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Noncommutative tensor triangulated categories and coherent frames

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Abstract. We develop a point-free approach for constructing the Nakano–Vashaw–Yakimov–Balmer spectrum of a noncommutative tensor triangulated category under certain assumptions. In particular, we provide a conceptual way of classifying radical thick tensor ideals of a noncommutative tensor triangulated category using frame theoretic methods, recovering the universal support data in the process. We further show that there is a homeomorphism between the spectral space of radical thick tensor ideals of a noncommutative tensor triangulated category and the collection of open subsets of its spectrum in the Hochster dual topology.

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1. Introduction

The subject of tensor triangular geometry has been an active area of research for the past two decades and has touched a wide range of areas in mathematics including algebraic geometry, modular representation theory, stable homotopy theory, noncommutative topology, to name a few. The subject involves the study of triangulated categories with a given biexact symmetric monoidal functor called the tensor. Balmer [1, 2], reinterpreting a previous work of Thomason [20], showed that the triangulated category of perfect complexes over a scheme X along with the derived tensor functor contains enough data to reconstruct X, establishing that the subject is rich enough to be studied. Associated to a tensor triangulated category $\mathscr{C}$, Balmer [2] constructed a locally ringed space $\mathit{Spec}\,\mathscr{C}$ called the spectrum of $\mathscr{C}$ which carries the geometric essence of the tensor triangulated category. For example, $\mathit{Spec}\big(\mathrm{Dperf}(X)\big) \cong X$ for a quasi-compact, quasi-separated scheme X (see [2, 8]). The construction of the spectrum involves constructing a space out of prime thick tensor ideals of the tensor triangulated category. Balmer showed that this satisfies the correct universal property among all spaces which act as targets for support data. In the case of modular representation theory, where the relevant tensor triangulated category is the

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stable module category of modules over a finite group scheme G, the spectrum recovers the projective support variety.

In addition to capturing the underlying scheme, the subject of tensor triangular geometry further lifts to this abstraction, the notions of finite étale maps [5], the Chow group and intersection theory [4, 14, 15], Grothendieck–Neeman duality, Wirthmüller isomorphism [6] among others. The theory also detects (the failure of) Gersten conjecture for singular schemes [3]. This demonstrates the richness of the theory.

However, all tensor structures on triangulated categories need not be symmetric. The basic examples being stable module categories over Hopf algebras. To study these Nakano, Vashaw and Yakimov [17] introduced a noncommutative version of tensor triangulated categories and, extending Balmer's theory, constructed a topological space in terms of prime thick tensor ideals. This construction is also universal in a manner parallel to Balmer's construction. Nakano et al prove that in various cases, where the spectrum can be computed, the space corresponds to either the projective space of the cohomology ring of the Hopf algebra, or a quotient of that.

Following the success of the theories a natural direction of exploration will be to understand the construction of spectrum itself to gain a better insight into the structure of the tensor triangulated category. A conceptual and formal way of constructing the spectrum was described by Kock and Pitsch [16] using the language of frames and locales (point free topology). Frames are complete lattices where finite meet distributes over arbitrary joins (see [12] or Section 2.2). A typical example of a frame is the lattice of open subsets of a topological space. The essence of point free approach to topology is to reduce the study of topology to the study of these frames. In this approach, one constructs the topological space in terms of the frame of open sets instead of starting with a set of points. Spectral spaces are topological spaces homeomorphic to the spectrum of a commutative ring. The spectral spaces correspond to coherent frames (see Definition 6) in the point free approach. Kock and Pitsch gave a point free description of the Balmer spectrum of a (commutative) tensor triangulated category $\mathscr{C}$ as the Hochster dual of the spectral space associated to the *coherent* frame of radical thick tensor ideals of the tensor triangulated category. They also define a notion of support taking values in a frame and prove that the coherent frame of radical thick tensor ideals is universal as a target for such supports. Kock and Pitsch's paper shows that one can arrive at various results about the Balmer spectrum (including the sheaf of rings) from this viewpoint.

In this paper, we explore the noncommutative Balmer spectrum studied by Nakano et al from a frame theoretic viewpoint. However, in this case the arguments have to be modified substantially due to the lack of commutativity of tensor and consequently the failure of the construction of the radical of an ideal by adjoining n^{th} roots (where n is a natural number). For the modified arguments to work, one needs to restrict to a class of noncommutative tensor triangulated categories, where the prime ideals (defined in terms of thick tensor ideals as $IJ \subseteq P \implies (I \subseteq P)$ or $(J \subseteq P)$) are complete primes (defined in terms of objects, i.e., $A \otimes B \in P \implies (A \in P)$ or $(B \in P)$). As Nakano et al [19] shows there is a rich class of examples where this holds, for instance the stable module category of any finite dimensional unipotent Hopf algebra (see [18, example 5.1.2]). Following Kock and Pitsch, we extend the notion of a frame theoretic support data to the noncommutative setup and prove the relevant universal properties to recover the spectrum. Let $\mathbf{K}$ be a (noncommutative) tensor triangulated category. The radical thick tensor ideals form a coherent frame (Theorem 16) and the association of $a \in Ob(\mathbf{K})$ to the smallest radical thick tensor ideal containing a gives a universal frame-theoretic support datum (Theorem 20) giving us a classification of radical thick tensor ideals (Theorem 21). The relation of this construction with the Nakano–Vashaw–Yakimov–Balmer spectrum is clarified in Corollary 22. Nakano et al's construction of the universal support data taking values in the Balmer spectrum is recovered in a frame theoretic way in Proposition 30. Finally, extending a result by Banerjee [7], we also show that there is a homeomorphism between the set of radical thick tensor ideals and the set of closed subsets of the spectrum with quasi-compact complements under the proper notions of topologies on these sets (see Theorem 32).

2. Preliminaries

2.1. *Noncommutative tensor triangulated category and support*

A general noncommutative theory of tensor triangular geometry was introduced by Nakano, Vashaw and Yakimov in [17]. They further studied support maps and its connection with tensor product in the setup of noncommutative tensor triangular geometry in their next paper [19]. In this section, we recall some definitions and results from [17] and [19].

A noncommutative tensor triangulated category, as introduced in [17], is a triangulated category $\mathbf{K}$ with a biexact monoidal structure. Throughout this paper $\mathbf{K}$ will denote an essentially small noncommutative tensor triangulated category.

Definition 1 ([17, § 1.2]).

(1) *A* thick tensor *ideal of* $\mathbf{K}$ *is a full triangulated subcategory* $\mathbf{I}$ *of* $\mathbf{K}$ *such that it contains all direct summands of its objects and for any* $A \in Ob(\mathbf{I})$, *we have* $A \otimes B, B \otimes A \in Ob(\mathbf{I})$ *for all* $B \in Ob(\mathbf{K})$.

(2) *A prime ideal of* $\mathbf{K}$ *is a proper thick tensor ideal* $\mathbf{P}$ *such that for all thick tensor ideals* $\mathbf{I}$ *and* $\mathbf{J}$ *of* $\mathbf{K}$, *we have* $\mathbf{I} \otimes \mathbf{J} \subseteq \mathbf{P} \implies \mathbf{I} \subseteq \mathbf{P}$ *or* $\mathbf{J} \subseteq \mathbf{P}$. *We denote by* $Spc(\mathbf{K})$ *the collection of all prime ideals of* $\mathbf{K}$.

