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Complex analysis and geometry / *Analyse et géométrie complexes*

On the Eneström–Kakeya theorem for quaternionic polynomials

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Abstract. In this paper, we present certain results concerning the distribution of zeros of polynomials of a quaternionic variable and with quaternionic coefficients. We obtain ring shaped regions of Eneström–Kakeya type for the zeros of these polynomials and also extend some classical results from the complex to quaternionic setting.

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1. Introduction

The task of finding the regions containing all the zeros of a polynomial on using various methods of the geometric function theory is a classical topic in analysis. In addition to having great importance in the geometric function theory, this study is equally important in the application areas such as physical systems. We can see a large body of research concerning the regions, mostly circular or annular, containing all the zeros of a polynomial in terms of the coefficients of the polynomial. One of the most known results about the distribution of zeros of a complex polynomial and is particularly important in the study of the stability of numerical methods for differential equations is the following Eneström–Kakeya theorem [10].

Theorem 1. *If $T(z)=\sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 \geq 0,$$

then all the zeros of $T(z)$ lie in

$$|z| \leq 1.$$

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In the literature (for example, see [7, 8]) there exist several extensions of the Eneström–Kakeya theorem. An exhaustive survey on the Eneström–Kakeya theorem and its various generalizations is given in the comprehensive books of Marden [10] and Milovanović et al. [11]. In 1967, Joyal, Labelle and Rahman [8] extended Theorem 1 to the polynomials whose coefficients are monotonic but not necessarily non-negative in the form of following result.

Theorem 2. *If $T(z)=\sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n (where z is a complex variable) with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0,$$

then all the zeros of $T(z)$ lie in

$$|z| \leq \frac{a_n - a_0 + |a_0|}{|a_n|}.$$

The extension of Theorem 1 to complex coefficients was established by Govil and Rahman [7] in the form of following result.

Theorem 3. *If $T(z)=\sum_{v=0}^{n} a_v z^v$, is a polynomial of degree n with complex coefficients where* $\operatorname{Re}(a_v)=\alpha_v$, $\operatorname{Im}(a_v)=\beta_v$, *for $0 \leq v \leq n$, and satisfying*

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 \geq 0, \alpha_n \neq 0,$$

then all the zeros of $T(z)$ lie in

$$|z| \leq 1 + \frac{2}{\alpha_n} \sum_{v=0}^{n} |\beta_v|.$$

Various estimates for the location of zeros in terms of coefficients, with emphasis on the distribution of zeros of the algebraic polynomials with restricted coefficients has been intensively studied since the second half of the nineteenth century, and substantial breakthroughs have been achieved. The Eneström–Kakeya theorem and its various generalizations as mentioned above are the classic and significant examples of this kind. Provided such a richness of the complex setting, a natural question is to ask what kind of results in the quaternionic setting can be obtained. The goal of this paper is to present extensions to the quaternionic setting of some classical results of Eneström–Kakeya type as discussed above.

2. Preliminary knowledge

In order to introduce the framework in which we will work, let us introduce some preliminaries on quaternions and regular functions of a quaternionic variable which will be useful in the sequel. Quaternions are essentially a generalization of complex numbers to four dimensions (one real and three imaginary parts) and were first studied and developed by Sir Rowan William Hamilton in 1843. This number system of quaternions is denoted by $\mathbb{H}$ in honor of Hamilton. This theory of quaternions is by now very well developed in many different directions, and we refer the reader to [15] for the basic features of quaternionic functions. Before we proceed further, we need to introduce some preliminaries on quaternions and quaternionic polynomials. The set of quaternions denoted by $\mathbb{H}$ is a noncommutative division ring. It consists of elements of the form $q=\alpha+\beta i+\gamma j+\delta k$, $\alpha,\beta,\gamma,\delta \in \mathbb{R}$, where the imaginary units i, j, k satisfy $i^2=j^2=k^2=-1$, $ij=-ji=k$, $jk=-kj=i$, $ki=-ik=j$. Every element $q=\alpha+\beta i+\gamma j+\delta k \in \mathbb{H}$ is composed by the real part $\operatorname{Re}(q)=\alpha$ and the imaginary part $\operatorname{Im}(q)=\beta i+\gamma j+\delta k$. The conjugate of q is denoted by $\bar{q}$ and is defined as $\bar{q}=\alpha-\beta i-\gamma j-\delta k$ and the norm of q is $|q|=\sqrt{q\bar{q}}=\sqrt{\alpha^2+\beta^2+\gamma^2+\delta^2}$.

