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Fractional Gagliardo–Nirenberg interpolation inequality and bounded mean oscillation

Jean Van Schaftingen [a]

[a] Université catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du Cyclotron 2 bte L7.01.01, 1348 Louvain-la-Neuve, Belgium
E-mail: Jean.VanSchaftingen@UCLouvain.be

Abstract. We prove Gagliardo–Nirenberg interpolation inequalities estimating the Sobolev semi-norm in terms of the bounded mean oscillation semi-norm and of a Sobolev semi-norm, with some of the Sobolev semi-norms having fractional order.

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1. Introduction

The homogeneous Gagliardo–Nirenberg interpolation inequality for Sobolev space states that if $d \in \mathbb{N} \setminus \{0\}$ and if $0 \le s_0 < s < s_1$, $1 \le p, p_0, p_1 \le \infty$ and $0 < \theta < 1$ fulfil the condition

$$\left(s, \tfrac{1}{p}\right) = (1-\theta)\left(s_0, \tfrac{1}{p_0}\right) + \theta\left(s_1, \tfrac{1}{p_1}\right), \tag{1}$$

then, for every function $f \in \dot{W}^{s_0,p_0}(\mathbb{R}^d) \cap \dot{W}^{s_1,p_1}(\mathbb{R}^d)$, one has $f \in \dot{W}^{s,p}(\mathbb{R}^d)$, and

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} \le C \|f\|^{1-\theta}_{\dot{W}^{s_0,p_0}(\mathbb{R}^d)} \|f\|^{\theta}_{\dot{W}^{s_1,p_1}(\mathbb{R}^d)}, \tag{2}$$

unless s_1 is an integer, $p_1 = 1$ and $s_1 - s_0 \le 1 - \frac{1}{p_0}$.

When $s = 0$, we use the convention that $\dot{W}^{0,p}(\mathbb{R}^d) = L^p(\mathbb{R}^d)$, and when $s \in \mathbb{N} \setminus \{0\}$ is a positive integer, $\dot{W}^{s,p}(\mathbb{R}^d)$ is the classical integer-order *homogeneous Sobolev space* of s times weakly differentiable functions $f : \mathbb{R}^d \to \mathbb{R}$ such that $D^s f \in L^p(\mathbb{R}^d)$ and

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |D^s f|^p\right)^{\frac{1}{p}}. \tag{3}$$

For $s_0, s_1, s \in \mathbb{N}$ the inequality (2) was proved by Gagliardo [15] and Nirenberg [26] (see also [14]).

When $s \notin \mathbb{N}$, the *homogeneous fractional Sobolev–Slobodeckiĭ space* $\dot{W}^{s,p}(\mathbb{R}^d)$ can be defined as the set of measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ which are k times weakly differentiable with a finite Gagliardo semi-norm:

$$\|f\|_{\dot{W}^{s,p}(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|D^k f(y) - D^k f(x)|^p}{|y-x|^{d+\sigma p}} \,\mathrm{d}y\,\mathrm{d}x\right)^{\frac{1}{p}} < \infty, \tag{4}$$

with $k \in \mathbb{N}$, $\sigma \in (0,1)$ and $s = k + \sigma$; the characterisation of the range in which the Gagliardo–Nirenberg interpolation inequality (2) holds was performed in a series of works [4, 9–11] up to the final complete settlement by Brezis and Mironescu [5].

We focus on the endpoint case where $s_0 = 0$ and $p_0 = \infty$. In this case, the inequality (2) becomes

$$\|f\|^p_{\dot{W}^{s,p}(\mathbb{R}^d)} \le C\|f\|^{p-p_1}_{L^\infty(\mathbb{R}^d)}\|f\|^{p_1}_{\dot{W}^{s_1,p_1}(\mathbb{R}^d)}, \tag{5}$$

and holds under the assumption that $sp = s_1 p_1$ and either $s_1 \neq 1$ or $p_1 > 1$. It is natural to ask whether the inequality (5) can be strengthened by replacing the uniform norm $\|\cdot\|_{L^\infty(\mathbb{R}^d)}$ by John and Nirenberg's *bounded mean oscillation* (BMO) semi-norm $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^d)}$, which plays an important role in harmonic analysis, calculus of variations and partial differential equations [18], that is, whether we have the inequality

$$\|f\|^p_{\dot{W}^{s,p}(\mathbb{R}^d)} \le C\|f\|^{p-p_1}_{\mathrm{BMO}(\mathbb{R}^d)}\|f\|^{p_1}_{\dot{W}^{s_1,p_1}(\mathbb{R}^d)}, \tag{6}$$

where the bounded mean oscillation semi-norm $\|\cdot\|_{\mathrm{BMO}(\mathbb{R}^d)}$ is defined for any measurable function $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^d)} := \sup_{\substack{x\in\mathbb{R}^d\\ r>0}} \fint_{B_r(x)} \fint_{B_r(x)} |f(y) - f(z)| \,\mathrm{d}y\,\mathrm{d}z. \tag{7}$$