(3) *A completely prime ideal of* $\mathbf{K}$ *is a proper thick tensor ideal* $\mathbf{P}$ *such that* $A \otimes B \in \mathbf{P} \implies A \in \mathbf{P}$ *or* $B \in \mathbf{P}$ *for all* $A, B \in Ob(\mathbf{K})$.

Definition 2 ([17, § 1.2]). *The noncommutative Balmer spectrum* $Spc(\mathbf{K})$ *of* $\mathbf{K}$ *is the topological space of prime ideals of* $\mathbf{K}$ *endowed with the Zariski topology which is given by closed sets of the form*

$$V(S) = \{\mathbf{P} \in Spc(\mathbf{K}) \,|\, \mathbf{P} \cap S = \emptyset\}$$

for all subsets S *of* $\mathbf{K}$.

Let X be a topological space and let $\mathscr{X}_{cl}(X)$ denote the collection of all closed subsets of X.

Definition 3 ([17, Definition 4.1.1]). *A map* $\sigma : \mathbf{K} \longrightarrow \mathscr{X}_{cl}(X)$ *is called a (noncommutative) support datum if the following conditions are satisfied:*

(1) $\sigma(0) = \emptyset$ *and* $\sigma(1) = X$
(2) $\sigma(A \oplus B) = \sigma(A) \cup \sigma(B), \quad \forall\, A, B \in Ob(\mathbf{K})$
(3) $\sigma(\sum A) = \sigma(A), \quad \forall\, A \in Ob(\mathbf{K})$
(4) *If* $A \longrightarrow B \longrightarrow C \longrightarrow \sum A$ *is a distinguished triangle, then* $\sigma(A) \subseteq \sigma(B) \cup \sigma(C)$
(5) $\bigcup_{C \in Ob(\mathbf{K})} \sigma(A \otimes B \otimes C) = \sigma(A) \cap \sigma(B), \quad \forall\, A, B \in Ob(\mathbf{K})$

We recall (see, [17, Lemma 4.1.2]) that the restriction of the map V (as in Definition 2) to the objects of $\mathbf{K}$ gives a support datum $\mathbf{K} \longrightarrow \mathscr{X}_{cl}(Spc(\mathbf{K}))$.

Theorem 4 ([17, Theorem 4.2.2]). *The support* V *is final among all the support data* σ *of* $\mathbf{K}$ *such that* $\sigma(A)$ *is closed for each* $A \in Ob(\mathbf{K})$. *Equivalently, for any support datum* σ *satisfying the above condition, there is a unique continuous map* $f_\sigma : X \longrightarrow Spc(\mathbf{K})$ *such that* $\sigma(A) = f_\sigma^{-1}(V(A))$. *This map is precisely given by*

$$f_\sigma(x) = \{A \in Ob(\mathbf{K}) \,|\, x \notin \sigma(A)\}.$$

2.2. *Coherent frames and support*

In this section, we recall some definition and results from [12, 13] and [16].

Definition 5. *A* frame *is a complete lattice which satisfies the infinite distributive law:*

$$a \wedge \bigvee_{s \in S} s = \bigvee_{s \in S} (a \wedge s).$$

A frame map is a lattice map that preserves arbitrary joins. The category of frames and frame maps is denoted by **Frm**.

There is a pair of adjoint functors between the category of topological spaces **Top** and the opposite category of frames $\mathbf{Frm}^{op}$ which we now recall ([12, § II.1.4]). The open sets of any topological space form a frame with join operation given by union of open sets and finite meet given by intersection. This gives a functor $\mathbf{Top} \longrightarrow \mathbf{Frm}^{op}$ which has a right adjoint, the functor of *points*. A *point* of a frame is a frame map $x : F \longrightarrow \{0,1\}$ where $\{0,1\}$ is the Boolean algebra of two elements (with $0 < 1$). The set of points of any frame form a topological space whose open sets are given by sets of the form $\{x : F \longrightarrow \{0,1\} \mid x(u) = 1\}$ for any $u \in F$ and this gives the functor $\mathbf{Frm}^{op} \longrightarrow \mathbf{Top}$.

We recall from [12, § II.3.1] that an element a of a frame F is called *finite* if for every subset $S \subseteq F$ with $a \leq \bigvee_{s \in S} s$, there exists a finite subset $S' \subseteq S$ with $a \leq \bigvee_{s \in S'} s$.

Definition 6 ([12, § II.3.2]). *A frame is called* coherent *if every element of the frame can be expressed as a join of finite elements and the finite elements form a sublattice (equivalently, 1 is finite and the meet of two finite elements is finite).*

Spectral spaces, introduced by Hochster in [11], are topological spaces homeomorphic to the spectrum of a commutative ring. A *spectral map* between spectral spaces is a continuous map such that the inverse image of a quasi-compact open is quasi-compact. Every coherent frame corresponds uniquely to a spectral space. In fact, we have the following theorem:

Theorem 7 ([13]). *The category of spectral spaces and spectral maps is contravariantly equivalent to the category of coherent frames and coherent maps.*

For a spectral space X, Hochster [11] considered a new topology on X by taking as basic open subsets the closed sets with quasi-compact complements. The space so obtained is called the *Hochster dual* of X and it is denoted by $X^\vee$. He showed that the Hochster dual $X^\vee$ of any spectral space X is also a spectral space and that $X^{\vee\vee} = X$. Motivated by this, the Hochster dual of a coherent frame is defined as follows:

Definition 8 ([16, Definition 1.2.4]). *The Hochster dual of a coherent frame F is its join completion.*

We recall that an ideal of a frame (in general for any lattice) is a down-set, closed under finite joins. An ideal $\mathscr{I}$ of a frame F is called *prime* if $1 \notin \mathscr{I}$ and if $a \wedge b \in \mathscr{I}$ implies either $a \in \mathscr{I}$ or $b \in \mathscr{I}$. The points of a frame F correspond bijectively to prime ideals of F. Indeed, any point $x : F \longrightarrow \{0,1\}$ corresponds to the prime ideal $x^{-1}(0)$. Moreover, in any frame, every prime ideal $\mathscr{P}$ is principal and the generating element is $u_{\mathscr{P}} := \bigvee_{b \in \mathscr{P}} b$. We have

$$\mathscr{P} = (u_{\mathscr{P}}) = \{b \in F \mid b \leq u_{\mathscr{P}}\}.$$

The generating element $u_{\mathscr{P}}$ of a prime ideal $\mathscr{P}$ is called a *prime element*. Therefore, we have the following natural bijections

$$\text{points} \leftrightarrow \text{prime ideals} \leftrightarrow \text{prime elements}.$$

Let $(\mathscr{T}, \otimes, \mathbf{1})$ be a (commutative) tensor triangulated category. We recall the definition of support on $(\mathscr{T}, \otimes, \mathbf{1})$ from [16, § 3.2]:

Definition 9. *A support on* $(\mathscr{T},\otimes,\mathbf{1})$ *is a pair* (F,d) *where* F *is a frame and* $d : Ob(\mathscr{T}) \longrightarrow F$ *is a map satisfying*

(1) $d(0)=0$ *and* $d(\mathbf{1})=1$,
(2) $d(\sum a) = d(a), \quad \forall\, a \in Ob(\mathscr{T})$
(3) $d(a\oplus b) = d(a)\vee d(b), \quad \forall\, a,b\in\mathscr{T}$
(4) $d(a\otimes b) = d(a)\wedge d(b), \quad \forall\, a,b\in\mathscr{T}$
(5) *if* $a \longrightarrow b \longrightarrow c \longrightarrow \sum a$ *is a triangle in* $\mathscr{T}$, *then* $d(b)\le d(a)\vee d(c)$.