The inverse of each non zero element q of $\mathbb{H}$ is given by $q^{-1} = |q|^{-2}\overline{q}$. For $r > 0$, we define the ball $B(0,r) = \{q \in \mathbb{H}; |q| < r\}$. By $\mathbb{B}$ we denote the open unit ball in $\mathbb{H}$ centered at the origin, i.e.,

$$\mathbb{B} = \{q = \alpha + \beta i + \gamma j + \delta k : \alpha^2 + \beta^2 + \gamma^2 + \delta^2 < 1\},$$

and by $\mathbb{S}$ the unit sphere of purely imaginary quaternions, i.e.,

$$\mathbb{S} = \{q = \beta i + \gamma j + \delta k : \beta^2 + \gamma^2 + \delta^2 = 1\}.$$

We represent the indeterminate for a quaternionic polynomial as q. A quaternionic polynomial T of degree n in the variable q, namely a polynomial with coefficients on the right and indeterminate on the left is given by $T(q) =: \sum_{\nu=0}^{n} q^{\nu} a_{\nu}$, $a_{\nu} \in \mathbb{H}$, $\nu = 0,1,2,\ldots,n$. These polynomials satisfy the regularity conditions and their behavior resembles very closely to that of holomorphic functions of a complex variable. In the theory of polynomials of this kind over skew-fields, one defines a different product (we use the symbol $*$ to denote such a product) which guarantees that the product of regular functions is regular. For polynomials, for example, this product is defined as follows:

Two quaternionic polynomials of this kind can be multiplied according to the convolution product (Cauchy multiplication rule): given $T_1(q) = \sum_{i=0}^{n} q^i a_i$, $T_2(q) = \sum_{j=0}^{m} q^j b_j$, we define

$$(T_1 * T_2)(q) := \sum_{\substack{i=0,1,\ldots,n \\ j=0,1,\ldots,m}} q^{i+j} a_i b_j.$$

If T_1 has real coefficients, the so called $*$ multiplication coincides with the usual point wise multiplication. Notice that the $*$ product is associative and not, in general, commutative. The absence of commutativity leads to a behavior of polynomials rather unlike their behavior in the real or complex setting. It is observed that the zeros of the aforementioned polynomial of a quaternionic variable are either isolated or spherical. In the quaternionic setting, for example, the second degree polynomial $q^2 + 1$ vanishes for every $q \in \mathbb{S}$. These regular functions of a quaternionic variable have been introduced and intensively studied in the past decade, and they have proven to be a fertile topic in analysis, and their rapid development has been largely driven by the applications to operator theory. In the recent study (for example, see [2–6,9]), a new theory of regularity for functions, particularly for polynomials of a quaternionic variable was developed, and is extremely useful in replicating many important properties of holomorphic functions. One of the basic properties of holomorphic functions of a complex variable is the discreteness of their zero sets (except when the function vanishes identically). Given a regular function of a quaternionic variable, all its restrictions to complex lines are holomorphic and hence either have a discrete zero set or vanishes identically. In the preliminary steps, the structure of the zero sets of a quaternionic regular function and the factorization property of zeros was described. In this regard, Gentili and Stoppato [4] (see also [6]) gave a necessary and sufficient condition for a quaternionic regular function to have a zero at a point in terms of the coefficients of the power series expansion of the function. This extends to quaterniomic power series the theory presented in [9] for polynomials. The following result which completely describes the zero sets of a regular product of two polynomials in terms of the zero sets of the two factors is from [9] (see also [4] and [6]).

Theorem 4. *Let f and g be given quaternionic polynomials. Then $(f * g)(q_0) = 0$ if and only if $f(q_0) = 0$ or $f(q_0) \neq 0$ implies $g\big(f(q_0)^{-1} q_0 f(q_0)\big) = 0$.*

Gentili and Struppa [5] introduced a maximum modulus theorem for regular functions, which includes convergent power series and polynomials in the form of the following result.

