alpha_{n_k}\}$ of $\{\alpha_n\}$ such that $\alpha_{n_k}^2$ converges to α^2 for some $\alpha \geq 0$. Hence $T_{\varphi_{n_k}}^* T_{\varphi_{n_k}} - \alpha_{n_k}^2 I = K_{n_k} - F_{n_k}$ converges to some $K \in \mathcal{K}(H^2)$. Therefore $T_\psi^* T_\psi = \alpha^2 I + K$. Now by [7, Exercise 7.7], we get $|\psi| = \alpha$, where $\alpha \in \sigma_{\text{ess}}(|T_\psi|)$. Hence by Theorem 12, $T_\psi \in \mathcal{AM}(H^2)$. □

Remark 14. In [20], it was proved that the operator norm closure of \mathcal{AM} -operators is the same as the operator norm closure of \mathcal{AN} -operators. Here by Theorems 8 and 12, we observe that the set of all \mathcal{AN} -Toeplitz operators is contained in the set of all \mathcal{AM} -Toeplitz operators. So by Proposition 13, we conclude that the operator norm closure of the set of \mathcal{AN} -Toeplitz operators is the set of \mathcal{AM} -Toeplitz operators. In other words, set of all \mathcal{AN} -Toeplitz operators is dense in the set of all \mathcal{AM} -Toeplitz operators with respect to the operator norm.

The next theorem shows that there is no non-compact absolutely norm attaining Hankel operator. For a detailed study on compact Hankel operators we refer to [10].

Theorem 15. *Let $\varphi \in L^\infty$. Then $H_\varphi \in \mathcal{AN}(H^2)$ if and only if $H_\varphi \in \mathcal{K}(H^2)$.*

Proof. Clearly if $H_\varphi \in \mathcal{K}(H^2)$, then $H_\varphi \in \mathcal{AN}(H^2)$.

On the other hand, if $H_\varphi \in \mathcal{AN}(H^2)$, then by [14, Corollary 2.11], $H_\varphi^* H_\varphi \in \mathcal{AN}(H^2)$. Hence by [14, Theorem 2.5], $H_\varphi^* H_\varphi = \alpha I - F + K$ for some $\alpha \geq 0$, $F \in \mathcal{F}(H^2)_+$, $K \in \mathcal{K}(H^2)_+$ with $F \leq \alpha I$ and $KF = 0$. By Proposition 5, we get $\alpha I - F + K = T_{|\varphi|^2} - T_\varphi^* T_\varphi$ or $T_{|\varphi|^2 - \alpha} - T_{\bar{\varphi}} T_\varphi = K - F \in \mathcal{K}(H^2)$. So by [7, Exercise 7.7], $|\varphi|^2 - \alpha = \bar{\varphi}\varphi$, which implies $\alpha = 0$. Hence $H_\varphi^* H_\varphi \in \mathcal{K}(H^2)_+$ which in turn results in $H_\varphi \in \mathcal{K}(H^2)$. □

Corollary 16. *Let $\varphi \in L^\infty$. Then $H_\varphi \in \mathcal{AN}(H^2)$ if and only if $\varphi \in H^\infty + C(\mathbb{T})$.*

Proof. The proof follows by Theorems 7 and 15. □

Remark 17.

- (1) By Theorem 15, we have that the set of all \mathcal{AN} -Hankel operators is a closed linear subspace of the space of all Hankel operators on H^2 .
- (2) By [19, Theorem 3.3], we have that the set of all \mathcal{AM} -Hankel operators are exactly the finite rank Hankel operators. Hence by Theorem 15, we conclude that the set of all \mathcal{AM} -Hankel operators is a dense subspace of the set of all \mathcal{AN} -Hankel operators.

We summarize the results of this paper.

For any $\varphi \in L^\infty$,

- (1) $T_\varphi \in \mathcal{AN}(H^2)$ if and only if $|\varphi| = \|\varphi\|_\infty$ a.e. (μ) and $(I - P)\varphi$ is a rational function.
- (2) $T_\varphi \in \mathcal{AM}(H^2)$ if and only if $|\varphi| = \alpha$ a.e. (μ) and $\varphi \in H^\infty + C(\mathbb{T})$, where $\alpha \in \sigma_{\text{ess}}(|T_\varphi|)$.
- (3) $H_\varphi \in \mathcal{AN}(H^2)$ if and only if $\varphi \in H^\infty + C(\mathbb{T})$.

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