A morphism of supports from (F,d) *to* (F',d') *is a frame map* $F\longrightarrow F'$ *compatible with the maps* d *and* d'.

3. Frames, Hochster duality and noncommutative tensor triangulated category

Assumption 10. *All primes of* $\mathbf{K}$ *are complete primes.*

One has a vast repertoire of examples where this assumption holds, and a detailed description of the current knowledge about this can be found in the introduction of [19].

Definition 11. *Let* S *be a set of objects in a noncommutative tensor triangulated category* $\mathbf{K}$. *We define* $G(S)$ *to be the set of objects of* $\mathbf{K}$ *which are of the following forms:*

(1) *an iterated suspension or desuspension of an object in* S,
(2) *or a finite sum of objects in* S,
(3) *or objects of the form* $s\otimes t$ *and* $t\otimes s$ *with* $s\in S$ *and* $t\in\mathbf{K}$,
(4) *or an extension of two objects in* S,
(5) *or a direct summand of an object in* S.

If $\mathbf{I}$ *is a thick tensor ideal containing* S, *then clearly* $G(S)\subseteq\mathbf{I}$. *Hence, by induction,* $G^{\omega}(S) := \bigcup_{n\in\mathbb{N}} G^n(S)\subseteq\mathbf{I}$. *It may be easily verified that* $G^{\omega}(S)$ *is itself a thick tensor ideal and therefore it is the smallest thick tensor ideal containing* S. *We will denote it by* $\langle S\rangle$.

Recall that the radical of an ideal of a noncommutative ring is defined as the intersection of all the prime ideals containing it. In the same spirit, we give the following definition.

Definition 12. *We define the* radical closure *of a thick tensor ideal* $\mathbf{I}$ *of a noncommutative tensor triangulated category* $\mathbf{K}$ *by*

$$\sqrt{\mathbf{I}} := \bigcap_{\mathbf{I}\subseteq\mathbf{P}} \mathbf{P}$$

where $\mathbf{P}$ *denotes the prime ideals of* $\mathbf{K}$. *If* $\mathbf{I}$ *is a thick tensor ideal such that* $\mathbf{I}=\sqrt{\mathbf{I}}$, *we call* $\mathbf{I}$ *radical.*

Clearly, any prime ideal is radical. It is also clear that if $\mathbf{I}$ is a thick tensor ideal, then $\sqrt{\mathbf{I}}$ is a radical thick tensor ideal. For any set of objects S, let $\sqrt{S}$ denote the radical of the thick tensor ideal $\langle S\rangle$.

Lemma 13. *Let* $\mathbf{I}$ *and* $\mathbf{J}$ *be two thick tensor ideals and let* S *be a set of objects of* $\mathbf{K}$. *Then,* $\{t\otimes s \mid t\in\mathbf{I}, s\in S\}\subseteq\mathbf{J}$ *implies* $\mathbf{I}\otimes\langle S\rangle\subseteq\mathbf{J}$.

Proof. By definition, $\langle S\rangle = \bigcup_{n\in\mathbb{N}} G^n(S)$ and $G^0(S)=S$. So, by Assumption 10, $\mathbf{I}\otimes G^0(S)\subseteq\mathbf{J}$. Suppose $\mathbf{I}\otimes G^m(S)\subseteq\mathbf{J}$ for some $0\neq m\in\mathbb{N}$. We will now show that $\mathbf{I}\otimes G^{m+1}(S)\subseteq\mathbf{J}$ i.e.,

$$\{t\otimes x \mid t\in\mathbf{I}\}\subseteq\mathbf{J} \tag{1}$$

for any $x\in G^{m+1}(S)$. Obviously, (1) is satisfied if x is a finite sum of objects in $G^m(S)$. If x is an iterated suspension or desuspension of an object in $G^m(S)$ or if x is an extension of two objects in $G^m(S)$, then (1) holds since $\otimes$ is biexact. If x is a direct summand of an object in $G^m(S)$, then (1) holds since $\mathbf{J}$ is thick. If x is of the form $s\otimes k$ or $k\otimes s$ for any $s\in G^m(S)$ and $k\in\mathbf{K}$, then clearly (1) holds since $\mathbf{I}$ and $\mathbf{J}$ are ideals. Thus, by induction, we obtain $\mathbf{I}\otimes G^n(S)\subseteq\mathbf{J}$ for all $n\in\mathbb{N}$. It follows that $\mathbf{I}\otimes\langle S\rangle\subseteq\mathbf{J}$. □

Theorem 14. *Let* $\mathbf{I}$ *be a thick tensor ideal of* $\mathbf{K}$*. Then,* $\sqrt{\mathbf{I}} = \langle\{k \in \mathbf{K} \mid k^{\otimes n} \in \mathbf{I} \text{ for some } n \in \mathbb{N}\}\rangle$.

Proof. Let $S := \{k \in \mathbf{K} \mid k^{\otimes n} \in \mathbf{I}$ for some $n \in \mathbb{N}\}$. Clearly, $S \subseteq \mathbf{P}$ for all prime ideals of $\mathbf{K}$ such that $\mathbf{P} \supseteq \mathbf{I}$. Hence, $\langle S\rangle \subseteq \sqrt{\mathbf{I}}$. Given $t \notin \langle S\rangle$, consider the collection Ω of all ideals $\mathbf{J} \supseteq \mathbf{I}$ such that $\mathbf{J} \cap \{t^{\otimes n} \mid n \in \mathbb{N}\} = \emptyset$. Clearly, $\mathbf{I} \in \Omega$ and the set Ω can be partially ordered by inclusion and any chain in Ω has an upper bound in Ω. Therefore, by Zorn's Lemma there exists a maximal element, say $\mathbf{M}$, in Ω. Thus, $\mathbf{M} \supseteq \mathbf{I}$ and $\mathbf{M} \cap \{t^{\otimes n} \mid n \in \mathbb{N}\} = \emptyset$. It is now enough to show that $\mathbf{M}$ is prime to prove that $t \notin \sqrt{\mathbf{I}}$.