Theorem 5 (Maximum Modulus Theorem). *Let $B = B(0,r)$ be a ball in $\mathbb{H}$ with centre 0 and radius $r > 0$, and let $f : B \to \mathbb{H}$ be a regular function. If $|f|$ has a relative maximum at a point $a \in B$, then f is a constant on B.*

In [4–6] the structure of the zeros of polynomials was used and a proof of the Fundamental Theorem of Algebra was established. We point out that the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ was already proved by Niven (for reference, see [12, 13]), by using different techniques. This led to the complete identification of the zeros of polynomials in terms of their factorization, for reference see [14]. Thus it became an interesting perspective to think about the regions containing some or all the zeros of a regular polynomial of quaternionic variable. Very recently, Carney et al. [1] extended the Eneström–Kakeya theorem and its various generalizations from complex polynomials to quaternionic polynomials by making use of Theorems 4 and 5. Firstly, they established the following quaternionic analogue of Theorem 1.

Theorem 6. *If $T(q) = \sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0 \ge 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \le 1.$$

In the same paper, Carney et al. [1] also established the following quaternionic analogue of Theorem 3 in the form of the following result.

Theorem 7. *If $T(q) = \sum_{v=0}^{n} q^v a_v$, is a quaternionic polynomial of degree n, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0, 1, 2, \dots, n$, satisfying*

$$\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_1 \ge \alpha_0 \ge 0, \quad \alpha_n \ne 0,$$

then all the zeros of $T(q)$ lie in

$$|q| \le 1 + \frac{2}{\alpha_n} \sum_{v=0}^{n} (|\beta_v| + |\gamma_v| + |\delta_v|).$$

Recently, Tripathi ([16, Corollary 3.3]) established the following generalization of Theorem 6 in the form of the following result.

Theorem 8. *If $T(q) = \sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n (where q is a quaternionic variable) with real coefficients satisfying*

$$a_n \ge a_{n-1} \ge \cdots \ge a_1 \ge a_0,$$

then all the zeros of $T(q)$ lie in

$$|q| \le \frac{1}{|a_n|}\left(|a_0| + \sum_{v=1}^{n} |a_v - a_{v-1}|\right) = \frac{1}{|a_n|}\left(|a_0| - a_0 + a_n\right).$$

In the literature, we could not find much except the above mentioned papers and the results therein, about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. The main purpose of this paper is to extend various results of Eneström–Kakeya type from the complex to quaternionic setting by making use of a recently established maximum modulus theorem (Theorem 5) and the structure of the zero sets of regular functions (Theorem 4) of a quaternionic variable. The obtained results also produce various generalizations of Theorems 6, 7 and 8.

3. Main results

In this section, we state our main results. Their proofs are given in the next section. We start with the following generalization of Theorem 6. As a consequence, it also provides a generalization of Theorem 7.

Theorem 9. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a quaternionic polynomial of degree n with quaternionic coefficients, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\ldots,n$, satisfying*

$$\lambda+\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0-\mu \geq 0, \alpha_n>0,$$

for some non-negative real numbers λ and μ, then all the zeros of $T(q)$ lie in

$$\left|q+\frac{\lambda}{a_n}\right| \leq \frac{1}{|a_n|}[\alpha_n+\lambda+2\mu+M_0],$$

$$\text{where} \quad M_0=\sum_{v=0}^{n}\left[|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\right], \quad \beta_{-1}=\gamma_{-1}=\delta_{-1}=0.$$

Taking $\lambda=(R-1)\alpha_n$, with $R \geq 1$ and $\mu=(1-r)\alpha_0$ with $0<r \leq 1$ in the above theorem, we get the following result.

Corollary 10. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a quaternionic polynomial of degree n with quaternionic coefficients, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\ldots,n$, satisfying*

$$R\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq r\alpha_0>0,$$

for some $R \geq 1$ and $0<r \leq 1$, then all the zeros of $T(q)$ lie in

$$\left|q+\frac{(R-1)\alpha_n}{a_n}\right| \leq \frac{1}{|a_n|}[R\alpha_n+2(1-r)\alpha_0+M_0],$$

where M_0 is as defined in Theorem 9.

Taking $r=1$ and $\beta_v=\gamma_v=\delta_v=0$, in Corollary 10, we get the following generalization of Theorem 6.

Corollary 11. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n (where q is a quaternionic variable), with real coefficients satisfying*

$$Ra_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0>0,$$

for some $R \geq 1$, then all the zeros of $T(q)$ lie in

$$\left|q+R-1\right| \leq R.$$

Remark 12. Taking $R=1$ in Corollary 11, we get Theorem 6.