The estimate (6) was proved indeed when $s = 1$, $p = 4$, $s_1 = 2$ and $p_1 = 2$ via a Littlewood–Paley decomposition by Meyer and Rivière [24, Theorem 1.4], and for $s, s_1 \in \mathbb{N}$ via the duality between $\mathrm{BMO}(\mathbb{R}^d)$ and the real Hardy space $\mathscr{H}^1(\mathbb{R}^d)$ by Strezelecki [28]; a direct proof was been given recently by Miyazaki [25] (in the limiting case $s_0 = s_1 = 0$, see [20, Theorem 2.2], [8]); when $s_1 < 1$, the estimate (6) has been proved by Brezis and Mironescu through a Littlewood–Paley decomposition [6, Lemma 15.7] (see also [2, 20] for similar estimates in Riesz potential spaces).

The main result (Theorem 1) of the present work is the estimate (6) when $s_1 = 1$ and $0 < s < 1$, with a proof which is quite elementary: the main analytical tool is the classical maximal function theorem. We also show how the same ideas can be used to give a direct proof of (6) when $s_1 < 1$, depending only on the definitions of the Gagliardo and bounded mean oscillation semi-norms (Theorem 7). Finally, we show how a last interpolation result (Theorem 10) allows one to obtain the full range of interpolation between $\mathrm{BMO}(\mathbb{R}^d)$ and higher-order fractional Sobolev–Slobodeckiĭ spaces $\dot{W}^{s,p}(\mathbb{R}^d)$ with $s \in (1,\infty)$.

Our proofs can be considered as fractional counterparts of Miyazaki's direct proof in the integer-order case [25]. We also refer to Dao's recent work [12] for an alternative approach via negative-order Besov spaces to the results in the present paper.

2. Interpolation between first-order Sobolev semi-norm and mean oscillation

We prove the following interpolation inequality between the fist-order Sobolev semi-norm and the mean oscillation seminorm into fractional Sobolev spaces.

Theorem 1. *For every $d \in \mathbb{N} \setminus \{0\}$ and every $p \in (1,\infty)$, there exists a constant $C(p) > 0$ such that for every $s \in (1/p,1)$, every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\kappa(\Omega) < \infty$ and every function $f \in \dot{W}^{1,sp}(\Omega) \cap \mathrm{BMO}(\Omega)$, one has $f \in \dot{W}^{s,p}(\Omega)$ and*

$$\iint_{\Omega\times\Omega} \frac{|f(y)-f(x)|^p}{|y-x|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x \le \frac{C(p)\kappa(\Omega)^{sp}}{(sp-1)(1-s)} \|f\|_{\mathrm{BMO}(\Omega)}^{(1-s)p} \int_\Omega |Df|^{sp}. \tag{8}$$

We define here for a domain $\Omega \subseteq \mathbb{R}^d$, the *bounded mean oscillation semi-norm* of a measurable function $f : \Omega \to \mathbb{R}$ as

$$\|f\|_{\mathrm{BMO}(\Omega)} \coloneqq \sup_{\substack{x\in\Omega\\ r>0}} \fint_{\Omega\cap B_r(x)} \fint_{\Omega\cap B_r(x)} |f(y)-f(z)|\,\mathrm{d}y\,\mathrm{d}z, \tag{9}$$

and the geometric quantity

$$\kappa(\Omega) \coloneqq \sup\left\{ \frac{\mathscr{L}^d(B_r(x))}{\mathscr{L}^d(\Omega\cap B_r(x))} \;\middle|\; x\in\Omega \text{ and } r\in(0,\operatorname{diam}(\Omega)) \right\}. \tag{10}$$

For the latter quantity, one has for example

$$\kappa(\mathbb{R}^d) = 1 \tag{11}$$

and

$$\kappa(\mathbb{R}^d_+) = 2. \tag{12}$$

If the set Ω is convex and bounded, we have $\Omega \subseteq B_{\operatorname{diam}(\Omega)}(x)$ and $t\Omega + (1-t)x \subseteq \Omega \cap B_r(x)$, with $t \coloneqq r/\operatorname{diam}(\Omega)$, so that

$$\mathscr{L}^d(\Omega\cap B_r(x)) \ge t^d \mathscr{L}^d(\Omega) = \frac{\mathscr{L}^d(\Omega)}{\operatorname{diam}(\Omega)^d} r^d,$$

and thus

$$\kappa(\Omega) \le \frac{\mathscr{L}^d(B_1)}{\mathscr{L}^d(\Omega)} \operatorname{diam}(\Omega)^d. \tag{13}$$

The quantity $\kappa(\Omega)$ can be infinite for some unbounded convex sets such as $\Omega = (0,1)\times\mathbb{R}^{d-1}$ and $\Omega = \{(x', x_d) \in \mathbb{R}^d \mid x_d \ge |x'|^2\}$.

