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Examples of non-flat bundles of rank one

Exemples de fibrés en droites qui ne sont pas plats

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Abstract. It is expected that there exist line bundles on a quasi-affine non-singular surface which do not admit a flat connection. However, to the best of our knowledge there is no known example of such a line bundle. In this article we give several explicit examples of line bundles on certain non-singular, quasi-affine surfaces that cannot be equipped with a flat connection.

Résumé. On s'attend à ce qu'il existe des fibrés en droites sur une surface non-singulière quasi-affine qui n'admettent pas de connexion plate mais, à notre connaissance, aucun exemple d'un tel fibré n'est connu. Dans cet article, nous en donnons plusieurs exemples explicites.

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1. Introduction

Existence of connections on modules defined over isolated surface singularities have been extensively studied (see [1, 3–6]). However, in these literatures the authors stress that they cannot produce a single example of a maximal Cohen–Macaulay (MCM) module over a surface singularity that does not admit a flat connection (see [5, p. 106], [4, p. 1562], [1, p. 903]), even with the help of a computer (see [3, p. 322]). By flat connection, we mean a connection with zero curvature. In this article we produce numerous examples of maximal Cohen–Macaulay modules over certain isolated surface singularities that cannot be equipped with a flat connection. In particular, we prove:

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Theorem 1. *Let (X, x) be the germ of a normal surface singularity such that the fundamental group of the link is perfect. Suppose that (X, x) contains a smooth curve passing through x. Then, there exists a line bundle on $X \setminus \{x\}$ that cannot be equipped with a flat connection. Moreover, one can associate to any such smooth curve (i.e., passing through x), an unique (up to isomorphism) line bundle on $X \setminus \{x\}$ that cannot be equipped with a flat connection.*

Note that, given a line bundle $\mathscr{L}$ on $X \setminus \{x\}$, the pushforward $i_*\mathscr{L}$ is a reflexive sheaf on X, where i is the inclusion of $X \setminus \{x\}$ into X (see [7, Proposition 1.6]). Furthermore, if X is a normal, integral surface, then reflexive sheaves on it are maximal Cohen–Macaulay. Therefore, as a corollary we produce explicit examples of maximal Cohen–Macaulay modules that cannot be equipped with a flat connection.

Corollary 2. *Let (p, q, r) be a triple of positive integers that are pairwise coprime and $r > pq$. Denote by $G(p, q, r)$ the surface in $\mathbb{C}^3$ defined by the polynomial $X^p + Y^q + Z^r$ and by $U(p, q, r)$ the regular locus of $G(p, q, r)$. Then, there exists line bundles on $U(p, q, r)$ that cannot be equipped with a flat connections.*

The study of the obstruction to the existence of flat connection on MCM modules has applications in Lie–Rinehart cohomology (see [6]) and Chern–Simmons theory (see [1]).

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2. Example of non-flat invertible sheaves

We give examples of rank 1 invertible sheaves which cannot be equipped with a flat connection. We will assume familiarity with reflexive sheaves. See [7] for basic properties of reflexive sheaves.

2.1. *Brieskorn–Pham surfaces*

Given positive integers (p, q, r), denote by $G(p, q, r) \subset \mathbb{C}^3$ the zero locus of the polynomial $X^p + Y^q + Z^r$. The resulting surface is the *Brieskorn–Pham type surface*. The origin 0 is the only singularity of the surface. Denote by

$$\mathscr{S} := \{C \subset G(p, q, r) \mid C \text{ is a non-singular curve passing through the origin } 0\}.$$

The following theorem proves the existence of smooth curves through the singularity of $G(p, q, r)$.

Theorem 3. *Assume $(p, q, r) = 1$ and $p \leq q \leq r$. Then, $\mathscr{S} \neq \emptyset$ if and only if at least one of the following conditions hold:*

(1) *two of the three integers (p, q, r) are coprime and the other one is divisible by at least one of the two coprime numbers,*
(2) *the inequality $r > pq/\gcd(p, q)$ holds.*

Proof. See [8, Theorem 3]. □

2.2. *Proof of Theorem 1*

Denote by U the regular locus of (X,x), i.e. $U = X\setminus\{x\}$. Note that the fundamental group $\pi_1(U)$ is the same as the fundamental group of the link L of (X,x). By hypothesis, the fundamental group of L is perfect. This means that the abelianization $\pi_1(U)^{\mathrm{ab}}$ of the fundamental group $\pi_1(U)$ is trivial. Since $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*$ is abelian, any 1-dimensional group representation of $\pi_1(U)$ factors through $\pi_1(U)^{\mathrm{ab}}$, which is trivial. Therefore, every 1-dimensional representation of $\pi_1(U)$ is trivial. By the Riemann–Hilbert correspondence, this implies that there does not exist any non-trivial line bundle on U that can be equipped with a flat connection. Therefore, it suffices to show the existence of non-trivial line bundles on U.