It is easy to verify that $M_0 \leq 2\sum_{v=0}^{n}(|\beta_v|+|\gamma_v|+|\delta_v|)$. Using this and take $\mu=0$ in Theorem 9, we get the following generalization of Theorem 7.

Corollary 13. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a quaternionic polynomial of degree n with quaternionic coefficients, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\ldots,n$, satisfying*

$$\lambda+\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 \geq 0, \quad \alpha_n>0,$$

for some $\lambda \geq 0$, then all the zeros of $T(q)$ lie in

$$\left|q+\frac{\lambda}{a_n}\right| \leq 1+\frac{1}{\alpha_n}\left[\lambda+2\sum_{v=0}^{n}(|\beta_v|+|\gamma_v|+|\delta_v|)\right],$$

Remark 14. For $\lambda=0$, Corollary 13 reduces to Theorem 7.

By taking $\mu=0$ and assume $\beta_v \geq \beta_{v-1}, \gamma_v \geq \gamma_{v-1}, \delta_v \geq \delta_{v-1}$ for $v=1,2,3,\ldots,n$, in Theorem 9, we get the following extension of Theorem 6 to quaternionic coefficients.

Corollary 15. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n with quaternionic coefficients, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\dots,n$, satisfying*

$$\lambda+\alpha_n\ge\alpha_{n-1}\ge\cdots\ge\alpha_1\ge\alpha_0\ge 0,\quad \alpha_n>0,$$
$$\beta_n\ge\beta_{n-1}\ge\cdots\ge\beta_1\ge\beta_0\ge 0,$$
$$\gamma_n\ge\gamma_{n-1}\ge\cdots\ge\gamma_1\ge\gamma_0\ge 0,$$
$$\delta_n\ge\delta_{n-1}\ge\cdots\ge\delta_1\ge\delta_0\ge 0,$$

where $\lambda\ge 0$, then all the zeros of $T(q)$ lie in

$$\left|q+\frac{\lambda}{a_n}\right|\le 2+\frac{\lambda}{|a_n|}.$$

Remark 16. For $\lambda=0$, Corollary 15 represents an extension of Theorem 6 to quaternionic coefficients.

Next, we shall obtain the following more general result giving a ring shaped region containing all the zeros of a quaternionic polynomial with quaternionic coefficients. The interest of this theorem lies in its flexibility. It weakens the hypothesis of Theorem 8 and is applicable to a larger class of polynomials of a quaternionic variable.

Theorem 17. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n with quaternionic coefficients, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\dots,n$, satisfying*

$$\alpha_n\le\alpha_{n-1}\le\cdots\le\alpha_{\lambda+1}\le\alpha_\lambda\ge\alpha_{\lambda-1}\ge\cdots\ge\alpha_1\ge\alpha_0,$$

where $0\le\lambda\le n$, then all the zeros of $T(q)$ lie in

$$\min(1,R_1)\le|q|\le\max(1,R_2),$$

where

$$R_1=\frac{|a_0|}{2\alpha_\lambda+|a_n|-\alpha_n-\alpha_0+M_1},$$
$$R_2=\frac{2\alpha_\lambda-\alpha_n-\alpha_0+|a_0|+M_1}{|a_n|},$$
$$\text{and}\quad M_1=\sum_{v=1}^{n}\left(|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\right).$$

If in Theorem 17, we take $\lambda=n$ and assume $\beta_v\ge\beta_{v-1},\gamma_v\ge\gamma_{v-1},\delta_v\ge\delta_{v-1}$ for $v=1,2,\dots,n$, we get the following result.

Corollary 18. *If $T(q)=\sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n with quaternionic coefficients, where $a_v=\alpha_v+\beta_v i+\gamma_v j+\delta_v k$ for $v=0,1,2,\dots,n$, satisfying*

$$\alpha_n\ge\alpha_{n-1}\ge\cdots\ge\alpha_1\ge\alpha_0,$$
$$\beta_n\ge\beta_{n-1}\ge\cdots\ge\beta_1\ge\beta_0,$$
$$\gamma_n\ge\gamma_{n-1}\ge\cdots\ge\gamma_1\ge\gamma_0,$$
$$\delta_n\ge\delta_{n-1}\ge\cdots\ge\delta_1\ge\delta_0,$$

then all the zeros of $T(q)$ lie in

$$r_1\le|q|\le r_2,$$

where

$$r_1=\frac{|a_0|}{(\alpha_n+\beta_n+\gamma_n+\delta_n)+|a_n|-(\alpha_0+\beta_0+\gamma_0+\delta_0)},$$
$$r_2=\frac{|a_0|-(\alpha_0+\beta_0+\gamma_0+\delta_0)+(\alpha_n+\beta_n+\gamma_n+\delta_n)}{|a_n|}.$$

Remark 19. The above Corollary 18 subsumes a result of Carney et al. ([1, Theorem 9]).