Our first tool to prove Theorem 1 is an estimate by the maximal function of the derivative of the average distance of values on a ball to a fixed value; this formula is related to the *Lusin–Lipschitz inequality* [21, Lemma 2], [1, Lemma II.1], [3], [16, p. 404], [17, (3.3)].

Lemma 2. *If the set $\Omega \subseteq \mathbb{R}^d$ is open and convex and if $f \in \dot{W}^{1,1}_{\mathrm{loc}}(\Omega)$, then for every $r \in (0,\operatorname{diam}(\Omega))$ and almost every $x \in \Omega$,*

$$\fint_{\Omega\cap B_r(x)} |f(z)-f(x)|\,\mathrm{d}z \le \kappa(\Omega)\, r\, \mathscr{M}|Df|(x). \tag{14}$$

Here $\mathscr{M}g : \mathbb{R}^d \to [0,+\infty]$ denotes the classical *Hardy–Littlewood maximal function* of the function $g : \Omega \to \mathbb{R}$, defined for each $x \in \mathbb{R}^d$ by

$$\mathscr{M}g(x) \coloneqq \sup_{r>0} \frac{1}{\mathscr{L}^d(B_r(x))} \int_{\Omega\cap B_r(x)} |g|. \tag{15}$$

Proof of Lemma 2. Since Ω is convex and $f \in \dot{W}^{1,1}(\Omega)$, for almost every $x \in \Omega$ and every $r \in (0,\infty)$, we have

$$\int_{\Omega\cap B_r(x)} |f(z)-f(x)|\,\mathrm{d}z \le \int_{\Omega\cap B_r(x)} \int_0^1 |Df((1-t)x+tz)[z-x]|\,\mathrm{d}t\,\mathrm{d}z. \tag{16}$$

By convexity of the set Ω, for every $z \in \Omega \cap B_r(x)$ and $t \in [0,1]$ we have $(1-t)x + tz \in \Omega \cap B_{tr}(x)$. We deduce from (15) and (16) through the change of variable $y = (1-t)x + tz$ that

$$\begin{aligned}\int_{\Omega\cap B_r(x)} |f(z) - f(x)|\,\mathrm{d}z &\le \int_0^1 \int_{\Omega \cap B_{tr}(x)} \frac{|Df(y)[y-x]|}{t^{d+1}}\,\mathrm{d}y\,\mathrm{d}t\\ &\le r\mathscr{M}|Df|(x)\int_0^1 \frac{\mathscr{L}^d(B_{tr}(x))}{t^d}\,\mathrm{d}t \le r\mathscr{L}^d(B_r(0))\mathscr{M}|Df|(x),\end{aligned} \tag{17}$$

in view of the definition (9) of the maximal function, and the conclusion (14) then follows from the definition of the geometric quantity $\kappa(\Omega)$ in (10). □

Our second tool to prove Theorem 1 is the following property of averages of functions of bounded mean oscillation (see [7, §3]).

Lemma 3. *If the set $\Omega \subseteq \mathbb{R}^d$ is open and convex, if $f \in \mathrm{BMO}(\Omega)$ and if $r_0 < r_1$, then*

$$\fint_{\Omega\cap B_{r_0}(x)} \fint_{\Omega\cap B_{r_1}(x)} |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z \le e\big(1 + d\ln(r_1/r_0)\big)\|f\|_{\mathrm{BMO}(\Omega)}. \tag{18}$$

In (18), e denotes Euler's number.

The proof of Lemma 3 will use the following triangle inequality for averages

Lemma 4. *Let $\Omega \subseteq \mathbb{R}^d$. If the function $f : \Omega \to \mathbb{R}$ is measurable, and the sets $A, B, C \subseteq \mathbb{R}^d$ are measurable and have positive measure, then*

$$\fint_A \fint_B |f(y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x \le \fint_A \fint_C |f(z) - f(x)|\,\mathrm{d}z\,\mathrm{d}x + \fint_C \fint_B |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z.$$

Proof. We have successively, in view of the triangle inequality,

$$\begin{aligned}\fint_A \fint_B |f(y) - f(x)|\,\mathrm{d}y\,\mathrm{d}x &= \fint_A \fint_B \fint_C |f(y) - f(x)|\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &\le \fint_A \fint_B |f(z) - f(x)| + |f(y) - f(z)|\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x\\ &= \fint_A \fint_C |f(z) - f(x)|\,\mathrm{d}z\,\mathrm{d}x + \fint_C \fint_B |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z.\end{aligned}$$ □