Let $k, k' \in \mathbf{K}$ be such that $k \otimes k' \in \mathbf{M}$. It may be easily verified using Lemma 13 that $\langle \mathbf{M}, k\rangle \otimes k' \subseteq \mathbf{M}$ and obviously we have $\langle \mathbf{M}, k\rangle\mathbf{M} \subseteq \mathbf{M}$. Therefore, again applying Lemma 13 we obtain $\langle \mathbf{M}, k\rangle\langle \mathbf{M}, k'\rangle \subseteq \mathbf{M}$. Suppose, if possible, $k, k' \notin \mathbf{M}$. Since $\mathbf{M}$ is maximal, we must have $t^{\otimes n} \in \langle \mathbf{M}, k\rangle$ and $t^{\otimes m} \in \langle \mathbf{M}, k'\rangle$ for some $n, m \in \mathbb{N}$. This implies $t^{\otimes(n+m)} \in \langle \mathbf{M}, k\rangle\langle \mathbf{M}, k'\rangle \subseteq \mathbf{M}$ which gives the required contradiction. □

Lemma 15. *Let* $\mathbf{Rad_K}$ *denote the poset of radical ideals of a noncommutative tensor triangulated category* $\mathbf{K}$ *satisfying Assumption 10. Then,* $\mathbf{Rad_K}$ *is a frame with the following meet and join operations:*

$$\mathbf{I}_1 \bigwedge \mathbf{I}_2 := \mathbf{I}_1 \bigcap \mathbf{I}_2$$
$$\bigvee_{j\in J} \mathbf{I}_j := \sqrt{\bigcup_{j\in J} \mathbf{I}_j}$$

for any two radical thick tensor ideals $\mathbf{I}_1$ *and* $\mathbf{I}_2$ *and for any set of radical thick tensor ideals* $\{\mathbf{I}_j\}_{j\in J}$.

Proof. By definition, $\bigvee_{j\in J}\mathbf{I}_j$ is a radical thick tensor ideal. Also, given a set of radical thick tensor ideals $\{\mathbf{I}_i\}_{i\in I}$, we have

$$\mathbf{I}_i = \sqrt{\mathbf{I}_i} = \bigcap_{\mathbf{I}_i\subseteq\mathbf{P}} \mathbf{P} \supseteq \bigcap_{\bigcap_{i\in I}\mathbf{I}_i\subseteq\mathbf{P}} \mathbf{P} = \sqrt{\bigcap_{i\in I}\mathbf{I}_i}$$

for every $i \in I$. Thus, we have $\bigcap_{i\in I}\mathbf{I}_i = \sqrt{\bigcap_{i\in I}\mathbf{I}_i}$. Hence, $\mathbf{Rad_K}$ is a complete lattice. Let us now verify that

$$\bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i) = \mathbf{J}\bigcap\left(\bigvee_{i\in I}\mathbf{I}_i\right).$$

We clearly have $\bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i) \subseteq \mathbf{J}\bigcap(\bigvee_{i\in I}\mathbf{I}_i)$. Now, let $x \in \mathbf{J}\bigcap(\bigvee_{i\in I}\mathbf{I}_i)$. We define

$$C_x := \left\{k \in \bigvee_{i\in I}\mathbf{I}_i \;\middle|\; x\otimes s\otimes k,\ k\otimes s\otimes x \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i) \text{ for all } s \in \mathbf{K}\right\}.$$

We claim that $C_x = \bigvee_{i\in I}\mathbf{I}_i$. If our claim holds, then $x \in C_x$ which implies $x\otimes x \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i) = \bigcap_{\bigcup_{i\in I}\mathbf{J}\cap\mathbf{I}_i\subseteq\mathbf{P}}\mathbf{P}$. Since, by assumption, all primes are complete, we have $x \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)$ and this completes the proof. We will now give a proof of our claim.

First, we show that C_x is a thick subcategory of $\mathbf{K}$. Note that it is clear that C_x is triangulated. Let $k_1, k_2 \in \mathbf{K}$ be such that $k_1 \oplus k_2 \in C_x$. Then since $\bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)$ is a thick subcategory, we have

$$\begin{aligned}
&(k_1\oplus k_2)\otimes s\otimes x,\ x\otimes s\otimes(k_1\oplus k_2) \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)\\
&\qquad\implies (k_1\otimes s\otimes x)\oplus(k_2\otimes s\otimes x),\ (x\otimes s\otimes k_1)\oplus(x\otimes s\otimes k_2) \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)\\
&\qquad\implies k_1\otimes s\otimes x,\ k_2\otimes s\otimes x,\ x\otimes s\otimes k_1,\ x\otimes s\otimes k_2 \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i) \implies k_1,\ k_2 \in C_x.
\end{aligned}$$

Next, we show that C_x is a two-sided ideal. Let $k \in C_x$ and let $t \in \mathbf{K}$. Since $x\otimes s\otimes k,\ k\otimes s\otimes x \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)$ for all $s \in \mathbf{K}$, we have $x\otimes s\otimes(k\otimes t) = (x\otimes s\otimes k)\otimes t \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)$ and $(k\otimes t)\otimes s\otimes x = k\otimes(t\otimes s)\otimes x \in \bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)$. This implies $k\otimes t \in C_x$. Similarly, $t\otimes k \in C_x$.

We will now show that C_x is radical. Clearly, $C_x \subseteq \sqrt{C_x}$. Now, if possible, let $t \in \sqrt{C_x} = \bigcap_{C_x\subseteq\mathbf{P}}\mathbf{P}$ be such that $t \notin C_x$. This implies either $x\otimes s\otimes t$ or $t\otimes s\otimes x$ does not belong to $\bigvee_{i\in I}(\mathbf{J}\bigcap\mathbf{I}_i)$ for some

$s \in \mathbf{K}$. Without loss of generality, suppose $x \otimes s \otimes t \notin \bigvee_{i \in I}(\mathbf{J} \bigcap \mathbf{I}_i) = \bigcap_{\cup_{i \in I}(\mathbf{J} \cap \mathbf{I}_i) \subseteq \mathbf{P}} \mathbf{P}$. Then, there exists some prime ideal $\mathbf{P}_0 \supseteq \bigcup_{i \in I}(\mathbf{J} \bigcap \mathbf{I}_i)$ such that $x \otimes s \otimes t \notin \mathbf{P}_0$. This implies $x \notin \mathbf{P}_0$ and $t \notin \mathbf{P}_0$. For $k \in C_x$, we have

$$x \otimes k,\ k \otimes x \in \bigvee_{i \in I} (\mathbf{J} \bigcap \mathbf{I}_i) \subseteq \mathbf{P}_0.$$

Hence $k \in \mathbf{P}_0$ showing that $C_x \subseteq \mathbf{P}_0$. Therefore, $t \in \sqrt{C_x} \subseteq \mathbf{P}_0$. This gives the required contradiction. Therefore, C_x is radical. It is clear that $\mathbf{I}_i \subseteq C_x$ for all $i \in I$. Hence, $\bigcup_{i \in I} \mathbf{I}_i \subseteq C_x$ and since C_x is radical, we must have $C_x = \bigvee_{i \in I} \mathbf{I}_i$. □

Theorem 16. *The poset of radical thick tensor ideals* $\mathbf{Rad_K}$ *of a noncommutative tensor triangulated category* $\mathbf{K}$ *satisfying Assumption 10 forms a coherent frame.*

Proof. By Lemma 15, we know that $\mathbf{Rad_K}$ forms a frame. It is now enough to check that an element of the frame $\mathbf{Rad_K}$ is finite if and only if it is a principal radical thick tensor ideal i.e., of the form $\sqrt{k}$ for some $k \in \mathbf{K}$. Let $\mathbf{I}$ be a finite element of the frame $\mathbf{Rad_K}$. Then, clearly we have

$$\mathbf{I} \subseteq \sqrt{\bigcup_{k \in \mathbf{I}} \sqrt{k}} = \bigvee_{k \in \mathbf{I}} \sqrt{k}.$$

Since $\mathbf{I}$ is radical, $k \in \mathbf{I}$ implies $\sqrt{k} \subseteq \mathbf{I}$. Thus, $\mathbf{I} = \bigvee_{k \in \mathbf{I}} \sqrt{k}$. Since $\mathbf{I}$ is finite, there exists $k_1, \ldots, k_n \in \mathbf{I}$ such that $\mathbf{I} \subseteq \sqrt{k_1} \vee \ldots \vee \sqrt{k_n}$. We observe that $\mathbf{I} \subseteq \sqrt{\sqrt{k_1} \cup \cdots \cup \sqrt{k_n}} \subseteq \sqrt{k_1 \oplus \cdots \oplus k_n} \subseteq \mathbf{I}$. Thus,

$$\mathbf{I} = \sqrt{\sqrt{k_1} \cup \cdots \cup \sqrt{k_n}} = \sqrt{k_1 \oplus \cdots \oplus k_n}.$$

Therefore, $\mathbf{I}$ is of the form $\sqrt{k}$.