In particular, if all the coefficients a_v, $0 \le v \le n$, are real, that is, $\beta_v = \gamma_v = \delta_v = 0$ for $0 \le v \le n$, the above Corollary 18 gives that all the zeros of $T(q)$ lie in

$$\frac{|a_0|}{a_n + |a_n| - a_0} \le |q| \le \frac{|a_0| - a_0 + a_n}{|a_n|},$$

which is a refinement of Theorem 8. It also strengthens another result of Tripathi ([16, Corollary 3.8]). Further, if $a_0 > 0$, then it implies a refinement to Theorem 6. By taking $\lambda = 0$ and assume $\beta_v \le \beta_{v-1}, \gamma_v \le \gamma_{v-1}, \delta_v \le \delta_{v-1}$ for $v = 1,2,\dots,n$, in Theorem 17, we get the following result which is also of interest.

Corollary 20. *If $T(q) = \sum_{v=0}^{n} q^v a_v$, is a polynomial of degree n with quaternionic coefficients, where $a_v = \alpha_v + \beta_v i + \gamma_v j + \delta_v k$ for $v = 0,1,2,\dots,n$, satisfying*

$$\begin{aligned}
\alpha_0 &\ge \alpha_1 \ge \cdots \ge \alpha_n,\\
\beta_0 &\ge \beta_1 \ge \cdots \ge \beta_n,\\
\gamma_0 &\ge \gamma_1 \ge \cdots \ge \gamma_n,\\
\delta_0 &\ge \delta_1 \ge \cdots \ge \delta_n,
\end{aligned}$$

then all the zeros of $T(q)$ lie in

$$r_3 \le |q| \le r_4,$$

where

$$\begin{aligned}
r_3 &= \frac{|a_0|}{|a_n| + (\alpha_0 + \beta_0 + \gamma_0 + \delta_0) - (\alpha_n + \beta_n + \gamma_n + \delta_n)},\\
r_4 &= \frac{|a_0| + (\alpha_0 + \beta_0 + \gamma_0 + \delta_0) - (\alpha_n + \beta_n + \gamma_n + \delta_n)}{|a_n|}.
\end{aligned}$$

4. Proofs of the main results

Proof of Theorem 9. Consider the polynomial

$$\begin{aligned}
T(q) * (1-q) &= a_0 + q(a_1 - a_0) + q^2(a_2 - a_1) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1} a_n\\
&= \sum_{v=0}^{n-1} q^v (a_v - a_{v-1}) + q^n(\lambda + a_n - a_{n-1}) - q^n \lambda - q^{n+1} a_n, \quad (a_{-1} = 0)\\
&= f(q) - q^n \lambda - q^{n+1} a_n,\\
&\qquad \text{where } f(q) = \sum_{v=0}^{n-1} q^v (a_v - a_{v-1}) + q^n(\lambda + a_n - a_{n-1})\\
&= \alpha_0 - q\mu + q(\alpha_1 + \mu - \alpha_0) + \sum_{v=2}^{n-1} q^v(\alpha_v - \alpha_{v-1}) + q^n(\lambda + \alpha_n - \alpha_{n-1})\\
&\quad + \sum_{v=0}^{n} q^v \Big\{(\beta_v - \beta_{v-1})i + (\gamma_v - \gamma_{v-1})j + (\delta_v - \delta_{v-1})k\Big\}, \quad (\beta_{-1} = \gamma_{-1} = \delta_{-1} = 0).
\end{aligned}$$

By Theorem 4, $T(q) * (1-q) = 0$ if and only if either $T(q) = 0$ or $T(q) \ne 0$ implies $T(q)^{-1} q T(q) - 1 = 0$, that is $T(q)^{-1} q T(q) = 1$. Thus, if $T(q) \ne 0$, this implies $q = 1$, so the only zeros of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$.