By hypothesis, there exists $C\subset X$ a smooth curve contained in the surface X passing through the singular point. Denote by $\mathscr{I}_C$ the ideal sheaf of C. We now show that the restriction M_U of $M := \mathscr{H}om_X(\mathscr{I}_C,\mathscr{O}_X)$ to U is a non-trivial line bundle. Indeed, M_U is trivial if and only if $i_*(M_U)\cong M$ is trivial (M is reflexive, and reflexive sheaves are uniquely determined by their restriction to the open subset U), where $i : U\to X$ is the natural inclusion. Furthermore, since M is a reflexive module, M is trivial if and only if $M^\vee$ is trivial (double dual of a reflexive module is isomorphic to itself). Consider the short exact sequence:

$$0\longrightarrow\mathscr{I}_C\longrightarrow\mathscr{O}_X\longrightarrow\mathscr{O}_C\longrightarrow 0 \tag{1}$$

By the depth comparison in exact sequences (see [2, Proposition 1.2.9]), we have that $\mathscr{I}_C$ is a reflexive $\mathscr{O}_X$-module. This implies that

$$M^\vee\cong\mathscr{I}_C^{\vee\vee}\cong\mathscr{I}_C.$$

This implies, M_U is trivial if and only if $\mathscr{I}_C$ is trivial. By (1), $\mathscr{I}_C$ is trivial if and only if C is a Cartier divisor. So, it suffices to show that C is a Weil divisor which is not Cartier.

We prove this by contradiction. In particular, suppose that C is a Cartier divisor. We are going to give a contradiction to the non-smoothness of X. Let $f\in\mathscr{O}_X$ be a function defining C (the existence of f is guaranteed as C is Cartier) and $\mathfrak{m}$ (resp. $\mathfrak{m}'$) the maximal ideal of $\mathscr{O}_X$ (resp. $\mathscr{O}_X/(f)$). We then have the following short exact sequence of $\mathscr{O}_X$-modules:

$$0\longrightarrow(f)\longrightarrow\mathfrak{m}\longrightarrow\mathfrak{m}'\longrightarrow 0. \tag{2}$$

Since C is a smooth curve, the dimension as a complex vector space of $\mathfrak{m}'/(\mathfrak{m}')^2$ is one. Let $c'\in\mathfrak{m}'$ be a generator of $\mathfrak{m}'/(\mathfrak{m}')^2$. By the surjectivity of (2), there exists an element $c\in\mathfrak{m}$ that is mapped onto c'. We will now show that $\mathfrak{m}/(\mathfrak{m})^2$ is generated as a $\mathbb{C}$-vector space by f and c. Indeed, let g be any element in $\mathfrak{m}$. If g maps to 0 in $\mathfrak{m}'$ then by the exact sequence (2), $g=\beta f$ for some $\beta\in\mathscr{O}_X$. If g maps to a non-zero element in $\mathfrak{m}'$, then there exists $\alpha\in\mathscr{O}_X$ such that $g-\alpha c$ is mapped to zero in $\mathfrak{m}'$. Hence by the exactness of (2), there exists $\beta\in\mathscr{O}_X$ such that $\beta f = g-\alpha c$. Therefore, g is a $\mathscr{O}_X$-linear combination of f and c in both cases. Since g was arbitrary, this implies that the dimension as a complex vector space of $\mathfrak{m}/(\mathfrak{m})^2$ is two (generated by f and c). But this will mean X is smooth at the origin, which is a contradiction. Hence, C cannot by a Cartier divisor. This proves the theorem. □

Proof of Corollary 2. Let (p,q,r) be a triple of positive integers that are pairwise coprime. By [9, p. 7], the link of $(G(p,q,r),0)$ is an integer homology sphere. Hence, the fundamental group of the link is perfect. By Theorem 3 there exists a smooth curve in $G(p,q,r)$ passing through the origin. The corollary then follows from Theorem 1. □

Example 4. Let $f = x^2+y^3+z^7$. By Corollary 2, the hypersurface in $\mathbb{C}^3$ defined by f admits MCM modules of rank one that cannot be equipped with a flat connection.

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