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Remarks on the L^p convergence of Bessel–Fourier series on the disc

Remarques sur la convergence L^p des séries de Bessel–Fourier sur le disque

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Abstract. The L^p convergence of eigenfunction expansions for the Laplacian on planar domains is largely unknown for $p \neq 2$. After discussing the classical Fourier series on the 2-torus, we move onto the disc, whose eigenfunctions are explicitly computable as products of trigonometric and Bessel functions. We summarise a result of Balodis and Córdoba regarding the L^p convergence of the Bessel–Fourier series in the mixed norm space $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ on the disk for the range $\frac{4}{3} < p < 4$. We then describe how to modify their result to obtain $L^p(\mathbb{D}, r\mathrm{d}r\mathrm{d}t)$ norm convergence in the subspace $L^p_{\mathrm{rad}}(L^q_{\mathrm{ang}})$ $(\frac{1}{p} + \frac{1}{q} = 1)$ for the restricted range $2 \leq p < 4$.

Résumé. La convergence L^p des développements en fonctions propres du Laplacien dans des domaines du plan est largement inconnue lorsque $p \neq 2$. Après avoir discuté des séries de Fourier classiques sur le tore, nous passons au disque, dont les fonctions propres sont explicitement calculables comme étant le produit des fonctions trigonométriques et de Bessel. Nous résumons un résultat de Balodis et Córdoba concernant la convergence L^p de la série de Bessel–Fourier dans l'espace de norme mixte $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ dans le disque pour l'intervalle $\frac{4}{3} < p < 4$. Nous décrivons ensuite comment on peut modifier leur résultat pour obtenir la convergence dans la norme $L^p(\mathbb{D}, r\mathrm{d}r\mathrm{d}t)$ dans le sous-espace $L^p_{\mathrm{rad}}(L^q_{\mathrm{ang}})$ $(\frac{1}{p} + \frac{1}{q} = 1)$ pour l'intervalle $2 \leq p < 4$.

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1. Introduction

For a function $f \in L^2(\mathbb{T}^n)$, we can truncate its Fourier series by "spherical modes"

$$S_N f := \sum_{|k| \leq N} \widehat{f}(k) \mathrm{e}^{2\pi \mathrm{i} k \cdot x}, \tag{1}$$

or by "cubic modes"

$$S_{[N]}f := \sum_{|k_j|\le N} \widehat{f}(k)\mathrm{e}^{2\pi \mathrm{i} k\cdot x}, \tag{2}$$

where

$$k = (k_1,\dots,k_n)\in\mathbb{Z}^n \quad\text{and}\quad |k|^2 = \sum_{j=1}^{n}|k_j|^2.$$

It is well known that $S_N f$ from (1) fails in general to converge to f in L^p when $p\neq 2$. This follows, by standard transference arguments (see [7]), from Fefferman's result [6] that the indicator function of the ball is not an L^p-bounded Fourier multiplier for any $p\neq 2$. (See [8] for a detailed discussion and related results.) On the other hand, the square truncations from (2) are perfectly well-behaved for all $1<p<\infty$ (see again [7]).

This behaviour is not restricted to the torus and related domains. For consider the disk $\mathbb{D}\subset\mathbb{R}^2$. The eigenfunctions for the Laplacian on $\mathbb{D}$ are of the form

$$\mathrm{e}^{2\pi \mathrm{i}\theta m} J_m(j_m^n r) \quad\text{for}\quad (r,\theta)\in[0,1]^2, (m,n)\in\mathbb{Z}\times\mathbb{N},$$

corresponding to the respective eigenvalues $4\pi^2 m^2 + (j_n^m)^2$. Here $J_m := J_{|m|}$ denotes a Bessel function of the first kind and $j_n^m := j_n^{|m|}$ its non-negative zeros (see [10]).