For $|q|=1$, we have

$$
\begin{aligned}
|f(q)| &\le |\alpha_0|+|q\mu|+|q(\alpha_1+\mu-\alpha_0)|+\sum_{v=2}^{n-1}\left|q^v(\alpha_v-\alpha_{v-1})\right|+|q^n(\lambda+\alpha_n-\alpha_{n-1})| \\
&\quad+\sum_{v=0}^{n}\left|q^v\left\{(\beta_v-\beta_{v-1})i+(\gamma_v-\gamma_{v-1})j+(\delta_v-\delta_{v-1})k\right\}\right| \\
&\le \alpha_0+\mu+\alpha_1+\mu-\alpha_0+\sum_{v=2}^{n-1}(\alpha_v-\alpha_{v-1})+(\lambda+\alpha_n-\alpha_{n-1})+M_0 \\
&= \alpha_n+\lambda+2\mu+M_0,
\end{aligned}
$$

$$\text{where } M_0=\sum_{v=0}^{n}\left[|\beta_v-\beta_{v-1}|+|\gamma_v-\gamma_{v-1}|+|\delta_v-\delta_{v-1}|\right].$$

Notice that, we have

$$\max_{|q|=1}\left|q^n*f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}\left|q^nf\left(\frac{1}{q}\right)\right|=\max_{|q|=1}\left|f\left(\frac{1}{q}\right)\right|=\max_{|q|=1}|f(q)|,$$

it is clear that $q^n*f\left(\frac{1}{q}\right)$ has the same bound on $|q|=1$ as f, that is

$$\left|q^n*f\left(\frac{1}{q}\right)\right|\le \alpha_n+\lambda+2\mu+M_0 \quad \text{for } |q|=1.$$

Since $q^n*f\left(\frac{1}{q}\right)$ is a polynomial and so is regular in $|q|\le 1$, it follows by the Maximum Modulus Theorem (Theorem 5), that

$$\left|q^n*f\left(\frac{1}{q}\right)\right|=\left|q^nf\left(\frac{1}{q}\right)\right|\le \alpha_n+\lambda+2\mu+M_0 \quad \text{for } |q|\le 1.$$

Hence

$$\left|f\left(\frac{1}{q}\right)\right|\le\frac{\alpha_n+\lambda+2\mu+M_0}{|q|^n} \quad \text{for } |q|\le 1.$$

Replacing q by $\frac{1}{q}$, we see that

$$|f(q)|\le(\alpha_n+\lambda+2\mu+M_0)|q|^n \quad \text{for } |q|\ge 1. \tag{1}$$

For $|q|\ge 1$, we have

$$
\begin{aligned}
|T(q)*(1-q)| &= |f(q)-q^n\lambda-q^{n+1}a_n| \\
&\ge |q^n||qa_n+\lambda|-|f(q)| \\
&\ge |q|^n[|qa_n+\lambda|-(\alpha_n+\lambda+2\mu+M_0)] \quad \text{(by (1))}.
\end{aligned}
$$

Hence, if

$$\left|q+\frac{\lambda}{a_n}\right|>\frac{\alpha_n+\lambda+2\mu+M_0}{|a_n|},$$

then $|T(q)*(1-q)|>0$, that is $T(q)*(1-q)\neq 0$. Since the only zeros of $T(q)*(1-q)$ are $q=1$ and the zeros of $T(q)$, therefore, $T(q)\neq 0$ for

$$\left|q+\frac{\lambda}{a_n}\right|>\frac{\alpha_n+\lambda+2\mu+M_0}{|a_n|}.$$

In other words, all the zeros of $T(q)$ lie in

$$\left|q+\frac{\lambda}{a_n}\right|\le\frac{\alpha_n+\lambda+2\mu+M_0}{|a_n|}.$$

This completes the proof of Theorem 9. □

Proof of Corollary 15. By using the given hypothesis, it is easy to see that

$$M_0 = \sum_{v=0}^{n} \left[|\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\right] = \beta_n + \gamma_n + \delta_n.$$

Also, since $|a_n| = \sqrt{\alpha_n^2 + \beta_n^2 + \gamma_n^2 + \delta_n^2}$, and by the Cauchy–Schwarz inequality, it follows that

$$\frac{\alpha_n + \beta_n + \gamma_n + \delta_n}{\sqrt{\alpha_n^2 + \beta_n^2 + \gamma_n^2 + \delta_n^2}} \le 2.$$