Conversely, let $\mathbf{I} = \sqrt{k_0}$ for some $k_0 \in \mathbf{K}$. We need to check that $\mathbf{I}$ is a finite radical thick tensor ideal. Assume that

$$\sqrt{k_0} \subseteq \bigvee_{\lambda \in \Lambda} \mathbf{J}_\lambda,$$

where the $\mathbf{J}_\lambda$ are radical thick tensor ideals. Then in particular $k_0 \in \sqrt{\bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda}$. Let us denote $\bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda$ by S. Thus, by Proposition 14,

$$k_0 \in \sqrt{\langle S \rangle} = \sqrt{\left\langle \bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda \right\rangle} = \left\langle \left\{ k \in \mathbf{K} \,\middle|\, k^{\otimes n} \in \left\langle \bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda \right\rangle \text{ for some } n \in \mathbb{N} \right\} \right\rangle.$$

Let the finitely many elements of $\{k \in \mathbf{K} \mid k^{\otimes n} \in \langle \bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda \rangle$ for some $n \in \mathbb{N}\}$ involved in the iterative construction of k_0 be $k_1, \ldots, k_r$ i.e.,

$$k_0 \in \langle k_1, \ldots, k_r \rangle \subseteq \sqrt{\langle S \rangle}. \tag{2}$$

Then, we have $k_i^{\otimes n_i} \in \langle \bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda \rangle$ for some $n_i \in \mathbb{N}$ for each $i = 1, \ldots, r$. Let the finitely many elements of $\bigcup_{\lambda \in \Lambda} \mathbf{J}_\lambda$ involved in the iterative construction of $k_i^{\otimes n_i}, i = 1, \ldots, r$ be $x_1, \ldots, x_m$. Suppose $\{x_1, \ldots, x_m\} \subseteq \mathbf{J}_{\lambda_1} \cup \ldots \cup \mathbf{J}_{\lambda_v}$ for some $v \in \mathbb{N}$. Thus, for each $i = 1, \ldots, r$, we have

$$k_i^{\otimes n_i} \in \langle \{x_1, \ldots, x_m\} \rangle \subseteq \langle \mathbf{J}_{\lambda_1} \cup \cdots \cup \mathbf{J}_{\lambda_v} \rangle \implies k_i \in \sqrt{\langle \mathbf{J}_{\lambda_1} \cup \cdots \cup \mathbf{J}_{\lambda_v} \rangle} = \mathbf{J}_{\lambda_1} \vee \cdots \vee \mathbf{J}_{\lambda_v}$$

and hence by (2)

$$k_0 \in \langle k_1, \ldots, k_r \rangle \subseteq \mathbf{J}_{\lambda_1} \vee \cdots \vee \mathbf{J}_{\lambda_v}.$$

Thus, $\sqrt{k_0} \subseteq \mathbf{J}_{\lambda_1} \vee \cdots \vee \mathbf{J}_{\lambda_v}$ which proves that $\sqrt{k_0}$ is finite. □

Definition 17. *We call the frame of radical thick tensor ideals of a noncommutative tensor triangulated category* $\mathbf{K}$ *satisfying Assumption 10 the Zariski frame of* $\mathbf{K}$ *and we denote it by* $\mathbf{Zar}(\mathbf{K})$*. By the Zariski spectrum of* $\mathbf{K}$ *we mean the spectral space associated to* $\mathbf{Zar}(\mathbf{K})$ *and we denote it by* $Spec_{Zar}(\mathbf{K})$*.*

Next, we introduce a notion of support for a noncommutative tensor triangulated category.

Definition 18. *A support on* $\mathbf{K}$ *is a pair* (F,d) *where* F *is a frame and* $d : Ob(\mathbf{K}) \longrightarrow F$ *is a map satisfying:*

(1) $d(0) = 0$ *and* $d(\mathbf{1}) = 1$
(2) $d(\sum k) = d(k)\ \ \forall\ k \in \mathbf{K}$
(3) $d(k \oplus t) = d(k) \vee d(t)\ \ \forall\ k,t \in \mathbf{K}$
(4) $d(k \otimes t) = d(k) \wedge d(t) = d(t \otimes k)\ \ \forall\ k,t \in \mathbf{K}$
(5) *If* $k \longrightarrow t \longrightarrow r \longrightarrow \sum k$ *is a triangle in* $\mathbf{K}$, *then* $d(t) \le d(k) \vee d(r)$

A morphism $\varphi\colon (F,d) \longrightarrow (F',d')$ *is a morphism of frames* $\varphi\colon F \longrightarrow F'$ *such that* $d'(a) = \varphi(d(a))$ *for all* $a \in Ob(\mathbf{K})$.

Remark 19. This definition of support is motivated by the one in [16], recalled here in Definition 9. We shall see that under Assumption 10 this continues to correspond to a notion of support on noncommutative tensor triangulated categories as described by Nakano, Vashaw and Yakimov.

Theorem 20. *Let* $\mathbf{K}$ *be a noncommutative tensor triangulated category satisfying Assumption 10. Then the assignment* $s : Ob(\mathbf{K}) \longrightarrow \mathbf{Zar}(\mathbf{K}),\ k \mapsto \sqrt{k}$ *is a support. Moreover, it is initial among all supports.*

Proof. We have

$$\sqrt{\mathbf{1}} = \bigcap_{\mathbf{1}\in\mathbf{P}} \mathbf{P} = \bigcap_{\emptyset} = \langle \mathbf{1} \rangle \quad \text{and} \quad \mathbf{I} \vee \sqrt{0} = \bigcap_{\mathbf{I}=\mathbf{I}\cup\sqrt{0}\subseteq\mathbf{P}} \mathbf{P} = \mathbf{I}$$

for any $\mathbf{I} \in \mathbf{Zar}(\mathbf{K})$. Thus, clearly condition (1) in Definition 17 is satisfied. Also, conditions (2) and (5) are clearly satisfied using the fact that prime ideals are triangulated subcategories of $\mathbf{K}$. Also, since prime ideals are thick, we have $\sqrt{k} \subseteq \sqrt{k\oplus t}$ and $\sqrt{t} \subseteq \sqrt{k\oplus t}$ for any $k,t \in \mathbf{K}$ and so $\sqrt{k} \vee \sqrt{t} \subseteq \sqrt{k\oplus t}$. Conversely, we clearly have $k \oplus t \in \sqrt{k} \vee \sqrt{t}$. Hence, condition (3) is also satisfied. Let us now check condition (4). It is clear that $\sqrt{k\otimes t} \subseteq \sqrt{k}$ and $\sqrt{k\otimes t} \subseteq \sqrt{t}$ and therefore $\sqrt{k\otimes t} \subseteq \sqrt{k} \cap \sqrt{t}$. Finally, we have $\sqrt{k} \cap \sqrt{t} \subseteq \sqrt{k\otimes t}$ since all primes are complete, by Assumption 10. This shows that s is a support.