Proof of Lemma 3. We first note that since $r_1 > r_0$, we have in view of (9)

$$\begin{aligned}&\fint_{\Omega\cap B_{r_0}(x)} \fint_{\Omega\cap B_{r_1}(x)} |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z\\ &\qquad\le \frac{\mathscr{L}^d(\Omega\cap B_{r_1}(x))}{\mathscr{L}^d(\Omega\cap B_{r_0}(x))} \fint_{\Omega\cap B_{r_1}(x)} \fint_{\Omega\cap B_{r_1}(x)} |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z \le \Big(\frac{r_1}{r_0}\Big)^d \|f\|_{\mathrm{BMO}(\Omega)},\end{aligned} \tag{19}$$

since by convexity $r_0/r_1(\Omega\cap B_{r_1}(x)) \subseteq \Omega\cap B_{r_0}(x)$ and thus $\mathscr{L}^d(\Omega\cap B_{r_1}(x))/r_1^d \le \mathscr{L}^d(\Omega\cap B_{r_0}(x))/r_0^d$. Applying $k \in \mathbb{N}\setminus\{0\}$ times the inequality (19), we get thanks to the triangle inequality for mean oscillation of Lemma 4,

$$\begin{aligned}&\fint_{\Omega\cap B_{r_0}(x)} \fint_{\Omega\cap B_{r_1}(x)} |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z\\ &\qquad\le \sum_{j=0}^{k-1} \fint_{\Omega\cap B_{r_0(r_1/r_0)^{j/k}}(x)} \fint_{\Omega\cap B_{r_0(r_1/r_0)^{(j+1)/k}}(x)} |f(y) - f(z)|\,\mathrm{d}y\,\mathrm{d}z \le k\Big(\frac{r_1}{r_0}\Big)^{d/k} \|f\|_{\mathrm{BMO}(\Omega)}.\end{aligned} \tag{20}$$

Taking $k \in \mathbb{N}\setminus\{0\}$ such that $k-1 < d\ln(r_1/r_0) \le k$, we obtain the conclusion (18). □

Our last tool to prove Theorem 1 is the following integral identity.

Lemma 5. *For every $p \in (1,\infty)$ and $\alpha \in (0,\infty)$, one has*

$$\int_1^\infty \frac{(\ln r)^p}{r^{1+\alpha}}\,\mathrm{d}r = \frac{\Gamma(p+1)}{\alpha^{p+1}}.$$

Proof. One performs the change of variable $r = \exp(t/\alpha)$ in the left-hand side integral and uses the classical integral definition of the Gamma function. □

We now proceed to the proof of Theorem 1.

Proof of Theorem 1. For every $x, y \in \Omega$, we have by the triangle inequality and the domain monotonicity of the integral

$$\begin{aligned} |f(y)-f(x)| &\le \fint_{\Omega\cap B_{|y-x|/2}(\frac{x+y}{2})} |f(y)-f| + \fint_{\Omega\cap B_{|y-x|/2}(\frac{x+y}{2})} |f-f(x)| \\ &\le 2^d \fint_{\Omega\cap B_{|y-x|}(y)} |f(y)-f| + 2^d \fint_{\Omega\cap B_{|y-x|}(x)} |f-f(x)|, \end{aligned} \tag{21}$$

since by convexity $\Omega\cap B_{|y-x|/2}\big(\frac{x+y}{2}\big) \subseteq \frac{1}{2}(B_{|y-x|}(x)\cap\Omega)+\frac{y}{2}$. It follows thus from (21) by integration and by symmetry that

$$\begin{aligned} \iint_{\Omega\times\Omega} \frac{|f(y)-f(x)|^p}{|y-x|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x &\le C_1 \iint_{\Omega\times\Omega} \bigg(\fint_{\Omega\cap B_{|y-x|}(x)} |f-f(x)|\bigg)^p \frac{\mathrm{d}y\,\mathrm{d}x}{|y-x|^{d+sp}} \\ &\le C_2 \int_\Omega \int_0^{\operatorname{diam}\Omega} \bigg(\fint_{\Omega\cap B_r(x)} |f-f(x)|\bigg)^p \frac{\mathrm{d}r}{r^{1+sp}}\,\mathrm{d}x. \end{aligned} \tag{22}$$

If $\varrho \in (0,\operatorname{diam}(\Omega))$, we first have by Lemma 2, for almost every $x\in\Omega$,

$$\begin{aligned} \int_0^\varrho \bigg(\fint_{\Omega\cap B_r(x)} |f-f(x)|\bigg)^p \frac{\mathrm{d}r}{r^{1+sp}} &\le \big(\kappa(\Omega)\,\mathcal{M}|Df|(x)\big)^p \int_0^\varrho r^{(1-s)p-1}\,\mathrm{d}r \\ &= \frac{\varrho^{(1-s)p}\big(\kappa(\Omega)\,\mathcal{M}|Df|(x)\big)^p}{(1-s)p}. \end{aligned} \tag{23}$$