Using $\mu = 0$ and the fact that $\alpha_v, \beta_v, \gamma_v, \delta_v \ge 0$, for $v = 0, 1, \dots, n$, in Theorem 9, we get as claimed in Corollary 15. □

Proof of Theorem 17. Consider the polynomial

$$\begin{aligned} T(q) * (1-q) &= a_0 + q(a_1 - a_0) + \cdots + q^n(a_n - a_{n-1}) - q^{n+1}a_n \\ &= \phi(q) - q^{n+1}a_n, \\ &\quad \text{where} \quad \phi(q) = a_0 + q(a_1 - a_0) + \cdots + q^n(a_n - a_{n-1}). \end{aligned}$$

By Theorem 4, $T(q) * (1-q) = 0$ if and only if either $T(q) = 0$ or $T(q) \neq 0$ implies $T(q)^{-1}qT(q) - 1 = 0$, that is $T(q)^{-1}qT(q) = 1$. Thus, if $T(q) \neq 0$, this implies $q = 1$, so the only zero of $T(q) * (1-q)$ are $q = 1$ and the zeros of $T(q)$. We first note that

$$\begin{aligned} |a_v - a_{v-1}| &= |(\alpha_v - \alpha_{v-1}) + (\beta_v - \beta_{v-1})i + (\gamma_v - \gamma_{v-1})j + (\delta_v - \delta_{v-1})k| \\ &\le |\alpha_v - \alpha_{v-1}| + |\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|. \end{aligned}$$

For $|q| = 1$, we have

$$\begin{aligned} |\phi(q)| &= \left|a_0 + \sum_{v=1}^{n} q^v(a_v - a_{v-1})\right| \\ &\le |a_0| + \sum_{v=1}^{n} |a_v - a_{v-1}| \\ &= |a_0| + \sum_{v=1}^{\lambda} (\alpha_v - \alpha_{v-1}) + \sum_{v=\lambda+1}^{n} (\alpha_{v-1} - \alpha_v) \\ &\quad + \sum_{v=1}^{n} \left(|\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\right) \\ &= |a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1 \\ &\quad \text{where } M_1 = \sum_{v=1}^{n} \left(|\beta_v - \beta_{v-1}| + |\gamma_v - \gamma_{v-1}| + |\delta_v - \delta_{v-1}|\right). \end{aligned}$$

Notice that, we have

$$\max_{|q|=1}\left|q^n * \phi\left(\frac{1}{q}\right)\right| = \max_{|q|=1}\left|q^n \phi\left(\frac{1}{q}\right)\right| = \max_{|q|=1}\left|\phi\left(\frac{1}{q}\right)\right| = \max_{|q|=1}|\phi(q)|,$$

it is clear that $q^n * \phi\left(\frac{1}{q}\right)$ has the same bound on $|q| = 1$ as ϕ, that is

$$\left|q^n * \phi\left(\frac{1}{q}\right)\right| \le |a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1 \quad \text{for} \quad |q| = 1.$$

Therefore, by the same reasoning as in the proof of Theorem 9, it follows by the Maximum Modulus Theorem (Theorem 5), that

$$\left|q^n * \phi\left(\frac{1}{q}\right)\right| = \left|q^n \phi\left(\frac{1}{q}\right)\right| \le |a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1 \quad \text{for} \quad |q| \le 1.$$

Hence

$$\left|\phi\left(\frac{1}{q}\right)\right| \le \frac{|a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1}{|q|^n} \text{ for } |q| \le 1.$$

Replacing q by $\frac{1}{q}$, we see that

$$|\phi(q)| \le (|a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1)|q|^n \text{ for } |q| \ge 1. \tag{2}$$

For $|q| \ge 1$, we have

$$\begin{aligned} |T(q) * (1-q)| &= |\phi(q) - q^{n+1} a_n| \\ &\ge |q|^{n+1}|a_n| - |\phi(q)| \\ &\ge |q|^n \left[|q||a_n| - (|a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1)\right] \text{ (by (2))}. \end{aligned}$$