Consider now the function $f(r) = r^{-3/2}$, which lies in the space $L^p([0,1], r\,\mathrm{d}r)$ for $1\le p<4/3$. Wing [11] proved that, for any choice of J_m, the 1-dimensional Bessel series of f fails to converge in $L^p([0,1], r\,\mathrm{d}r)$. By letting $g(r,t) := f(r)$ in $L^p(\mathbb{D})$, it follows that the 2-dimensional Bessel–Fourier series of g is

$$\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{N}} a_{m,n} J_m(j_n^m r)\mathrm{e}^{2\pi \mathrm{i} m t} = \sum_{n\in\mathbb{N}} a_n J_m(j_n^m r)$$

and so does not converge to g for any $m\ge 0$, whether we truncate the 2-dimensional series by cubic or spherical modes.

Recently, Fefferman et al. [5] have asked whether, given a differential operator with an orthonormal family w_k of eigenfunctions, there is a choice $\{w\in E_N : N\in\mathbb{N}\}$ of eigenfunctions such that the "truncations"

$$S_{E_N} f = \sum_{w\in E_N}\langle f, w\rangle w$$

are "well-behaved" in L^p for all $1<p<\infty$. To our knowledge, this conjecture is still open. For instance, Wing's counterexample could be summable if we were to find a more cunning grouping of terms to exploit further cancellations to which the coarser truncations are not sensitive. We have, however, obtained some new results for a certain class of triangular domains in [1].

Returning to the disk, Wing's result tells us that we should expect restrictions on the range of p. A natural range is $4/3<p<4$, since this is precisely the range that works for the 1-dimensional Bessel series [11]. (It is instructive to compare this to the ranges of L^p convergence for the Bochner–Riesz means on $\mathbb{R}^2$; see [7] for details. We will return to this question later.)

The best result known at this time is due to Balodis and Córdoba [2], who reduced the problem of convergence on the disc to extant results on the convergence of Fourier and 1-dimensional Bessel series, albeit with a modified norm. We will exploit their argument to obtain L^p convergence in a certain subspace of L^p.

2. Mixed-norm convergence

Define the space $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ by the inequality

$$\|f\|_{p,2} := \left[\int_0^1\left(\sum_m |f_m(r)|^2\right)^{p/2} r\,\mathrm{d}r\right]^{1/p} \equiv \big\|\,\|(f_m(r))\|_{\ell^2}\big\|_{L^p(r\,\mathrm{d}r)} < \infty, \tag{3}$$

where $f_m(r)$ are the Fourier coefficients of the angular function $t \mapsto f(r,t)$:

$$f_m(r) := \int_0^1 f(r,\theta)\mathrm{e}^{-2\pi \mathrm{i} m\theta}\,\mathrm{d}\theta.$$

Denote by $S_{N,M}f$ the partial sums of the Bessel–Fourier series of $f : \mathbb{D} \to \mathbb{C}$:

$$S_{N,M}f(r,t) := \sum_{m=-M}^{M}\sum_{n=1}^{N} a_{m,n} J_m(j_n^m r)\mathrm{e}^{2\pi \mathrm{i} m t}. \tag{4}$$

We drop the superscript "(d)" present in [2], since $d = 2$ will remain fixed in our discussion. This simplification is imposed by our use of the Hausdorff–Young inequality below. In higher dimensions, the expansion of the spherical part involves spherical harmonics, so the device of the Hausdorff–Young inequality is no longer available to us.

Theorem 1 ([2]). *The operators $S_{N,M}$ are uniformly bounded on $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ if, and only if, $\frac{4}{3} < p < 4$ when $N \geq AM + 1$ for an absolute constant $A > 0$.*

The norm convergence of the series to f follows by the usual uniform boundedness argument. (See [2]; cf. the analogous Fourier series argument in [7].)