Next we have by the triangle inequality, by Lemma 2 again and by Lemma 3, for every $r \in (\varrho, \operatorname{diam}(\Omega))$,

$$\begin{aligned} \fint_{\Omega\cap B_r(x)} |f-f(x)| &\le \fint_{\Omega\cap B_\varrho(x)} |f-f(x)| + \fint_{\Omega\cap B_r(x)} \fint_{\Omega\cap B_\varrho(x)} |f(y)-f(z)|\,\mathrm{d}y\,\mathrm{d}z \\ &\le \big(\varrho\kappa(\Omega)\,\mathcal{M}|Df|(x) + e(1+d\ln(r/\varrho))\|f\|_{\mathrm{BMO}(\Omega)}\big), \end{aligned} \tag{24}$$

and hence, integrating (24), we get

$$\begin{aligned} &\int_\varrho^{\operatorname{diam}(\Omega)} \bigg(\fint_{\Omega\cap B_r(x)} |f-f(x)|\bigg)^p \frac{\mathrm{d}r}{r^{1+sp}} \\ &\quad\le C_3\bigg(\int_\varrho^\infty \frac{\varrho^p \mathcal{M}|Df|(x)^p}{r^{1+sp}}\,\mathrm{d}r + \int_\varrho^\infty \kappa(\Omega)^p \|f\|^p_{\mathrm{BMO}(\Omega)} \frac{(1+d\ln(r/\varrho))^p}{r^{1+sp}}\,\mathrm{d}r\bigg) \\ &\quad\le C_4\bigg(\frac{\varrho^{(1-s)p}\kappa(\Omega)^p\mathcal{M}|Df|(x)^p}{s} + \frac{\Gamma(p+1)\|f\|^p_{\mathrm{BMO}(\Omega)}}{(sp)^{p+1}\varrho^{sp}}\bigg), \end{aligned} \tag{25}$$

in view of Lemma 5. Putting (23) and (25) together, we get, since $sp>1$,

$$\int_0^{\operatorname{diam}(\Omega)} \bigg(\fint_{\Omega\cap B_r(x)} \big|f-f(x)\big|\bigg)^p \frac{\mathrm{d}r}{r^{1+sp}} \le C_5\bigg(\frac{\varrho^{(1-s)p}\kappa(\Omega)^p\mathcal{M}|Df|(x)^p}{1-s} + \frac{\|f\|^p_{\mathrm{BMO}(\Omega)}}{\varrho^{sp}}\bigg). \tag{26}$$

If $\|f\|_{\mathrm{BMO}(\Omega)} \le \operatorname{diam}(\Omega)\kappa(\Omega)\mathcal{M}|Df|(x)$, taking $\varrho := \|f\|_{\mathrm{BMO}(\Omega)}/(\kappa(\Omega)\mathcal{M}|Df|(x))$ in (26), we obtain

$$\int_0^{\operatorname{diam}(\Omega)} \bigg(\fint_{\Omega\cap B_r(x)} \big|f-f(x)\big|\bigg)^p \frac{\mathrm{d}r}{r^{1+sp}} \le \frac{C_6}{1-s}\big(\kappa(\Omega)\mathcal{M}|Df|(x)\big)^{sp}\,\|f\|^{(1-s)p}_{\mathrm{BMO}(\Omega)}; \tag{27}$$

otherwise we take $\varrho := \operatorname{diam}(\Omega) \le \|f\|_{\mathrm{BMO}(\Omega)}/(\kappa(\Omega)\mathcal{M}|Df|(x))$ in (23) and also obtain (27). Integrating the inequality (27), we reach the conclusion (8) by the quantitative version of the classical maximal function theorem in $L^{sp}(\mathbb{R}^d)$ since $sp > 1$ (see for example [27, Theorem I.1]). □

We conclude this section by pointing out that Theorem 1 admits a *localised version* in terms of Fefferman and Stein's *sharp maximal function* $f^\sharp : \Omega \to [0,\infty]$ which is defined for every $x \in \Omega$ (see [13, (4.1)]) as

$$f^\sharp(x) := \sup_{r>0} \fint_{\Omega\cap B_r(x)} \fint_{\Omega\cap B_r(x)} |f(y) - f(z)| \,\mathrm{d}y\,\mathrm{d}z; \tag{28}$$

noting that the proof of Lemma 3 yields in fact the estimate

$$\fint_{\Omega\cap B_{r_0}(x)} \fint_{\Omega\cap B_{r_1}(x)} |f(y) - f(z)| \,\mathrm{d}y\,\mathrm{d}z \le e\big(1 + d\ln(r_1/r_0)\big) f^\sharp(x) \tag{29}$$

and following then the proof of Theorem 1, we reach the following local counterpart of (27).

Proposition 6. *For every $d \in \mathbb{N}\setminus\{0\}$ and for every $p \in (1,\infty)$, there exists a constant $C > 0$ such that for every $s \in (1/p,1)$, for every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\kappa(\Omega) < \infty$, for every function $f \in \dot W^{1,1}_{\mathrm{loc}}(\Omega)$ and for almost every $x \in \Omega$, we have*

$$\int_0^{\operatorname{diam}(\Omega)} \left(\fint_{\Omega\cap B_r(x)} |f - f(x)|\right)^p \frac{\mathrm{d}r}{r^{1+sp}} \le \frac{C}{1-s}\big(f^\sharp(x)\big)^{(1-s)p}\big(\kappa(\Omega)\mathcal{M}|Df|(x)\big)^{sp}. \tag{30}$$

Proposition 6 is stronger than Theorem 1 in the sense that the integration of the estimate (30) yields (8).