We will now show that s is initial among all supports. Let $d : Ob(\mathbf{K}) \longrightarrow F$ be an arbitrary support. Since $\mathbf{Zar}(\mathbf{K})$ is coherent, every element is a join of finite elements and so any frame map $\mathbf{Zar}(\mathbf{K}) \longrightarrow F$ is completely determined by its value on finite elements. So consider the frame map $u : \mathbf{Zar}(\mathbf{K}) \longrightarrow F$ given by $\sqrt{k} \mapsto d(k)$. Clearly, we have $u \circ s = d$. In fact, it is also clear that there cannot be any other choice of map $\mathbf{Zar}(\mathbf{K}) \longrightarrow F$ which is compatible with s and d. So there is at most one support map u. Let us now check that u is well defined. Let $k,t \in \mathbf{K}$ be such that $\sqrt{k} = \sqrt{t}$. We define $I(t) = \{r \in \mathbf{K} \mid d(r) \le d(t)\}$. It follows from the properties of the support d that $I(t)$ is a thick tensor ideal. Moreover, if $s \in \mathbf{K}$ be such that $s^{\otimes n} \in I(t)$ for some $n \in \mathbb{N}$, then $s \in I(t)$ since $d(s^{\otimes n}) = d(s)$. Therefore, it follows from Proposition 14 that $I(t)$ is a radical thick tensor ideal containing t and hence $\sqrt{t}$. Since $\sqrt{k} \subset \sqrt{t} \subseteq I(t)$ we have $d(k) \le d(t)$ and by symmetry we obtain our desired result. □

We now proceed to show that if $\mathbf{K}$ is a noncommutative tensor triangulated category satisfying Assumption 10, then the noncommutative Balmer spectrum $Spc(\mathbf{K})$ is the Hochster dual of the Zariski spectrum $Spec_{Zar}(\mathbf{K})$ of $\mathbf{K}$.

Theorem 21. *Let* $\mathbf{K}$ *be a noncommutative tensor triangulated category satisfying Assumption 10. Then,*

(1) *the frame-theoretic points of* $\mathbf{Zar}(\mathbf{K})$ *correspond bijectively to prime thick tensor ideals in* $\mathbf{K}$.
(2) *Under the above correspondence, a finite element* $\sqrt{k}$ *of* $\mathbf{Zar}(\mathbf{K})$ *corresponds to the set of prime thick tensor ideals* $\{\mathbf{P} \in Spc(\mathbf{K}) \mid k \notin \mathbf{P}\}$.

Proof.

(1) Recall that for any frame F, the frame-theoretic points of F correspond bijectively to the prime ideals of F and the prime ideals in turn are in natural bijection with the prime elements of F. Now, we put $F = \mathbf{Zar}(\mathbf{K})$. For any point $x : \mathbf{Zar}(\mathbf{K}) \longrightarrow \{0,1\}$ of $\mathbf{Zar}(\mathbf{K})$, the corresponding prime ideal of $\mathbf{Zar}(\mathbf{K})$ is given by $\mathfrak{p}_x := x^{-1}(0)$ and the corresponding prime element of $\mathbf{Zar}(\mathbf{K})$ is given by $\mathbf{I}_x := \bigvee_{\mathbf{I} \in \mathfrak{p}_x} \mathbf{I}$. We also know $\mathfrak{p}_x = (\mathbf{I})_x = \{\mathbf{I} \in \mathbf{Zar}(\mathbf{K}) \mid \mathbf{I} \subseteq \mathbf{I}_x\}$. We will now show that $\mathbf{I}_x$ is prime. Let $\mathbf{J}_1$ and $\mathbf{J}_2$ be thick tensor ideals such that $\mathbf{J}_1 \otimes \mathbf{J}_2 \subseteq \mathbf{I}_x$. Clearly, $(\mathbf{J}_1 \cap \mathbf{J}_2)^{\otimes 2} \subseteq \mathbf{J}_1 \otimes \mathbf{J}_2 \subseteq \mathbf{I}_x$. Since $\mathbf{I}_x$ is radical, we have $\mathbf{J}_1 \cap \mathbf{J}_2 \subseteq \mathbf{I}_x$ by Proposition 14. Therefore, we have $\mathbf{J}_1 \wedge \mathbf{J}_2 = \mathbf{J}_1 \cap \mathbf{J}_2 \in (\mathbf{I}_x) = \mathfrak{p}_x$ in the frame $\mathbf{Zar}(\mathbf{K})$. Since $\mathfrak{p}_x$ is a prime ideal of $\mathbf{Zar}(\mathbf{K})$ we must have $\mathbf{J}_1 \in \mathfrak{p}_x$ or $\mathbf{J}_2 \in \mathfrak{p}_x$. In other words, we have $\mathbf{J}_1 \subseteq \mathbf{I}_x$ or $\mathbf{J}_2 \subseteq \mathbf{I}_x$ which proves that $\mathbf{I}_x$ is prime. Thus, we obtain the following well defined map of sets

$$\{\text{frame-theoretic points of } \mathbf{Zar}(\mathbf{K})\} \longrightarrow \{\text{prime thick tensor ideals in } \mathbf{K}\} \qquad x \mapsto \mathbf{I}_x. \tag{3}$$

If $\mathbf{I}_x = \mathbf{I}_y$, then we have $x^{-1}(0) = \mathfrak{p}_x = (\mathbf{I}_x) = (\mathbf{I}_y) = \mathfrak{p}_y = y^{-1}(0)$. Clearly, this implies $x = y$ showing that (3) is an injection. To show surjection, let $\mathbf{P}$ be any prime thick tensor ideal of $\mathbf{K}$. It may be easily verified that $(\mathbf{P}) := \{\mathbf{I} \in \mathbf{Zar}(\mathbf{K}) \mid \mathbf{I} \subseteq \mathbf{P}\}$ defines a prime ideal of the frame $\mathbf{Zar}(\mathbf{K})$. Now, we define $y : \mathbf{Zar}(\mathbf{K}) \longrightarrow \{0,1\}$ by

$$y(\mathbf{I}) := \begin{cases} 0 & \text{if } \mathbf{I} \in (\mathbf{P}) \\ 1 & \text{otherwise} \end{cases}$$

It may be easily verified that $y : \mathbf{Zar}(\mathbf{K}) \longrightarrow \{0,1\}$ is a morphism of frames which shows that y is a frame-theoretic point of $\mathbf{Zar}(\mathbf{K})$. Since $\mathfrak{p}_y := y^{-1}(0) = (\mathbf{P})$, it follows that y maps to $\mathbf{P}$ under (3).