To attack the proof of Theorem 1, they exploited the presence of the Fourier coefficients in the norm (3). Indeed, $S_{N,M}f$ is a trigonometric polynomial whose m^{th}Fourier mode ($|m| \leq M$) is

$$S_{N,m}f_m(r) \equiv \sum_{n=1}^{N} a_{m,n} J_m(j_n^m r), \tag{5}$$

which is precisely the 1-dimensional Bessel series summation operator for the radial function $r \mapsto f_m(r)$ in terms of the m^{th}order Bessel function J_m. Thus,

$$\begin{aligned}\left\| S_{N,M}f \right\|_{p,2} &= \left[\int_0^1 \left(\sum_{m=-M}^{M} |S_{N,m}f_m(r)|^2 \right)^{p/2} r\,\mathrm{d}r \right]^{1/p} \\ &= \left\| (S_{N,m}f_m)_m \right\|_{L^p(r\,\mathrm{d}r;\ell^2)},\end{aligned}$$

so the boundedness of $S_{N,M}$ in $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ is reduced to a uniform bound for vector-valued inequalities. Such bounds must be independent of the length, $2M+1$, of the vector

$$\left(S_{N,m}f_{-m}, \ldots, S_{N,m}f_m \right).$$

Note that $S_{N,-m} = S_{N,m}$ by our convention that $J_m = J_{|m|}$ for $m \in \mathbb{Z}$. The functions f_m and f_{-m}, however, are distinct in general, as they correspond to distinct Fourier coefficients.

Let us now turn to L^p convergence on the disc, where the relevant norm is

$$\left\| f \right\|_{L^p(\mathbb{D})} = \left\| \left\| f(r,t) \right\|_{L^p(\mathrm{d}t)} \right\|_{L^p(r\,\mathrm{d}r)}.$$

For $p \neq 2$ we cannot replace the inner "angular" L^p norm by a sum of Fourier coefficients (see [9, Chapter IV]). However, for $p \geq 2$ we may use the following *"Reverse" Hausdorff–Young Inequality*: if $p \geq 2$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left\| g \right\|_{L^p(\mathbb{T})} \leq \left\| (\widehat{g}(k))_k \right\|_{\ell^q(\mathbb{Z})}.$$

for all $g \in L^p(\mathbb{T})$. To see this, simply interpolate between Plancherel's identity and the trivial bound

$$\left\| g \right\|_{L^\infty(\mathbb{T})} \leq \left\| (\widehat{g}(k))_k \right\|_{\ell^1(\mathbb{Z})}.$$

We therefore have

$$\begin{aligned}\|f\|_{L^p(\mathbb{D})} &= \left[\int_0^1 \|f(r,t)\|_{L^p(\mathbb{T},\mathrm{d}t)}^p \, r\,\mathrm{d}r\right]^{1/p}\\ &\le \left[\int_0^1 \left(\sum_k |f_k(r)|^q\right)^{p/q} r\,\mathrm{d}r\right]^{1/p}\\ &=: \|f\|_{p,q}.\end{aligned}$$

Using this norm, we define the space

$$L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}}) := \left\{f \in L^p(\mathbb{D}) : \|f\|_{p,q} < \infty\right\}.$$

Careful inspection of the proofs in [2] shows that the space ℓ^2 can be replaced by ℓ^q throughout when

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad 2 \le p < 4.$$

The kernels of the 1-dimensional summation operators $S_{N,m}$ in (5) are controlled by weighted, vector-valued norms on the operators

$$\int_0^1 \frac{f(t)}{2-x-t}\,\mathrm{d}t \quad \text{and} \quad \int_0^1 \frac{f(t)}{x+t}\,\mathrm{d}t, \tag{6}$$

the Hilbert Transform and the Hardy–Littlewood Maximal Functional. The weight $r^{1-p/2}$ satisfies the 1-dimensional Muckenhoupt A_p condition if, and only if,

$$-1 < 1 - \frac{p}{2} < p - 1 \quad \text{that is} \quad \frac{4}{3} < p < 4$$

(see [7, Example 7.1.7]) and, when this is the case, we have the inequalities

$$\left\|(\mathscr{M} f_k)_k\right\|_{L^p(r^{1-p/2};\ell^q)} + \left\|(H f_k)_k\right\|_{L^p(r^{1-p/2};\ell^q)} \lesssim_{p,q} \left\|(f_k)_k\right\|_{L^p(r^{1-p/2};\ell^q)}.$$

(See [4, (B), p. 25].) Furthermore, the kernels of the operators in (6) are nice enough that both are bounded on the space $L^p([0,1], r^{1-p/2}\,\mathrm{d}r)$ and, since they are positive, they admit vector-valued extensions to ℓ^q too [7, Theorem 5.5.10].