Hence, if $|q| \ge 1$ and

$$|q| > \frac{|a_0| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1}{|a_n|},$$

then $|T(q) * (1-q)| > 0$, that is $T(q) * (1-q) \neq 0$. Hence all the zeros of $T(q) * (1-q)$ whose norm is greater than or equal to one lie in

$$|q| \le \frac{2\alpha_\lambda - \alpha_n + |a_0| - \alpha_0 + M_1}{|a_n|}.$$

In other words, all the zeros of $T(q)$ lie in

$$|q| \le \max\left(1, \frac{2\alpha_\lambda - \alpha_n + |a_0| - \alpha_0 + M_1}{|a_n|}\right).$$

For the inner bound, consider the reciprocal polynomial

$$\begin{aligned} P(q) &= q^n * T\left(\frac{1}{q}\right) \\ &= q^n T\left(\frac{1}{q}\right) \\ &= q^n a_0 + q^{n-1} a_1 + \cdots + q a_{n-1} + a_n, \end{aligned}$$

and consider

$$\begin{aligned} P(q) * (1-q) &= a_n + q(a_{n-1} - a_n) + q^2(a_{n-2} - a_{n-1}) + \cdots + q^n(a_0 - a_1) - q^{n+1} a_0 \\ &= \psi(q) - q^{n+1} a_0, \end{aligned}$$

$$\text{where } \psi(q) = a_n + \sum_{v=1}^{n} q^{n-v+1}(a_{v-1} - a_v).$$

For $|q| = 1$, we have

$$\begin{aligned} |\psi(q)| &\le |a_n| + \sum_{v=1}^{n} |q^{n-v+1}(a_{v-1} - a_v)| \\ &\le |a_n| + \sum_{v=1}^{n} \left(|\alpha_{v-1} - \alpha_v| + |\beta_{v-1} - \beta_v| + |\gamma_{v-1} - \gamma_v| + |\delta_{v-1} - \delta_v|\right) \\ &= |a_n| + 2\alpha_\lambda - \alpha_0 - \alpha_n + M_1. \end{aligned}$$

Proceeding as in the first part of this theorem, we get for $|q| \ge 1$,

$$|\psi(q)| \le (2\alpha_\lambda + |a_n| - \alpha_n - \alpha_0 + M_1)|q|^n. \tag{3}$$

For $|q| \geq 1$, we have

$$\begin{aligned}|P(q)*(1-q)| &= |\psi(q) - q^{n+1}a_0| \\ &\geq |q|^{n+1}|a_0| - |\psi(q)| \\ &\geq |q|^n\left[|q||a_0| - (2\alpha_\lambda + |a_n| - \alpha_n - \alpha_0 + M_1)\right] \quad \text{(by (3))}.\end{aligned}$$

Hence, if $|q| \geq 1$ and

$$|q| > \frac{2\alpha_\lambda + |a_n| - \alpha_n - \alpha_0 + M_1}{|a_0|},$$

then $|P(q)*(1-q)| > 0$, that is $P(q)*(1-q) \neq 0$. Hence all the zeros of $P(q)*(1-q)$ where norm is greater than or equal to one lie in

$$|q| \leq \frac{2\alpha_\lambda + |a_n| - \alpha_n - \alpha_0 + M_1}{|a_0|},$$

i.e, all the zeros of $P(q)$ lie in

$$|q| \leq \max\left(1, \frac{2\alpha_\lambda + |a_n| - \alpha_n - \alpha_0 + M_1}{|a_0|}\right).$$

Therefore all the zeros of $T(q)$ lie in

$$|q| \geq \min\left(1, \frac{|a_0|}{2\alpha_\lambda + |a_n| - \alpha_n - \alpha_0 + M_1}\right).$$

This completes the proof of Theorem 17. □

Conclusion

The regular functions of a quaternionic variable have been introduced and intensively studied since 2006, and they have proven to be a fertile topic in analysis, and their rapid development has been largely driven by the applications to operator theory. We point out that the Fundamental Theorem of Algebra for regular polynomials with coefficients in $\mathbb{H}$ led to the complete identification of the zeros of polynomials in terms of their factorization. Thus it became an interesting perspective to think about the regions containing some or all the zeros of a regular polynomial of quaternionic variable. In the literature, we could not find much about the distribution of zeros of polynomials with quaternionic variable and quaternionic coefficients. Here, we obtain annuli containing all the zeros of a regular polynomial of quaternionic variable when the real and imaginary parts of its coefficients are restricted by virtue of a maximum modulus theorem and the structure of the zero sets for regular functions.

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