Proposition 6 is a counterpart of the interpolation involving maximal and sharp maximal function of derivatives [22, (4)], which generalised a priori estimates in terms of maximal functions [23, Theorem 1], [19]; Proposition 6 generalises the corresponding result for integer-order Sobolev spaces [25, Remark 2.2].

3. Interpolation between first-order Sobolev semi-norm and mean oscillation

We explain how the tools of the previous section can be used to prove the fractional BMO Gagliardo–Nirenberg interpolation inequality as persented by Brezis and Mironescu's [6, Lemma 15.7].

Theorem 7. *For every $d \in \mathbb{N}\setminus\{0\}$, every $s, s_1 \in (0,1)$ and every $p, p_1 \in (1,+\infty)$ satisfying $s < s_1$ and $s_1 p_1 = sp$, there exists a constant $C > 0$ such that for every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\kappa(\Omega) < \infty$ and for every function $f \in \dot W^{s_1,p_1}(\Omega) \cap \mathrm{BMO}(\Omega)$, one has $f \in \dot W^{s,p}(\Omega)$ and*

$$\iint_{\Omega\times\Omega} \frac{|f(y) - f(x)|^p}{|y-x|^{d+sp}} \,\mathrm{d}y\,\mathrm{d}x \le C\|f\|^{p-p_1}_{\mathrm{BMO}(\Omega)}\kappa(\Omega)^{p_1} \iint_{\Omega\times\Omega} \frac{|f(y) - f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y\,\mathrm{d}x. \tag{31}$$

The proof of Theorem 7 will follow essentially the proof of Theorem 1, the main difference being the replacement of Lemma 2 by its easier fractional counterpart.

Lemma 8. *For every $p \in (1,\infty)$, there exists a constant $C > 0$ such that if the set $\Omega \subseteq \mathbb{R}^d$ is open and convex, if $s \in (0,1)$ and if $f : \Omega \to \mathbb{R}$ is measurable, then for every $r \in (0, \operatorname{diam}(\Omega))$ and every $x \in \Omega$,*

$$\fint_{\Omega\cap B_r(x)} |f - f(x)| \le C\kappa(\Omega)\, r^s \left(\int_\Omega \frac{|f(y) - f(x)|^p}{|y-x|^{d+sp}} \,\mathrm{d}y\right)^{\frac{1}{p}}. \tag{32}$$

Proof. By Hölder's inequality we have for every $r \in (0, \operatorname{diam}(\Omega))$ and for every $x \in \Omega$,

$$\int_{\Omega\cap B_r(x)} |f - f(x)| \le \left(\int_\Omega \frac{|f(y) - f(x)|^p}{|y-x|^{d+sp}} \,\mathrm{d}y\right)^{\frac{1}{p}} \left(\int_{B_r(x)} |y-x|^{\frac{d+sp}{p-1}} \,\mathrm{d}y\right)^{1-\frac{1}{p}}. \tag{33}$$

Noting that

$$\int_{B_r(x)} |y-x|^{\frac{d+sp}{p-1}} \,\mathrm{d}y = C_7 \frac{p-1}{d+sp}\big(r^s \mathscr{L}^d(B_r(x))\big)^{\frac{p}{p-1}} \le C_8\big(r^s \mathscr{L}^d(B_r(x))\big)^{\frac{p}{p-1}}, \tag{34}$$

we reach the conclusion (32) thanks to the definition of the geometric quantity $\kappa(\Omega)$ in (10). □

Proof of Theorem 7. We begin as in the proof of Theorem 1. Instead of (23), we have by Lemma 8,

$$\begin{aligned}\int_0^{\varrho} \Big(\fint_{\Omega\cap B_r(x)} |f - f(x)|\Big)^p \frac{\mathrm{d}r}{r^{1+sp}} &\le C_9 \kappa(\Omega)^p \Big(\int_\Omega \frac{|f(y)-f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y\Big)^{\frac{p}{p_1}} \int_0^{\varrho} r^{(s_1-s)p-1} \,\mathrm{d}r\\ &\le \frac{C_{10}\kappa(\Omega)^p \varrho^{(s_1-s)p}}{(s_1-s)p} \Big(\int_\Omega \frac{|f(y)-f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y\Big)^{\frac{p}{p_1}}.\end{aligned} \tag{35}$$

Next instead of (25), we have

$$\begin{aligned}\int_{\varrho}^{\operatorname{diam}(\Omega)} &\Big(\fint_{\Omega\cap B_r(x)} |f - f(x)|\Big)^p \frac{\mathrm{d}r}{r^{1+sp}}\\ &\le C_{11}\Big(\frac{\kappa(\Omega)^p \varrho^{(s_1-s)p}}{sp} \Big(\int_\Omega \frac{|f(y)-f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y\Big)^{\frac{p}{p_1}} + \frac{\|f\|_{\mathrm{BMO}(\Omega)}^p}{(sp)^{p+1}\varrho^{sp}}\Big).\end{aligned} \tag{36}$$