There is one more operator to consider in [2, p. 280]:

$$T_{N,m} f(x) := \sqrt{x} f_\nu(A_N^\nu x) \int_0^1 \sqrt{t} f_\nu(A_N^\nu t) f(t)\,\mathrm{d}t$$

with f_ν as in [2, Lemma 1]. For the stated range of p, $p/q \ge 1$, so we can apply Jensen's Inequality as in [2, p. 280], and we obtain the $L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}})$-boundeness of $T_{N,\nu}$ too.

Theorem 2. *The operators $S_{N,M}$ are uniformly bounded on $L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}})$ if $2 \le p < 4$ and $\frac{1}{p} + \frac{1}{q} = 1$, when $N \ge AM + 1$ for an absolute constant $A > 0$.*

Corollary 3. *Let $2 \le p < 4$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, if N_k, M_k are sequences of natural numbers such that $N_k \ge AM_k + 1$ and $M_k \to \infty$, we have*

$$\lim_{k\to\infty} \left\|S_{N_k,M_k} f - f\right\|_{p,q} = 0$$

for all $f \in L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}})$. In particular,

$$\lim_{k\to\infty} \left\|S_{N_k,M_k} f - f\right\|_{L^p(\mathbb{D})} = 0$$

for all $f \in L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}})$.

Note, however, that this method does not allow us to conclude L^p norm convergence *for all* $f \in L^p$, but only for the smaller space $L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}}) \subsetneq L^p(\mathbb{D})$. The following function lies in L^p (since it is continuous [12]), but its Fourier coefficients are not ℓ^q summable for any $q > 2$:

$$g(t) := \sum_{k=2}^{\infty} \frac{\mathrm{e}^{ik\log k}}{\sqrt{k}(\log k)^2}\mathrm{e}^{2\pi \mathrm{i} t}.$$

In other words, $g \in L^p(\mathbb{D}) \setminus L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}})$.

The above example, which is a counterexample to Plancherel's theorem in L^p, hints at the underlying issue: we did not obtain L^p *bounds* for the partial sum operators, so we cannot apply the proof of Corollary 3 to $L^p(\mathbb{D})$ directly.

3. Concluding remarks

To summarise, we have the following pieces of the convergence puzzle:

Table 1. L^p convergence of Bessel–Fourier series for various ranges of p.

p range	$[1,4/3)$	$[4/3,2)$	2	$(2,4)$	4	$(4,\infty)$
$\|\cdot\|_{L^p(\mathbb{D})}$-convergence	No	?	Yes	$f \in L^p_{\mathrm{rad}}(\ell^q_{\mathrm{ang}})$	?	No

In light of the existing results [2, 3, 11], we might offer the following conjecture.

Conjecture 4. *For all* $4/3 < p < 4$ *and* $f \in L^p(\mathbb{D})$,

$$\lim_{k\to\infty} \left\| S_{N_k,M_k} f - f \right\|_{L^p(\mathbb{D})} = 0,$$

for some appropriate choice of $N_k, M_k \in \mathbb{N}$, $k \geq 1$. *For p outside of this range, convergence fails in general.*

However, it is not completely clear that we can expect this. Córdoba [4] proved that the ball multiplier is bounded on the mixed norm space $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ in the range $2n/(n+1) < p < 2n/(n-1)$ that was originally conjectured for $L^p(\mathbb{R}^n)$, but disproved by Fefferman [6]. As we saw above, the $L^p_{\mathrm{rad}}(L^2_{\mathrm{ang}})$ argument works by essentially "eliminating" the angular eigenfunctions and reducing the problem to bounds on the one-dimensional Bessel–Fourier series. This "trick" is not available in $L^p(\mathbb{D})$, owing to the failure of Plancherel's Theorem for $p \neq 2$, so obtaining uniform bounds on the operators $S_{N,M}$ is considerably more difficult.

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