(2) The open set corresponding to the finite element $\sqrt{k}$ of the coherent frame $\mathbf{Zar}(K)$ is $\{x : \mathbf{Zar}(\mathbf{K}) \longrightarrow \{0,1\} \mid x(\sqrt{k}) = 1\}$. Clearly, we have

$$x(\sqrt{k}) = 1 \iff \sqrt{k} \notin \mathfrak{p}_x = x^{-1}(0) = (\mathbf{I}_x) \iff k \notin \mathbf{I}_x.$$

Since (3) is a bijection, it follows that the set $\{x : \mathbf{Zar}(\mathbf{K}) \longrightarrow \{0,1\} \mid x(\sqrt{k}) = 1\}$ corresponds bijectively to the set of prime thick tensor ideals $\{\mathbf{P} \in Spc(\mathbf{K}) \mid k \notin \mathbf{P}\}$.

□

Corollary 22. *Let* $\mathbf{K}$ *be a noncommutative tensor triangulated category satisfying Assumption 10. The noncommutative Balmer's spectrum* $Spc(\mathbf{K})$ *of* $\mathbf{K}$ *is the Hochster dual of the Zariski spectrum* $Spec_{Zar}(\mathbf{K})$.

Proof. The topology of $Spc(\mathbf{K})$ is given by open sets which are complements of the sets of the form $\{\mathbf{P} \in Spc(\mathbf{K}) \mid k \notin \mathbf{P}\}$. The result therefore follows from Theorem 21. □

From the frame theoretic support datum, one can reconstruct the support datum $V : \mathbf{K} \longrightarrow \mathscr{X}_{cl}(Spc(\mathbf{K}))$ described by Nakano, Vashaw and Yakimov [19, Definition 2.3.1]. This is described below in terms of a functorial equivalence between the frame theoretic support datum and the support datum taking values in closed subsets of $Spc(\mathbf{K})$.

Construction 23. *We briefly recall the construction of a topological support datum corresponding to a frame theoretic support datum. Suppose that* $d : \mathbf{K} \to F$ *is a frame-theoretic support datum and that* F *is coherent. Let* X_F *be the spectral space corresponding to* F *(see Theorem 7). We know that the points in* X_F *correspond to frame maps* $p : F \to \{0,1\}$ *and the topology consists of open sets* $U_f = \{p \in X_F \mid p(f) = 1\}$. *Let* Y_F *be the Hochster dual of* X_F, *where the open sets are closed subsets of* X_F *with quasi-compact complement. Consider the assignment* $\sigma : \mathbf{K} \longrightarrow \mathscr{X}_{cl}(Y_F)$ *given by*

$$\sigma(a) = \left\{p \in X_F \,\middle|\, p(d(a)) = 1\right\}.$$

This is well defined (see Remark 24).

One also has a reverse construction. By the Lemma in [12, p. 41], the closed subsets of Y_F are in one-to-one correspondence with elements of F. Thus, given a support $\sigma\colon \mathbf{K} \longrightarrow \mathscr{X}_{cl}(Y_F)$, one can define for $a \in \mathbf{K}$, $d(a)$ to be the element of F corresponding to the closed subset $\sigma(a)$. When σ satisfies the tensor product property, d turns out to be a frame theoretic support in the sense of Definition 18.

Remark 24. In the argument below we shall need the fact that the support $\sigma(a)$ constructed above are closed subsets of Y_F. This is standard and can be seen as follows. The space Y_F is denoted by $(Spec\, F)_{\text{inv}}$ in [9]. The subsets of the form $\{p\colon F \longrightarrow \{0,1\} \mid p(a) = 1\}$ are quasi-compact by [9, 2.2.3 (c)] as $Spec\, F$ inherits the subspace topology from 2^F.

Notation 25. Let $\mathscr{F}$ be the category of support data for $\mathbf{K}$ taking values in coherent frames as in Definition 18.

Let $\mathscr{S}$ be the category of support data for $\mathbf{K}$ taking values in spectral topological spaces, along with the restriction that the support data should have the tensor product property. In other words, an object (X,σ) in $\mathscr{S}$ is a support datum $\sigma\colon \mathbf{K} \longrightarrow \mathscr{X}_{cl}(X)$, where X is a spectral topological space and σ in addition to being a support datum as defined in Definition 3, satisfies $\sigma(a \otimes b) = \sigma(a) \cap \sigma(b)$. A morphism $\psi\colon (X,\sigma) \longrightarrow (X',\sigma')$ in $\mathscr{S}$ is a continuous map $f\colon X \longrightarrow X'$ such that for all $a \in Ob(\mathbf{K})$, $\sigma(a) = f^{-1}(\sigma'(a))$.

Lemma 26. *Suppose (F,d) belongs to $\mathscr{F}$. Then the support datum (Y_F,σ) constructed in Construction 23 belongs to $\mathscr{S}$. Conversely, if (X,τ) belongs to $\mathscr{S}$, then the reverse construction gives an element (F_X,d) of $\mathscr{F}$.*

Proof. Suppose $(F,d) \in \mathscr{F}$. That $d(0) = 0$ and $d(1) = 1$ (property (1) in Definition 18) implies that $\sigma(0) = \emptyset$ and $\sigma(\mathbf{1}) = X$ and that $d(\Sigma k) = d(k)$ for all $k \in Ob(\mathbf{K})$ implies that $\sigma(\Sigma(k)) = \sigma(k)$ for all k is straightforward.

We know that $d(k \oplus t) = d(k) \vee d(t)$ for all $k, t \in Ob(\mathbf{K})$. Thus,

$$\begin{aligned}\sigma(k \oplus t) &= \{p \mid p(d(k \oplus t)) = 1\} = \{p \mid p(d(k) \vee d(t)) = 1\} = \{p \mid p(d(k)) \vee p(d(t)) = 1\}\\ &= \{p \mid p(d(k)) = 1 \text{ or } p(d(t)) = 1\} = \{p \mid p(d(k)) = 1\} \cup \{p \mid p(d(t)) = 1\}\\ &= \sigma(k) \cup \sigma(t).\end{aligned}$$

proving (2) of Definition 3.

Property (5) in Definition 18 implies that if $a \to b \to c \to \Sigma a$ is a distinguished triangle then $d(a) \le d(b) \vee d(c)$. Thus, for any point $p\colon F \to \{0,1\}$, $p(d(a)) \le p(d(b)) \vee p(d(c))$. We have

$$\begin{aligned}\sigma(a) &= \{p \mid p(d(a)) = 1\}\\ &\subseteq \{p \mid p(d(b)) \vee p(d(c)) = 1\} = \{p \mid p(d(b)) = 1 \text{ or } p(d(c)) = 1\}\\ &= \sigma(b) \cup \sigma(c).\end{aligned}$$

This establishes Definition 3 (4). It remains to check (5). Note that for any $a, b \in Ob(\mathbf{K})$,

$$\begin{aligned}\sigma(a \otimes b) &= \{p \mid p(d(a \otimes b)) = 1\} = \{p \mid p(d(a)) \wedge p(d(b)) = 1\} \quad \text{(by Definition 18 (4))}\\ &= \{p \mid p(d(a)) = 1\} \cap \{p \mid p(d(b)) = 1\} = \sigma(a) \cap \sigma(b).\end{aligned}$$

Thus such a support satisfies the tensor product property and hence also satisfies the property (5). This completes the proof that (Y,σ) constructed in c Construction 23 belongs to $\mathscr{S}$.