Taking $\varrho \in (0, \operatorname{diam}(\Omega))$ such that

$$\|f\|_{\mathrm{BMO}(\Omega)}^p = \varrho^{s_1p}\kappa(\Omega)^p \Big(\int_\Omega \frac{|f(y)-f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y\Big)^{\frac{p}{p_1}} \tag{37}$$

if possible, and otherwise taking $\varrho := \operatorname{diam}(\Omega)$, we obtain, since $s_1p_1 = sp$, by (35), (36) and (37)

$$\int_0^{\operatorname{diam}\Omega} \Big(\fint_{\Omega\cap B_r(x)} |f - f(x)|\Big)^p \frac{\mathrm{d}r}{r^{1+sp}} \le C_{12} \|f\|_{\mathrm{BMO}(\Omega)}^{p-p_1} \kappa(\Omega)^{p_1} \int_\Omega \frac{|f(y)-f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y. \tag{38}$$

We conclude by integration of (38). □

As previously, we point out that the estimate (38) admits a localised version, which is the fractional counterpart of Proposition 6.

Proposition 9. *For every $d \in \mathbb{N}\setminus\{0\}$, every $s, s_1 \in (0,1)$ and every $p, p_1 \in (1,+\infty)$ satisfying $s < s_1$ and $s_1p_1 = sp$, there exists a constant $C > 0$ such that for every open convex set $\Omega \subseteq \mathbb{R}^d$ satisfying $\kappa(\Omega) < \infty$, for every measurable function $f : \Omega \to \mathbb{R}$ and for every $x \in \Omega$,*

$$\int_0^{\operatorname{diam}(\Omega)} \Big(\fint_{\Omega\cap B_r(x)} |f - f(x)|\Big)^p \frac{\mathrm{d}r}{r^{1+sp}} \le C\big(f^\sharp(x)\big)^{p-p_1} \kappa(\Omega)^{p_1} \int_\Omega \frac{|f(y)-f(x)|^{p_1}}{|y-x|^{d+s_1p_1}} \,\mathrm{d}y. \tag{39}$$

The estimate (31) can be seen as a consequence of the integration of (39).

4. Higher-order fractional spaces estimates

The last ingredient to obtain the full scale of Gagliardo–Nirenberg interpolation inequalities between fractional Sobolev–Slobodeckiĭ spaces and the bounded mean oscillation space is the following estimate.

Theorem 10. *For every $d \in \mathbb{N}\setminus\{0\}$, every $k_1 \in \mathbb{N}\setminus\{0\}$, every $\sigma_1 \in (0,1)$ and every $p, p_1 \in (1,\infty)$ satisfying*

$$k_1 p = (k_1 + \sigma_1) p_1, \tag{40}$$

there exists a constant $C > 0$ such that for every function $f \in \dot{W}^{k_1+\sigma_1,p_1}(\mathbb{R}^d) \cap \mathrm{BMO}(\mathbb{R}^d)$, one has $f \in \dot{W}^{k_1,p}(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} |D^{k_1} f|^p \le C \|f\|_{\mathrm{BMO}(\mathbb{R}^d)}^{p-p_1} \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|D^{k_1}f(x) - D^{k_1}f(y)|^{p_1}}{|x-y|^{d+\sigma_1p_1}} \,\mathrm{d}x\,\mathrm{d}y. \tag{41}$$

As a consequence of Theorem 10, we have that $f \in \dot{W}^{k+\sigma,p}(\mathbb{R}^d)$ whenever $k \in \mathbb{N}$, $\sigma \in [0,1)$ and $p \in (1,\infty)$ satisfy $k+\sigma < k_1+\sigma_1$ and $(k+\sigma)p = (k_1+\sigma_1)p_1$. Indeed for $\sigma = 0$ and $k = k_1$, this follows from Theorem 10 and then for $k \in \{1,\dots,k_1-1\}$ by the Gagliardo–Nirenberg interpolation inequality for integer-order Sobolev space [25, 28]; for $0 < \sigma < 1$ and $k = 0$ one then uses Theorem 1 whereas for $0 < \sigma < 1$ and $k \in \mathbb{N}\setminus\{0\}$ one uses the classical fractional Gagliardo–Nirenberg interpolation inequality [5].