Conversely suppose $(X,\sigma) \in \mathscr{S}$. Let us denote the coherent frame corresponding to X by F_X, and $d(a)$ to be the element of F_X corresponding to the closed subset $\sigma(a) = \{p \in Y_{X_F} \mid p(d(a)) = 1\}$. Now observe the equalities:

$$\begin{aligned}\sigma(a) \cup \sigma(b) &= \{p \mid p(d(a)) = 1\} \cup \{p \mid p(d(b)) = 1\} \\ &= \{p \mid p(d(a)) = 1 \text{ or } p(d(b)) = 1\} = \{p \mid p(d(a) \vee d(b)) = 1\}. \\ \sigma(a) \cap \sigma(b) &= \{p \mid p(d(a)) = 1\} \cap \{p \mid p(d(b)) = 1\} \\ &= \{p \mid p(d(a)) = 1 \text{ and } p(d(b)) = 1\} = \{p \mid p(d(a) \wedge d(b)) = 1\}.\end{aligned}$$

These equalities, along with the fact that the subset $\{p \mid p(f) = 1\}$ uniquely determines f, and the computations above give us the fact that d satisfies all the properties listed in Definition 18. Thus $(F_X, d) \in \mathscr{F}$. □

Definition 27. *Let* $\Xi\colon \mathscr{F} \longrightarrow \mathscr{S}$ *and* $\Gamma\colon \mathscr{S} \longrightarrow \mathscr{F}$ *be the two maps constructed in Construction 23. These are well defined by Lemma 26.*

Lemma 28. Ξ *and* Γ *are contravariant functors inducing equivalences between* $\mathscr{F}$ *and* $\mathscr{S}$*.*

Proof. That $\Sigma \circ \Xi$ and $\Xi \circ \Sigma$ are naturally isomorphic to the identity functors follow from the correspondence between spectral spaces and coherent frames and the computations done in the proof of Lemma 26. □

Lemma 29. (F_0, d_0) *is an initial support datum in* $\mathscr{F}$ *if and only if the corresponding support datum* $(Y_{F_0}, \sigma_0) = \Xi((F_0, d_0))$ *is a final support datum.*

Proof. This follows from the contravariance of the functors involved. □

Theorem 30. *Let* $\mathbf{K}$ *be a noncommutative tensor triangulated category satisfying Assumption 10, and* $Spc(\mathbf{K})$ *be the corresponding noncommutative Balmer spectrum. The support datum given by* $k \mapsto \sqrt{k}$ *taking values in* $\mathbf{Zar}(\mathbf{K})$ *induces a support (in the sense of Definition 3) on* $Spc(\mathbf{K})$*. Moreover, this support datum matches with the one given by* $V\colon \mathbf{K} \longrightarrow \mathscr{X}_{cl}(Spc(\mathbf{K}))$*:*

$$V(A) = \{\mathbf{P} \in Spc(\mathbf{K}) \mid A \notin \mathbf{P}\}.$$

Proof. We know that the support datum $(\mathbf{Zar}(\mathbf{K}), \sqrt{\ })$ is initial in $\mathscr{F}$. Thus $\Xi\big((\mathbf{Zar}(\mathbf{K}), \sqrt{\ })\big)$ is a final support datum in $\mathscr{S}$, i.e. it is final among all support data of the form $(X, \sigma\colon \mathbf{K} \longrightarrow Spc(\mathbf{K}))$. But by [19, Theorem 2.3.2 (a)], V described above is also the final support datum in $\mathscr{S}$. Thus by universality of final objects, there is a natural isomorphism between $\Xi\big((\mathbf{Zar}(\mathbf{K}), \sqrt{\ })\big)$ and $(Spc(\mathbf{K}, V)$. □

Remark 31. Proposition 30 shows that the frame-theoretic methods reconstructs the support function V from a categorical viewpoint.

Next, we show that the bijective correspondence between the radical thick tensor ideals and the open subsets of $Spc(\mathbf{K})^\vee$ can be promoted to a homeomorphism of spectral spaces.

Theorem 32. *Let* $\mathbf{K}$ *be a noncommutative tensor triangulated category satisfying Assumption 10 and let* $Spc(\mathbf{K})^\vee$ *be the Hochster dual of the noncommutative Balmer's spectrum* $Spc(\mathbf{K})$*. Then, the following spaces are spectral and there is a homeomorphism between them:*

(1) *The frame* $\mathbf{Zar}(\mathbf{K})$ *of radical thick tensor ideals of* $\mathbf{K}$ *endowed with the topology generated by the open sets*

$$\{\mathbf{I} \in \mathbf{Zar}(\mathbf{K}) \mid k \notin \mathbf{I}\} \qquad \forall\ k \in \mathbf{K}. \tag{4}$$

(2) *The poset* $\Omega(Spc(\mathbf{K})^\vee)$ *of open subsets of* $Spc(\mathbf{K})^\vee$ *(or equivalently, open subsets of* $Spec_{Zar}(\mathbf{K})$*) endowed with the topology generated by the open sets*

$$\{V \in \Omega\big(Spc(\mathbf{K})^\vee\big) \mid V \not\supseteq U\} \qquad \forall\ U \in \Omega\big(Spc(\mathbf{K})^\vee\big). \tag{5}$$

Proof. By Theorem 16, we know that the collection of radical thick tensor ideals **Zar**(**K**) of a noncommutative tensor triangulated category **K** forms a coherent frame. Thus, by [7, Proposition 4.1], the set **Zar**(**K**) endowed with the lower interval topology is a spectral space. The lower interval topology on **Zar**(**K**) is generated by the open sets ($\forall\ \mathbf{I} \in \mathbf{Zar}(\mathbf{K})$)

$$L(\mathbf{I}) = \left\{\mathbf{J} \in \mathbf{Zar}(\mathbf{K}) \,\middle|\, \mathbf{J} \not\supseteq \mathbf{I}\right\} = \bigcup_{k \in \mathbf{I}} \{\mathbf{J} \in \mathbf{Zar}(\mathbf{K}) \mid k \notin \mathbf{J}\}.$$

In other words, the lower interval topology on **Zar**(**K**) is generated by the collection in (4). By Theorem 7 and Corollary 22, we know that there is an order-preserving bijective correspondence between the radical thick tensor ideals and the open subsets of $Spc(\mathbf{K})^\vee$. Clearly, the lower interval topology on the frame of open subsets of $Spc(\mathbf{K})^\vee$ is generated by the collection in (5). Thus, we have the required homeomorphism. □

Remark 33. Hilbert's Nullstellensatz is the most fundamental theorem in algebraic geometry which establishes a bridge between geometry and algebra by relating algebraic sets to ideals in polynomial rings over algebraically closed fields. Another classical fact is that the closed subspaces of the spectrum *Spec*(R) of a commutative ring R are in bijective correspondence with radical ideals of R which can be viewed as a nullstellensatz-like result. A topological enhancement of this nullstellensatz-like result was provided by Finocchiaro, Fontana and Spirito in [10] where they showed that this bijective correspondence can be promoted to a homeomorphism. In [7], Banerjee provided a similar "topological nullstellensatz"-like result for a (commutative) tensor triangulated category. Our Theorem 30 could be seen as a "topological nullstellensatz" for a noncommutative tensor triangulated category.

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