Proof of Theorem 10. Fixing a function $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}\eta = 1$ and $\operatorname{supp}\eta \subseteq B_1$, we have for every $x \in \mathbb{R}^d$ and every $\varrho \in (0,\infty)$,

$$D^{k_1}f(x) = \frac{1}{\varrho^d}\int_{\mathbb{R}^d}\eta\Big(\tfrac{x-y}{\varrho}\Big)\big(D^{k_1}f(x)-D^{k_1}f(y)\big)\,\mathrm{d}y + \frac{1}{\varrho^d}\int_{\mathbb{R}^d}\eta\Big(\tfrac{x-y}{\varrho}\Big)D^{k_1}f(y)\,\mathrm{d}y. \tag{42}$$

We estimate the first term in the right-hand side of (42) by Hölder's inequality

$$\begin{aligned}\Big|\frac{1}{\varrho^d}\int_{\mathbb{R}^d}&\eta\Big(\tfrac{x-y}{\varrho}\Big)\big(D^{k_1}f(x)-D^{k_1}f(y)\big)\,\mathrm{d}y\Big|\\ &\le \frac{C_{13}}{\varrho^d}\bigg(\int_{\mathbb{R}^d}\frac{|D^{k_1}f(x)-D^{k_1}f(y)|^{p_1}}{|x-y|^{d+\sigma_1 p_1}}\,\mathrm{d}x\bigg)^{\frac{1}{p_1}}\bigg(\int_{B_\varrho(x)}|x-y|^{\frac{d+\sigma_1 p_1}{p_1-1}}\,\mathrm{d}x\bigg)^{1-\frac{1}{p_1}}\\ &\le C_{14}\varrho^{\sigma_1}\bigg(\int_{\mathbb{R}^d}\frac{|D^{k_1}f(x)-D^{k_1}f(y)|^{p_1}}{|x-y|^{d+\sigma_1 p_1}}\,\mathrm{d}x\bigg)^{\frac{1}{p_1}}.\end{aligned} \tag{43}$$

For the second-term in the right-hand side of (42), for every $x \in \mathbb{R}^d$, we have by weak differentiability,

$$\begin{aligned}\frac{1}{\varrho^d}\int_{\mathbb{R}^d}\eta\Big(\tfrac{x-y}{\varrho}\Big)D^{k_1}f(y)\,\mathrm{d}y &= \frac{1}{\varrho^{d+k_1}}\int_{\mathbb{R}^d}D^{k_1}\eta\Big(\tfrac{x-y}{\varrho}\Big)f(y)\,\mathrm{d}y\\ &= \frac{1}{\varrho^{2d+k_1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}D^{k_1}\eta\Big(\tfrac{x-y}{\varrho}\Big)\eta\Big(\tfrac{x-z}{\varrho}\Big)\big(f(y)-f(z)\big)\,\mathrm{d}y\,\mathrm{d}z,\end{aligned} \tag{44}$$

and thus by (44) and by definition of bounded mean oscillation (7), we have

$$\Big|\frac{1}{\varrho^d}\int_{\mathbb{R}^d}\eta\Big(\tfrac{x-y}{\varrho}\Big)D^{k_1}f(y)\,\mathrm{d}y\Big| \le \frac{C_{15}}{\varrho^{k_1}}\|f\|_{\mathrm{BMO}(\mathbb{R}^d)}. \tag{45}$$

Choosing $\varrho \in (0,\infty)$ such that

$$\varrho^{k_1+\sigma_1}\bigg(\int_{\mathbb{R}^d}\frac{|D^{k_1}f(x)-D^{k_1}f(y)|^{p_1}}{|x-y|^{d+\sigma_1 p_1}}\,\mathrm{d}y\bigg)^{\frac{1}{p_1}} = \|f\|_{\mathrm{BMO}(\mathbb{R}^d)}, \tag{46}$$

we get from (42), (43) and (45), for every $x \in \mathbb{R}^d$,

$$|D^{k_1}f(x)| \le C_{16}\|f\|_{\mathrm{BMO}(\mathbb{R}^d)}^{1-\frac{k_1}{k_1+\sigma_1}}\bigg(\int_{\mathbb{R}^d}\frac{|D^{k_1}f(x)-D^{k_1}f(y)|^{p_1}}{|x-y|^{d+\sigma_1 p_1}}\,\mathrm{d}y\bigg)^{\frac{k_1}{(k_1+\sigma_1)p_1}}, \tag{47}$$

and thus in view of the condition (40), the estimate (41) follows by integration. □

Theorem 10 also admits a localised version involving the sharp maximal function which follows from the replacement of $\|f\|_{\mathrm{BMO}(\mathbb{R}^d)}$ by $f^\sharp(x)$ in (45).

Proposition 11. *For every $d \in \mathbb{N}\setminus\{0\}$, every $k_1 \in \mathbb{N}\setminus\{0\}$, every $\sigma_1 \in (0,1)$ and every $p_1 \in (1,\infty)$, there exists a constant $C > 0$ such that for every function $f \in \dot{W}^{k_1,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and every $x \in \mathbb{R}^d$,*

$$|D^{k_1}f(x)| \le C\big(f^\sharp(x)\big)^{1-\frac{k_1}{k_1+\sigma_1}}\bigg(\int_{\mathbb{R}^d}\frac{|D^{k_1}f(x)-D^{k_1}f(y)|^{p_1}}{|x-y|^{d+\sigma_1 p_1}}\,\mathrm{d}y\bigg)^{\frac{k_1}{(k_1+\sigma_1)p_1}}. \tag{48}$$

As previously, the integration of (48) yields (41).

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